The adiabatic limit of wave-map flow

Martin Speight[1]

November 5, 2009

[1] Joint work with Mark Haskins

Harmonic maps

$$\varphi : M \to N \subset \mathbb{R}^k, \qquad E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2$$

$$(\Delta\varphi)(x) \perp T_{\varphi(x)}N$$

- Let's choose $N = S^2 \subset \mathbb{R}^3$ hereafter:

$$\Delta\varphi - (\varphi \cdot \Delta\varphi)\varphi = 0$$

- $\varphi : M \to S^2$ (M=compact Riemann surface)

$$E(\varphi) \geq 4\pi n, \qquad \text{equality} \iff \varphi \text{ holomorphic}$$

(Belavin-Polyakov-Liechnerowicz)

Harmonic maps

$$\varphi : M \to N \subset \mathbb{R}^k, \qquad E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2$$

$$(\Delta\varphi)(x) \perp T_{\varphi(x)}N$$

- Let's choose $N = S^2 \subset \mathbb{R}^3$ hereafter:

$$\Delta\varphi - (\varphi \cdot \Delta\varphi)\varphi = 0$$

- $\varphi : M \to S^2$ (M=compact Riemann surface)

$$E(\varphi) \geq 4\pi n, \quad \text{equality} \iff \varphi \text{ holomorphic}$$

(Belavin-Polyakov-Liechnerowicz)

Harmonic maps

$$\varphi : M \to N \subset \mathbb{R}^k, \qquad E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2$$

$$(\Delta\varphi)(x) \perp T_{\varphi(x)}N$$

- Let's choose $N = S^2 \subset \mathbb{R}^3$ hereafter:

$$\Delta\varphi - (\varphi \cdot \Delta\varphi)\varphi = 0$$

- $\varphi : M \to S^2$ (M=compact Riemann surface)

$$E(\varphi) \geq 4\pi n, \quad \text{equality} \iff \varphi \text{ holomorphic}$$

(Belavin-Polyakov-Liechnerowicz)

Wave maps

Domain Lorentzian ($E(\varphi)$ now $S(\varphi)$).

Let's choose $(M, \eta) = (\mathbb{R} \times \Sigma, dt^2 - g_\Sigma)$

$$S(\varphi) = \int_{\mathbb{R}} dt \left\{ \frac{1}{2} \int_\Sigma |\varphi_t|^2 - \frac{1}{2} \int_\Sigma |d\varphi|^2 \right\}$$

$$(\Box\varphi)(t,x) \perp T_{\varphi(t,x)}N \qquad \text{for all } (t,x) \in M$$

where $\Box = \partial_t^2 - \Delta_\Sigma$.

- Obviously, static wave maps are harmonic maps $\Sigma \to N$
- $\dim \Sigma = 1$, wave map equation is an integrable system
- $\dim \Sigma \geq 3$, nonintegrable but well understood analytically. Good model of blow-up.

Wave maps

Domain Lorentzian ($E(\varphi)$ now $S(\varphi)$).
Let's choose $(M,\eta) = (\mathbb{R} \times \Sigma, dt^2 - g_\Sigma)$

$$S(\varphi) = \int_{\mathbb{R}} dt \left\{ \frac{1}{2} \int_\Sigma |\varphi_t|^2 - \frac{1}{2} \int_\Sigma |\mathrm{d}\varphi|^2 \right\}$$

$$(\Box\varphi)(t,x) \perp T_{\varphi(t,x)} N \qquad \text{for all } (t,x) \in M$$

where $\Box = \partial_t^2 - \Delta_\Sigma$.

- Obviously, static wave maps are harmonic maps $\Sigma \longrightarrow N$
- $\dim \Sigma = 1$, wave map equation is an integrable system
- $\dim \Sigma \geq 3$, nonintegrable but well understood analytically. Good model of blow-up.

Wave maps

Domain Lorentzian ($E(\varphi)$ now $S(\varphi)$).
Let's choose $(M,\eta) = (\mathbb{R}\times\Sigma, dt^2 - g_\Sigma)$

$$S(\varphi) = \int_{\mathbb{R}} dt \left\{ \frac{1}{2}\int_\Sigma |\varphi_t|^2 - \frac{1}{2}\int_\Sigma |\mathrm{d}\varphi|^2 \right\}$$

$$(\Box\varphi)(t,x) \perp T_{\varphi(t,x)}N \qquad \text{for all } (t,x)\in M$$

where $\Box = \partial_t^2 - \Delta_\Sigma$.

- Obviously, static wave maps are harmonic maps $\Sigma \to N$
- $\dim\Sigma = 1$, wave map equation is an integrable system
- $\dim\Sigma \geq 3$, nonintegrable but well understood analytically. Good model of blow-up.

Wave maps

Domain Lorentzian ($E(\varphi)$ now $S(\varphi)$).
Let's choose $(M,\eta) = (\mathbb{R}\times\Sigma, dt^2 - g_\Sigma)$

$$S(\varphi) = \int_{\mathbb{R}} dt \left\{ \frac{1}{2}\int_\Sigma |\varphi_t|^2 - \frac{1}{2}\int_\Sigma |\mathrm{d}\varphi|^2 \right\}$$

$$(\Box\varphi)(t,x) \perp T_{\varphi(t,x)}N \qquad \text{for all } (t,x)\in M$$

where $\Box = \partial_t^2 - \Delta_\Sigma$.

- Obviously, static wave maps are harmonic maps $\Sigma \to N$
- $\dim\Sigma = 1$, wave map equation is an integrable system
- $\dim\Sigma \geq 3$, nonintegrable but well understood analytically. Good model of blow-up.

Wave maps

Domain Lorentzian ($E(\varphi)$ now $S(\varphi)$).
Let's choose $(M,\eta) = (\mathbb{R}\times\Sigma, dt^2 - g_\Sigma)$

$$S(\varphi) = \int_{\mathbb{R}} dt \left\{ \frac{1}{2}\int_\Sigma |\varphi_t|^2 - \frac{1}{2}\int_\Sigma |\mathrm{d}\varphi|^2 \right\}$$

$$(\Box\varphi)(t,x) \perp T_{\varphi(t,x)}N \qquad \text{for all } (t,x)\in M$$

where $\Box = \partial_t^2 - \Delta_\Sigma$.

- Obviously, static wave maps are harmonic maps $\Sigma \to N$
- $\dim\Sigma = 1$, wave map equation is an integrable system
- $\dim\Sigma \geq 3$, nonintegrable but well understood analytically. Good model of blow-up.

Wave maps

- $\dim \Sigma = 2$, most interesting (and hardest) case for nonlinear analysts. Blow-up problem very hard (Rodnianski and Sterbenz).
- Also most interesting to (my sort of) theoretical physicists: Have large families of static solutions

$$M_n = \mathrm{hol}_n(\Sigma, S^2)$$

which saturate a topological energy bound, and satisfy a "Bogomolnyi" equation

$$\varphi_y = \varphi \times \varphi_x$$

Topological "solitons" (cf monopoles, vortices, instantons. . .)

Wave maps

- $\dim \Sigma = 2$, most interesting (and hardest) case for nonlinear analysts. Blow-up problem very hard (Rodnianski and Sterbenz).
- Also most interesting to (my sort of) theoretical physicists: Have large families of static solutions

$$\mathsf{M}_n = \mathsf{hol}_n(\Sigma, S^2)$$

which saturate a topological energy bound, and satisfy a "Bogomolnyi" equation

$$\varphi_y = \varphi \times \varphi_x$$

Topological "solitons" (cf monopoles, vortices, instantons. . .)

- $\dim\Sigma = 2$, most interesting (and hardest) case for nonlinear analysts. Blow-up problem very hard (Rodnianski and Sterbenz).
- Also most interesting to (my sort of) theoretical physicists: Have large families of static solutions

$$\mathsf{M}_n = \mathrm{hol}_n(\Sigma, S^2)$$

which saturate a topological energy bound, and satisfy a "Bogomolnyi" equation

$$\varphi_y = \varphi \times \varphi_x$$

Topological "solitons" (cf monopoles, vortices, instantons. . .)

Geodesic approximation (Ward, after Manton)

- Wave map flow conserves $E_{\text{total}} = \frac{1}{2}\int_\Sigma |\varphi_t|^2 + E(\varphi(t))$
- Cauchy problem: $\varphi(0) \in \mathsf{M}_n$, $\varphi_t(0) \in T_{\varphi(0)}\mathsf{M}_n$, small
- $E_{\text{total}}(t) = 4\pi n +$ small for all time: expect $\varphi(t)$ remains close (e.g. in H^1 norm) to M_n for all time.
- Consider much simpler **constrained** variational problem for S, where $\psi(t) \in \mathsf{M}_n$ for all t:

$$S|_{T\mathsf{M}_n} = \int dt \{\frac{1}{2}\int_\Sigma |\psi_t|^2 - 4\pi n\}$$

$\psi(t)$ follows a *geodesic* in (M_n, γ)

$$\gamma(X, Y) = \int_\Sigma X \cdot Y, \qquad X, Y \in T_\psi \mathsf{M}_n \subset \psi^{-1} TS^2.$$

- **Conjecture:** $\psi(t)$ with initial data $\psi(0) = \varphi(0)$, $\psi_t(0) = \varphi_t(0)$ is a “good approximation” to wave map $\varphi(t)$

Geodesic approximation (Ward, after Manton)

- Wave map flow conserves $E_{\text{total}} = \frac{1}{2}\int_\Sigma |\varphi_t|^2 + E(\varphi(t))$
- Cauchy problem: $\varphi(0) \in \mathsf{M}_n$, $\varphi_t(0) \in T_{\varphi(0)}\mathsf{M}_n$, small
- $E_{\text{total}}(t) = 4\pi n +$ small for all time: expect $\varphi(t)$ remains close (e.g. in H^1 norm) to M_n for all time.
- Consider much simpler **constrained** variational problem for S, where $\psi(t) \in \mathsf{M}_n$ for all t:

$$S|_{T\mathsf{M}_n} = \int dt \{\frac{1}{2}\int_\Sigma |\psi_t|^2 - 4\pi n\}$$

$\psi(t)$ follows a *geodesic* in (M_n, γ)

$$\gamma(X, Y) = \int_\Sigma X \cdot Y, \qquad X, Y \in T_\psi \mathsf{M}_n \subset \psi^{-1} TS^2.$$

- **Conjecture:** $\psi(t)$ with initial data $\psi(0) = \varphi(0)$, $\psi_t(0) = \varphi_t(0)$ is a “good approximation” to wave map $\varphi(t)$

Geodesic approximation (Ward, after Manton)

- Wave map flow conserves $E_{\text{total}} = \frac{1}{2}\int_\Sigma |\varphi_t|^2 + E(\varphi(t))$
- Cauchy problem: $\varphi(0) \in \mathsf{M}_n$, $\varphi_t(0) \in T_{\varphi(0)}\mathsf{M}_n$, small
- $E_{\text{total}}(t) = 4\pi n + \text{small}$ for all time: expect $\varphi(t)$ remains close (e.g. in H^1 norm) to M_n for all time.
- Consider much simpler **constrained** variational problem for S, where $\psi(t) \in \mathsf{M}_n$ for all t:

$$S|_{T\mathsf{M}_n} = \int dt \{\frac{1}{2}\int_\Sigma |\psi_t|^2 - 4\pi n\}$$

  $\psi(t)$ follows a *geodesic* in (M_n, γ)

$$\gamma(X, Y) = \int_\Sigma X \cdot Y, \qquad X, Y \in T_\psi \mathsf{M}_n \subset \psi^{-1} TS^2.$$

- **Conjecture:** $\psi(t)$ with initial data $\psi(0) = \varphi(0)$, $\psi_t(0) = \varphi_t(0)$ is a "good approximation" to wave map $\varphi(t)$

Geodesic approximation (Ward, after Manton)

- Wave map flow conserves $E_{\text{total}} = \frac{1}{2}\int_\Sigma |\varphi_t|^2 + E(\varphi(t))$
- Cauchy problem: $\varphi(0) \in \mathsf{M}_n$, $\varphi_t(0) \in T_{\varphi(0)}\mathsf{M}_n$, small
- $E_{\text{total}}(t) = 4\pi n + \text{small}$ for all time: expect $\varphi(t)$ remains close (e.g. in H^1 norm) to M_n for all time.
- Consider much simpler **constrained** variational problem for S, where $\psi(t) \in \mathsf{M}_n$ for all t:

$$S|_{T\mathsf{M}_n} = \int dt \{\frac{1}{2}\int_\Sigma |\psi_t|^2 - 4\pi n\}$$

$\psi(t)$ follows a *geodesic* in (M_n, γ)

$$\gamma(X, Y) = \int_\Sigma X \cdot Y, \qquad X, Y \in T_\psi \mathsf{M}_n \subset \psi^{-1} TS^2.$$

- **Conjecture:** $\psi(t)$ with initial data $\psi(0) = \varphi(0)$, $\psi_t(0) = \varphi_t(0)$ is a “good approximation” to wave map $\varphi(t)$

Geodesic approximation (Ward, after Manton)

- Wave map flow conserves $E_{\mathrm{total}} = \frac{1}{2}\int_\Sigma |\varphi_t|^2 + E(\varphi(t))$
- Cauchy problem: $\varphi(0) \in \mathsf{M}_n$, $\varphi_t(0) \in T_{\varphi(0)}\mathsf{M}_n$, small
- $E_{\mathrm{total}}(t) = 4\pi n + \mathrm{small}$ for all time: expect $\varphi(t)$ remains close (e.g. in H^1 norm) to M_n for all time.
- Consider much simpler **constrained** variational problem for S, where $\psi(t) \in \mathsf{M}_n$ for all t:

$$S|_{T\mathsf{M}_n} = \int dt \{ \frac{1}{2} \int_\Sigma |\psi_t|^2 - 4\pi n \}$$

$\psi(t)$ follows a *geodesic* in (M_n, γ)

$$\gamma(X, Y) = \int_\Sigma X \cdot Y, \qquad X, Y \in T_\psi \mathsf{M}_n \subset \psi^{-1} TS^2.$$

- **Conjecture:** $\psi(t)$ with initial data $\psi(0) = \varphi(0)$, $\psi_t(0) = \varphi_t(0)$ is a "good approximation" to wave map $\varphi(t)$

Geodesic approximation (Ward, after Manton)

- Wave map flow conserves $E_{\mathrm{total}} = \frac{1}{2}\int_\Sigma |\varphi_t|^2 + E(\varphi(t))$
- Cauchy problem: $\varphi(0) \in \mathsf{M}_n$, $\varphi_t(0) \in T_{\varphi(0)}\mathsf{M}_n$, small
- $E_{\mathrm{total}}(t) = 4\pi n + \mathrm{small}$ for all time: expect $\varphi(t)$ remains close (e.g. in H^1 norm) to M_n for all time.
- Consider much simpler **constrained** variational problem for S, where $\psi(t) \in \mathsf{M}_n$ for all t:

$$S|_{T\mathsf{M}_n} = \int dt \{\frac{1}{2}\int_\Sigma |\psi_t|^2 - 4\pi n\}$$

$\psi(t)$ follows a *geodesic* in (M_n, γ)

$$\gamma(X,Y) = \int_\Sigma X \cdot Y, \qquad X, Y \in T_\psi \mathsf{M}_n \subset \psi^{-1} TS^2.$$

- **Conjecture:** $\psi(t)$ with initial data $\psi(0) = \varphi(0)$, $\psi_t(0) = \varphi_t(0)$ is a "good approximation" to wave map $\varphi(t)$

Precise conjecture

Consider the following one-parameter family of Cauchy problems for the wave map flow $\mathbb{R} \times \Sigma \to S^2$:

$$\varphi(0) = \varphi_0, \qquad \varphi_t(0) = \varepsilon\varphi_1$$

where $\varphi_0 \in \mathsf{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathsf{M}_n$ and $\varepsilon > 0$.
There exist $T > 0$ and $\varepsilon_* > 0$ (depending on (φ_0, φ_1)) such that, for all $\varepsilon \in (0, \varepsilon_*]$, this Cauchy problem has a unique solution for $t \in [0, T/\varepsilon]$. Furthermore, the time re-scaled solution

$$\varphi_\varepsilon : [0, T] \times \Sigma \to S^2, \qquad \varphi_\varepsilon(\tau, x) = \varphi(\tau/\varepsilon, x)$$

converges uniformly in C^0 norm to $\psi : [0, T] \times \Sigma \to S^2$, the geodesic in M_n with the same initial data.

- Loosely: the geodesic approximation "works" for times of order $1/\varepsilon$ when the initial velocities are of order ε
- Can't do much better: (M_n, γ) incomplete (Sadun, JMS)!

Precise conjecture

Consider the following one-parameter family of Cauchy problems for the wave map flow $\mathbb{R} \times \Sigma \to S^2$:

$$\varphi(0) = \varphi_0, \qquad \varphi_t(0) = \varepsilon\varphi_1$$

where $\varphi_0 \in \mathsf{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathsf{M}_n$ and $\varepsilon > 0$.
There exist $T > 0$ and $\varepsilon_* > 0$ (depending on (φ_0, φ_1)) such that, for all $\varepsilon \in (0, \varepsilon_*]$, this Cauchy problem has a unique solution for $t \in [0, T/\varepsilon]$.
Furthermore, the time re-scaled solution

$$\varphi_\varepsilon : [0, T] \times \Sigma \to S^2, \qquad \varphi_\varepsilon(\tau, x) = \varphi(\tau/\varepsilon, x)$$

converges uniformly in C^0 norm to $\psi : [0, T] \times \Sigma \to S^2$, the geodesic in M_n with the same initial data.

- Loosely: the geodesic approximation "works" for times of order $1/\varepsilon$ when the initial velocities are of order ε
- Can't do much better: (M_n, γ) incomplete (Sadun, JMS)!

Precise conjecture

Consider the following one-parameter family of Cauchy problems for the wave map flow $\mathbb{R} \times \Sigma \to S^2$:

$$\varphi(0) = \varphi_0, \qquad \varphi_t(0) = \varepsilon\varphi_1$$

where $\varphi_0 \in \mathrm{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathrm{M}_n$ and $\varepsilon > 0$.
There exist $T > 0$ and $\varepsilon_* > 0$ (depending on (φ_0, φ_1)) such that, for all $\varepsilon \in (0, \varepsilon_*]$, this Cauchy problem has a unique solution for $t \in [0, T/\varepsilon]$.
Furthermore, the time re-scaled solution

$$\varphi_\varepsilon : [0, T] \times \Sigma \to S^2, \qquad \varphi_\varepsilon(\tau, x) = \varphi(\tau/\varepsilon, x)$$

converges uniformly in C^0 norm to $\psi : [0, T] \times \Sigma \to S^2$, the geodesic in M_n with the same initial data.

- Loosely: the geodesic approximation "works" for times of order $1/\varepsilon$ when the initial velocities are of order ε
- Can't do much better: (M_n, γ) incomplete (Sadun, JMS)!

Precise conjecture

Consider the following one-parameter family of Cauchy problems for the wave map flow $\mathbb{R} \times \Sigma \to S^2$:

$$\varphi(0) = \varphi_0, \qquad \varphi_t(0) = \varepsilon\varphi_1$$

where $\varphi_0 \in \mathrm{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathrm{M}_n$ and $\varepsilon > 0$.
There exist $T > 0$ and $\varepsilon_* > 0$ (depending on (φ_0, φ_1)) such that, for all $\varepsilon \in (0, \varepsilon_*]$, this Cauchy problem has a unique solution for $t \in [0, T/\varepsilon]$.
Furthermore, the time re-scaled solution

$$\varphi_\varepsilon : [0, T] \times \Sigma \to S^2, \qquad \varphi_\varepsilon(\tau, x) = \varphi(\tau/\varepsilon, x)$$

converges uniformly in C^0 norm to $\psi : [0, T] \times \Sigma \to S^2$, the geodesic in M_n with the same initial data.

- Loosely: the geodesic approximation “works” for times of order $1/\varepsilon$ when the initial velocities are of order ε
- Can’t do much better: (M_n, γ) incomplete (Sadun, JMS)!

Precise conjecture

Consider the following one-parameter family of Cauchy problems for the wave map flow $\mathbb{R} \times \Sigma \to S^2$:

$$\varphi(0) = \varphi_0, \qquad \varphi_t(0) = \varepsilon\varphi_1$$

where $\varphi_0 \in \mathrm{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathrm{M}_n$ and $\varepsilon > 0$.
There exist $T > 0$ and $\varepsilon_* > 0$ (depending on (φ_0, φ_1)) such that, for all $\varepsilon \in (0, \varepsilon_*]$, this Cauchy problem has a unique solution for $t \in [0, T/\varepsilon]$. Furthermore, the time re-scaled solution

$$\varphi_\varepsilon : [0, T] \times \Sigma \to S^2, \qquad \varphi_\varepsilon(\tau, x) = \varphi(\tau/\varepsilon, x)$$

converges uniformly in C^0 norm to $\psi : [0, T] \times \Sigma \to S^2$, the geodesic in M_n with the same initial data.

- Loosely: the geodesic approximation "works" for times of order $1/\varepsilon$ when the initial velocities are of order ε
- Can't do much better: (M_n, γ) incomplete (Sadun, JMS)!

Stuart's method

We'll sketch the proof in the case $\Sigma = T^2$. Ingredients:

1. Wave map eqn for $\varphi \leftrightarrow$ coupled ODE/PDE system for $\varphi = \psi + \varepsilon^2 Y$
2. Short time existence and uniqueness theorem for this system (in a suitable Sobolev space)
3. Coercivity of the Hessian (and "higher" Hessian)
4. Energy estimates for $Y(t)$
5. A priori bound

Stuart's method

We'll sketch the proof in the case $\Sigma = T^2$. Ingredients:

1. Wave map eqn for $\varphi \leftrightarrow$ coupled ODE/PDE system for $\varphi = \psi + \varepsilon^2 Y$
2. Short time existence and uniqueness theorem for this system (in a suitable Sobolev space)
3. Coercivity of the Hessian (and "higher" Hessian)
4. Energy estimates for $Y(t)$
5. A priori bound

Stuart's method

We'll sketch the proof in the case $\Sigma = T^2$. Ingredients:

1. Wave map eqn for $\varphi \leftrightarrow$ coupled ODE/PDE system for $\varphi = \psi + \varepsilon^2 Y$
2. Short time existence and uniqueness theorem for this system (in a suitable Sobolev space)
3. Coercivity of the Hessian (and "higher" Hessian)
4. Energy estimates for $Y(t)$
5. A priori bound

Stuart's method

We'll sketch the proof in the case $\Sigma = T^2$. Ingredients:

1. Wave map eqn for $\varphi \leftrightarrow$ coupled ODE/PDE system for $\varphi = \psi + \varepsilon^2 Y$
2. Short time existence and uniqueness theorem for this system (in a suitable Sobolev space)
3. Coercivity of the Hessian (and "higher" Hessian)
4. Energy estimates for $Y(t)$
5. A priori bound

Stuart's method

We'll sketch the proof in the case $\Sigma = T^2$. Ingredients:

1. Wave map eqn for $\varphi \leftrightarrow$ coupled ODE/PDE system for $\varphi = \psi + \varepsilon^2 Y$
2. Short time existence and uniqueness theorem for this system (in a suitable Sobolev space)
3. Coercivity of the Hessian (and "higher" Hessian)
4. Energy estimates for $Y(t)$
5. A priori bound

The moduli space

- Moduli space (stereographic coord on S^2)

$$\psi(z) = \lambda \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}, \qquad \begin{array}{c} \sum a_i = \sum b_i, \\ \{a_i\} \cap \{b_i\} = \emptyset \end{array}$$

$\dim_{\mathbb{C}} \mathsf{M}_n = 2n$

- Choose and fix initial data $\varphi_0 \in \mathsf{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathsf{M}_n$.
- Choose and fix real local coords $q : \mathbb{R}^{4n} \supset U \to \mathsf{M}_n$
  Denote by $\psi(q)$ the h-map corresponding to q.
  Convenient to demand that $\varphi_0 = \psi(0)$ and $U = \mathbb{R}^{4n}$.

The moduli space

- Moduli space (stereographic coord on S^2)

$$\psi(z) = \lambda \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}, \qquad \begin{array}{c}\sum a_i = \sum b_i, \\ \{a_i\}\cap\{b_i\} = \emptyset\end{array}$$

  $\dim_{\mathbb{C}} \mathrm{M}_n = 2n$

- Choose and fix initial data $\varphi_0 \in \mathrm{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathrm{M}_n$.
- Choose and fix real local coords $q : \mathbb{R}^{4n} \supset U \to \mathrm{M}_n$
  Denote by $\psi(q)$ the h-map corresponding to q.
  Convenient to demand that $\varphi_0 = \psi(0)$ and $U = \mathbb{R}^{4n}$.

The moduli space

- Moduli space (stereographic coord on S^2)

$$\psi(z) = \lambda \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)}, \qquad \begin{array}{c} \sum a_i = \sum b_i, \\ \{a_i\} \cap \{b_i\} = \emptyset \end{array}$$

$\dim_{\mathbb{C}} \mathrm{M}_n = 2n$

- Choose and fix initial data $\varphi_0 \in \mathrm{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathrm{M}_n$.
- Choose and fix real local coords $q : \mathbb{R}^{4n} \supset U \to \mathrm{M}_n$
  Denote by $\psi(q)$ the h-map corresponding to q.
  Convenient to demand that $\varphi_0 = \psi(0)$ and $U = \mathbb{R}^{4n}$.

Sobolev spaces

- Sobolev spaces:

$$\begin{aligned}
\mathscr{H}^k &= \{u : \Sigma \to \mathbb{R} \,|\, u \text{ and all partial derivs up to order } k \text{ are in } L^2\} \\
\|u\|_k^2 &= \int_\Sigma u^2 + \sum_{1\le|\alpha\le k} \int_\Sigma (\partial_\alpha u)^2 \\
H^k &= \{Y : \Sigma \to \mathbb{R}^3 \,|\, Y_1, Y_2, Y_3 \in \mathscr{H}^k\} \\
\|Y\|_k^2 &= \|Y_1\|_k^2 + \|Y_2\|_k^2 + \|Y_3\|_k^2
\end{aligned}$$

  Note $H^0 = L^2$.

- Fact: $\mathscr{H}^k$ is a Banach **algebra** for $k \geq 2$, that is,

$$u, v \in \mathscr{H}^k \Rightarrow uv \in \mathscr{H}^k, \qquad \text{and} \qquad \|uv\|_k \leq c\|u\|_k\|v\|_k$$

Sobolev spaces

- Sobolev spaces:

$$
\begin{aligned}
\mathscr{H}^k &= \{u : \Sigma \to \mathbb{R} \mid u \text{ and all partial derivs up to order } k \text{ are in } L^2\} \\
\|u\|_k^2 &= \int_\Sigma u^2 + \sum_{1 \le |\alpha \le k} \int_\Sigma (\partial_\alpha u)^2 \\
H^k &= \{Y : \Sigma \to \mathbb{R}^3 \mid Y_1, Y_2, Y_3 \in \mathscr{H}^k\} \\
\|Y\|_k^2 &= \|Y_1\|_k^2 + \|Y_2\|_k^2 + \|Y_3\|_k^2
\end{aligned}
$$

  Note $H^0 = L^2$.

- Fact: $\mathscr{H}^k$ is a Banach **algebra** for $k \geq 2$, that is,

$$u, v \in \mathscr{H}^k \Rightarrow uv \in \mathscr{H}^k, \qquad \text{and} \qquad \|uv\|_k \leq c\|u\|_k\|v\|_k$$

Sobolev spaces

- Sobolev spaces:

$$
\begin{aligned}
\mathscr{H}^k &= \{u : \Sigma \to \mathbb{R} \mid u \text{ and all partial derivs up to order } k \text{ are in } L^2\} \\
\|u\|_k^2 &= \int_\Sigma u^2 + \sum_{1 \le |\alpha \le k} \int_\Sigma (\partial_\alpha u)^2 \\
H^k &= \{Y : \Sigma \to \mathbb{R}^3 \mid Y_1, Y_2, Y_3 \in \mathscr{H}^k\} \\
\|Y\|_k^2 &= \|Y_1\|_k^2 + \|Y_2\|_k^2 + \|Y_3\|_k^2
\end{aligned}
$$

  Note $H^0 = L^2$.

- Fact: $\mathscr{H}^k$ is a Banach **algebra** for $k \geq 2$, that is,

$$
u, v \in \mathscr{H}^k \Rightarrow uv \in \mathscr{H}^k, \qquad \text{and} \qquad \|uv\|_k \leq c\|u\|_k\|v\|_k
$$

Projection to the moduli space

- Wave map equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} + (|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2)\varphi = 0$$

- Slow time $\tau = \varepsilon t$ (book-keeping device)
- Decompose $\varphi(t) = \psi(q(\tau)) + \varepsilon^2 Y(t)$.
  - **Section:** map $Z : \Sigma \to \mathbb{R}^3$
  - **Tangent section:** $Z : \Sigma \to \mathbb{R}^3$ s.t. $Z \cdot \psi = 0$ everywhere

  Y is **not** a tangent section (but it's close):

$$|\psi|^2 = |\varphi|^2 = 1 \quad \Rightarrow \quad \psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$$

- Choose q so that $\psi(q)$ (locally) minimizes $||Y||_0$:

$$\langle Y, Z \rangle = 0, \qquad \forall Z \in T_{\psi(q)} M_n.$$

Projection to the moduli space

- Wave map equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} + (|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2)\varphi = 0$$

- Slow time $\tau = \varepsilon t$ (book-keeping device)
- Decompose $\varphi(t) = \psi(q(\tau)) + \varepsilon^2 Y(t)$.
  - **Section:** map $Z : \Sigma \to \mathbb{R}^3$
  - **Tangent section:** $Z : \Sigma \to \mathbb{R}^3$ s.t. $Z \cdot \psi = 0$ everywhere

  Y is **not** a tangent section (but it's close):

$$|\psi|^2 = |\varphi|^2 = 1 \quad \Rightarrow \quad \psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$$

- Choose q so that $\psi(q)$ (locally) minimizes $||Y||_0$:

$$\langle Y, Z \rangle = 0, \qquad \forall Z \in T_{\psi(q)} \mathsf{M}_n.$$

Projection to the moduli space

- Wave map equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} + (|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2)\varphi = 0$$

- Slow time $\tau = \varepsilon t$ (book-keeping device)
- Decompose $\varphi(t) = \psi(q(\tau)) + \varepsilon^2 Y(t)$.
  - **Section:** map $Z : \Sigma \to \mathbb{R}^3$
  - **Tangent section:** $Z : \Sigma \to \mathbb{R}^3$ s.t. $Z \cdot \psi = 0$ everywhere

  Y is **not** a tangent section (but it's close):

$$|\psi|^2 = |\varphi|^2 = 1 \quad \Rightarrow \quad \psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$$

- Choose q so that $\psi(q)$ (locally) minimizes $||Y||_0$:

$$\langle Y, Z \rangle = 0, \qquad \forall Z \in T_{\psi(q)} \mathsf{M}_n.$$

Projection to the moduli space

- Wave map equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} + (|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2)\varphi = 0$$

- Slow time $\tau = \varepsilon t$ (book-keeping device)
- Decompose $\varphi(t) = \psi(q(\tau)) + \varepsilon^2 Y(t)$.
  - **Section:** map $Z : \Sigma \to \mathbb{R}^3$
  - **Tangent section:** $Z : \Sigma \to \mathbb{R}^3$ s.t. $Z \cdot \psi = 0$ everywhere

Y is **not** a tangent section (but it's close):

$$|\psi|^2 = |\varphi|^2 = 1 \quad \Rightarrow \quad \psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$$

- Choose q so that $\psi(q)$ (locally) minimizes $||Y||_0$:

$$\langle Y, Z \rangle = 0, \qquad \forall Z \in T_{\psi(q)} \mathsf{M}_n.$$

Projection to the moduli space

- Wave map equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} + (|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2)\varphi = 0$$

- Slow time $\tau = \varepsilon t$ (book-keeping device)
- Decompose $\varphi(t) = \psi(q(\tau)) + \varepsilon^2 Y(t)$.
    - **Section:** map $Z : \Sigma \to \mathbb{R}^3$
    - **Tangent section:** $Z : \Sigma \to \mathbb{R}^3$ s.t. $Z \cdot \psi = 0$ everywhere

Y is **not** a tangent section (but it's close):

$$|\psi|^2 = |\varphi|^2 = 1 \quad \Rightarrow \quad \psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$$

- Choose q so that $\psi(q)$ (locally) minimizes $\|Y\|_0$:

$$\langle Y, Z \rangle = 0, \qquad \forall Z \in T_{\psi(q)} M_n.$$

Projection to the moduli space

- Wave map equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} + (|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2)\varphi = 0$$

- Slow time $\tau = \varepsilon t$ (book-keeping device)
- Decompose $\varphi(t) = \psi(q(\tau)) + \varepsilon^2 Y(t)$.
    - **Section:** map $Z : \Sigma \to \mathbb{R}^3$
    - **Tangent section:** $Z : \Sigma \to \mathbb{R}^3$ s.t. $Z \cdot \psi = 0$ everywhere

  Y is **not** a tangent section (but it's close):

$$|\psi|^2 = |\varphi|^2 = 1 \quad \Rightarrow \quad \psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$$

- Choose q so that $\psi(q)$ (locally) minimizes $||Y||_0$:

$$\langle Y, Z \rangle = 0, \qquad \forall Z \in T_{\psi(q)} \mathsf{M}_n.$$

Projection to the moduli space

- Wave map equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} + (|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2)\varphi = 0$$

- Slow time $\tau = \varepsilon t$ (book-keeping device)
- Decompose $\varphi(t) = \psi(q(\tau)) + \varepsilon^2 Y(t)$.
  - **Section:** map $Z : \Sigma \to \mathbb{R}^3$
  - **Tangent section:** $Z : \Sigma \to \mathbb{R}^3$ s.t. $Z \cdot \psi = 0$ everywhere

  Y is **not** a tangent section (but it's close):

$$|\psi|^2 = |\varphi|^2 = 1 \quad \Rightarrow \quad \psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$$

- Choose q so that $\psi(q)$ (locally) minimizes $\|Y\|_0$:

$$\langle Y, Z \rangle = 0, \qquad \forall Z \in T_{\psi(q)} \mathsf{M}_n.$$

PDE for Y

$$Y_{tt} + J_\psi Y = k + \varepsilon j$$

where

$$\begin{aligned}
J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y - 2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\
k &= -\psi_{\tau\tau} - |\psi_\tau|^2\psi \\
j &= -2(\psi_\tau \cdot Y_t)\psi - \varepsilon(|Y_t|^2 - |Y_x|^2 - |Y_y|^2)\psi \\
&\quad -\varepsilon(|\psi_\tau|^2 - 2\psi_x \cdot Y_x - 2\psi_y \cdot Y_y)Y\} - 2\varepsilon^2(\psi_\tau \cdot Y_t)Y \\
&\quad -\varepsilon^3(|Y_t|^2 - |Y_x|^2 - |Y_y|^2)Y
\end{aligned}$$

- We'll need to bound $\|j\|_0$, so work with $Y \in H^3$, $Y_t \in H^2$

PDE for Y

$$Y_{tt} + J_\psi Y = k + \varepsilon j$$

where

$$\begin{aligned}
J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2) Y - 2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\
k &= -\psi_{\tau\tau} - |\psi_\tau|^2 \psi \\
j &= -2(\psi_\tau \cdot Y_t)\psi - \varepsilon(|Y_t|^2 - |Y_x|^2 - |Y_y|^2)\psi \\
&\quad -\varepsilon(|\psi_\tau|^2 - 2\psi_x \cdot Y_x - 2\psi_y \cdot Y_y) Y\} - 2\varepsilon^2(\psi_\tau \cdot Y_t) Y \\
&\quad -\varepsilon^3(|Y_t|^2 - |Y_x|^2 - |Y_y|^2) Y
\end{aligned}$$

- We'll need to bound $\|j\|_0$, so work with $Y \in H^3$, $Y_t \in H^2$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathrm{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned} J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y + AY \\ AY &= -2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\ A^\dagger Y &= 2\{(\psi\cdot Y)\Delta\psi + (\psi\cdot Y)_x\psi_x + (\psi\cdot Y_y)\psi_y\} \end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
Clearly self-adjoint, still elliptic
$L_\psi = J_\psi$ on tangent sections
$L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathrm{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
  Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned}
J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y + AY \\
AY &= -2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\
A^\dagger Y &= 2\{(\psi\cdot Y)\Delta\psi + (\psi\cdot Y)_x\psi_x + (\psi\cdot Y_y)\psi_y\}
\end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
  Clearly self-adjoint, still elliptic
  $L_\psi = J_\psi$ on tangent sections
  $L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
  Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

  Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathrm{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
  Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned} J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y + AY \\ AY &= -2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\ A^\dagger Y &= 2\{(\psi \cdot Y)\Delta\psi + (\psi \cdot Y)_x \psi_x + (\psi \cdot Y_y)\psi_y\} \end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
  Clearly self-adjoint, still elliptic
  $L_\psi = J_\psi$ on tangent sections
  $L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
  Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

  Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathrm{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
  Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned} J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2) Y + AY \\ AY &= -2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\ A^\dagger Y &= 2\{(\psi \cdot Y)\Delta\psi + (\psi \cdot Y)_x \psi_x + (\psi \cdot Y_y)\psi_y\} \end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
  Clearly self-adjoint, still elliptic
  $L_\psi = J_\psi$ on tangent sections
  $L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
  Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

  Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathsf{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
  Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned}
J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y + AY \\
AY &= -2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\
A^\dagger Y &= 2\{(\psi \cdot Y)\Delta\psi + (\psi \cdot Y)_x \psi_x + (\psi \cdot Y_y)\psi_y\}
\end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
  Clearly self-adjoint, still elliptic
  $L_\psi = J_\psi$ on tangent sections
  $L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
  Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

 Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathrm{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
 Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned} J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y + AY \\ AY &= -2(\psi_x\cdot Y_x + \psi_y \cdot Y_y)\psi \\ A^\dagger Y &= 2\{(\psi\cdot Y)\Delta\psi + (\psi\cdot Y)_x\psi_x + (\psi\cdot Y_y)\psi_y\} \end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
 Clearly self-adjoint, still elliptic
 $L_\psi = J_\psi$ on tangent sections
 $L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
 Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

  Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathsf{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
  Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned}
J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y + AY \\
AY &= -2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\
A^\dagger Y &= 2\{(\psi\cdot Y)\Delta\psi + (\psi\cdot Y)_x\psi_x + (\psi\cdot Y_y)\psi_y\}
\end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
  Clearly self-adjoint, still elliptic
  $L_\psi = J_\psi$ on tangent sections
  $L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
  Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- **Fact:** J_ψ acting on tangent sections is the **Jacobi** operator for h-map ψ

$$\mathrm{Hess}_\psi(Y,Y) = \langle Y, J_\psi Y\rangle$$

Self-adjoint elliptic operator on $\psi^{-1}TS^2$, $T_\varphi \mathsf{M}_n = \ker J_\psi$

- **Annoying fact:** J not self-adjoint on general sections
  Self-adjointness of J is crucial for Stuart's method

$$\begin{aligned} J_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2)Y + AY \\ AY &= -2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\ A^\dagger Y &= 2\{(\psi\cdot Y)\Delta\psi + (\psi\cdot Y)_x\psi_x + (\psi\cdot Y_y)\psi_y\} \end{aligned}$$

- Improved Jacobi operator $L_\psi Y = J_\psi Y + A^\dagger Y$
  Clearly self-adjoint, still elliptic
  $L_\psi = J_\psi$ on tangent sections
  $L_\psi(\alpha\psi) = -(\Delta\alpha)\psi - 4(\alpha_x\psi_x + \alpha_y\psi_y)$ on normal sections
  Hence $\ker L_\psi = \ker J_\psi \oplus \langle\psi\rangle$

The (improved) Jacobi operator

- For Y satisfying pointwise constraint $\psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$,

$$A^\dagger Y = -\varepsilon^2 \{ |Y|^2 \Delta\psi + |Y|_x^2 \psi_x + |Y|_y^2 \psi_y \}$$

- So, replace J by L and j by

$$j' = j + \varepsilon \{ |Y|^2 \Delta\psi + |Y|_x^2 \psi_x + |Y|_y^2 \psi_y \}$$

- Doesn't change analytic structure of error term

$$Y_{tt} + LY = k + \varepsilon j'$$

The (improved) Jacobi operator

- For Y satisfying pointwise constraint $\psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$,

$$A^\dagger Y = -\varepsilon^2 \{ |Y|^2 \Delta\psi + |Y|^2_x \psi_x + |Y|^2_y \psi_y \}$$

- So, replace J by L and j by

$$j' = j + \varepsilon \{ |Y|^2 \Delta\psi + |Y|^2_x \psi_x + |Y|^2_y \psi_y \}$$

- Doesn't change analytic structure of error term

$$Y_{tt} + LY = k + \varepsilon j'$$

The (improved) Jacobi operator

- For Y satisfying pointwise constraint $\psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2$,

$$A^\dagger Y = -\varepsilon^2 \{ |Y|^2 \Delta\psi + |Y|_x^2 \psi_x + |Y|_y^2 \psi_y \}$$

- So, replace J by L and j by

$$j' = j + \varepsilon \{ |Y|^2 \Delta\psi + |Y|_x^2 \psi_x + |Y|_y^2 \psi_y \}$$

- Doesn't change analytic structure of error term

$$Y_{tt} + LY = k + \varepsilon j'$$

Evolution of $q(\tau)$

- Recall L^2 orthogonality constraint

$$\langle Y, \frac{\partial \psi}{\partial q^i} \rangle = 0, \qquad i = 1, 2, \ldots, 4n$$

since $\frac{\partial \psi}{\partial q^i}$ span $\ker J_\psi$

- Differentiate w.r.t. t twice

$$\begin{aligned} \langle Y_{tt}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle -LY + k, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle \psi_{\tau\tau}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \end{aligned}$$

Geodesic flow (with $O(\varepsilon)$ correction).

Evolution of $q(\tau)$

- Recall L^2 orthogonality constraint

$$\langle Y, \frac{\partial \psi}{\partial q^i} \rangle = 0, \qquad i = 1, 2, \ldots, 4n$$

  since $\frac{\partial \psi}{\partial q^i}$ span $\ker J_\psi$

- Differentiate w.r.t. t twice

$$\begin{aligned} \langle Y_{tt}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle -LY + k, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle \psi_{\tau\tau}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \end{aligned}$$

  Geodesic flow (with $O(\varepsilon)$ correction).

Evolution of $q(\tau)$

- Recall L^2 orthogonality constraint

$$\langle Y, \frac{\partial \psi}{\partial q^i} \rangle = 0, \qquad i = 1, 2, \ldots, 4n$$

since $\frac{\partial \psi}{\partial q^i}$ span $\ker J_\psi$

- Differentiate w.r.t. t twice

$$\begin{aligned} \langle Y_{tt}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle -LY + k, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle \psi_{\tau\tau}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \end{aligned}$$

Geodesic flow (with $O(\varepsilon)$ correction).

Evolution of $q(\tau)$

- Recall L^2 orthogonality constraint

$$\langle Y, \frac{\partial \psi}{\partial q^i} \rangle = 0, \qquad i = 1, 2, \ldots, 4n$$

  since $\frac{\partial \psi}{\partial q^i}$ span $\ker J_\psi$

- Differentiate w.r.t. t twice

$$\begin{aligned} \langle Y_{tt}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle -LY + k, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle \psi_{\tau\tau}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \end{aligned}$$

  Geodesic flow (with $O(\varepsilon)$ correction).

Evolution of $q(\tau)$

- Recall L^2 orthogonality constraint

$$\langle Y, \frac{\partial \psi}{\partial q^i} \rangle = 0, \qquad i = 1, 2, \ldots, 4n$$

since $\frac{\partial \psi}{\partial q^i}$ span $\ker J_\psi$

- Differentiate w.r.t. t twice

$$\begin{aligned} \langle Y_{tt}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle -LY + k, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \\ \langle \psi_{\tau\tau}, \frac{\partial \psi}{\partial q^i} \rangle &= O(\varepsilon) \end{aligned}$$

Geodesic flow (with $O(\varepsilon)$ correction).

Summary: the ODE/PDE system

$$
\begin{aligned}
Y_{tt} + LY &= k + \varepsilon j' \\
q^i_{\tau\tau} + \Gamma(q)^i_{jk} q^j_\tau q^k_\tau &= \varepsilon f^i(q, q_\tau, Y, Y_t, \varepsilon)
\end{aligned}
$$

Short time existence theorem

There exist ε_*, $T > 0$, depending only on Γ, such that, for all $\varepsilon \in (0, \varepsilon_*]$ and any initial data

$$\|Y(0)\|_3^2 + \|Y_t(0)\|_2^2 + |q(0)|^2 + |q_\tau(0)|^2 \leq \Gamma^2$$

the system has a unique solution

$$(Y, q) \in C^0([0,T], H^3 \oplus \mathbb{R}^{4n}) \cap \cdots \cap C^3([0,T], H^0 \oplus \mathbb{R}^{4n})$$

Furthermore, if the initial data are tangent to the L^2 orthogonality constraint, and the pointwise constraint, the solution preserves these constraints for all t.

Proof: Picard's method.

Summary: the ODE/PDE system

$$\begin{aligned} Y_{tt} + LY &= k + \varepsilon j' \\ q^i_{\tau\tau} + \Gamma(q)^i_{jk} q^j_\tau q^k_\tau &= \varepsilon f^i(q, q_\tau, Y, Y_t, \varepsilon) \end{aligned}$$

Short time existence theorem

There exist ε_*, $T > 0$, depending only on Γ, such that, for all $\varepsilon \in (0, \varepsilon_*]$ and any initial data

$$\| Y(0) \|_3^2 + \| Y_t(0) \|_2^2 + |q(0)|^2 + |q_\tau(0)|^2 \leq \Gamma^2$$

the system has a unique solution

$$(Y, q) \in C^0([0,T], H^3 \oplus \mathbb{R}^{4n}) \cap \cdots \cap C^3([0,T], H^0 \oplus \mathbb{R}^{4n})$$

Furthermore, if the initial data are tangent to the L^2 orthogonality constraint, and the pointwise constraint, the solution preserves these constraints for all t.

Proof: Picard's method.

Summary: the ODE/PDE system

$$\begin{aligned} Y_{tt} + LY &= k + \varepsilon j' \\ q^i_{\tau\tau} + \Gamma(q)^i_{jk} q^j_\tau q^k_\tau &= \varepsilon f^i(q, q_\tau, Y, Y_t, \varepsilon) \end{aligned}$$

Short time existence theorem
There exist $\varepsilon_*, T > 0$, depending only on Γ, such that, for all $\varepsilon \in (0, \varepsilon_*]$ and any initial data

$$\|Y(0)\|_3^2 + \|Y_t(0)\|_2^2 + |q(0)|^2 + |q_\tau(0)|^2 \leq \Gamma^2$$

the system has a unique solution

$$(Y, q) \in C^0([0,T], H^3 \oplus \mathbb{R}^{4n}) \cap \cdots \cap C^3([0,T], H^0 \oplus \mathbb{R}^{4n})$$

Furthermore, if the initial data are tangent to the L^2 orthogonality constraint, and the pointwise constraint, the solution preserves these constraints for all t.

Proof: Picard's method.

Summary: the ODE/PDE system

$$
\begin{aligned}
Y_{tt} + LY &= k + \varepsilon j' \\
q^i_{\tau\tau} + \Gamma(q)^i_{jk} q^j_\tau q^k_\tau &= \varepsilon f^i(q, q_\tau, Y, Y_t, \varepsilon)
\end{aligned}
$$

Short time existence theorem

There exist $\varepsilon_*, T > 0$, depending only on Γ, such that, for all $\varepsilon \in (0, \varepsilon_*]$ and any initial data

$$\|Y(0)\|_3^2 + \|Y_t(0)\|_2^2 + |q(0)|^2 + |q_\tau(0)|^2 \le \Gamma^2$$

the system has a unique solution

$$(Y, q) \in C^0([0,T], H^3 \oplus \mathbb{R}^{4n}) \cap \cdots \cap C^3([0,T], H^0 \oplus \mathbb{R}^{4n})$$

Furthermore, if the initial data are tangent to the L^2 orthogonality constraint, and the pointwise constraint, the solution preserves these constraints for all t.

Proof: Picard's method.

Coercivity of the Hessian(s)

- Essence of Stuart's method: use quasi-conservation of $\|Y_t\|_0^2 + \langle Y, LY\rangle$ to bound growth of $\|Y_t\|_0^2 + \|Y\|_1^2$
- Key ingredient: show that $\langle Y, LY\rangle$ controls $\|Y\|_1$

Theorem (Haskins,JMS)

For all tangent *sections L^2 orthogonal to* $\ker J_\psi$*,*

$$\langle Y, J_\psi Y\rangle \geq c(\psi)\|Y\|_1^2.$$

The constant $c(\psi) > 0$ and depends continuously on ψ.

- Define tangent projection $P(Z) = Z - (\psi\cdot Z)\psi$. Then ($\alpha = \psi\cdot Z$),

$$\begin{aligned}\langle Z, LZ\rangle &= \langle P(Z) + \alpha\psi, JP(Z) + L(\alpha\psi)\rangle\\ &= \langle P(Z), JP(Z)\rangle + \|d\alpha\|^2 \geq \langle P(Z), JP(Z)\rangle.\end{aligned}$$

Coercivity of the Hessian(s)

- Essence of Stuart's method: use quasi-conservation of $\|Y_t\|_0^2 + \langle Y, LY\rangle$ to bound growth of $\|Y_t\|_0^2 + \|Y\|_1^2$
- Key ingredient: show that $\langle Y, LY\rangle$ controls $\|Y\|_1$

Theorem (Haskins,JMS)

For all tangent *sections* L^2 *orthogonal to* $\ker J_\psi$*,*

$$\langle Y, J_\psi Y\rangle \geq c(\psi)\|Y\|_1^2.$$

The constant $c(\psi) > 0$ *and depends continuously on* ψ*.*

- Define tangent projection $P(Z) = Z - (\psi \cdot Z)\psi$. Then ($\alpha = \psi \cdot Z$),

$$\begin{aligned} \langle Z, LZ\rangle &= \langle P(Z) + \alpha\psi, JP(Z) + L(\alpha\psi)\rangle \\ &= \langle P(Z), JP(Z)\rangle + \|d\alpha\|^2 \geq \langle P(Z), JP(Z)\rangle. \end{aligned}$$

Coercivity of the Hessian(s)

- Essence of Stuart's method: use quasi-conservation of $\|Y_t\|_0^2 + \langle Y, LY\rangle$ to bound growth of $\|Y_t\|_0^2 + \|Y\|_1^2$
- Key ingredient: show that $\langle Y, LY\rangle$ controls $\|Y\|_1$

Theorem (Haskins,JMS)

For all tangent *sections* L^2 *orthogonal to* $\ker J_\psi$,

$$\langle Y, J_\psi Y\rangle \geq c(\psi)\|Y\|_1^2.$$

The constant $c(\psi) > 0$ *and depends continuously on* ψ.

- Define tangent projection $P(Z) = Z - (\psi \cdot Z)\psi$.
  Then $(\alpha = \psi \cdot Z)$,

$$\begin{aligned}\langle Z, LZ\rangle &= \langle P(Z) + \alpha\psi, JP(Z) + L(\alpha\psi)\rangle \\ &= \langle P(Z), JP(Z)\rangle + \|\mathrm{d}\alpha\|^2 \geq \langle P(Z), JP(Z)\rangle.\end{aligned}$$

Coercivity of the Hessian(s)

- Essence of Stuart's method: use quasi-conservation of $\|Y_t\|_0^2 + \langle Y, LY\rangle$ to bound growth of $\|Y_t\|_0^2 + \|Y\|_1^2$
- Key ingredient: show that $\langle Y, LY\rangle$ controls $\|Y\|_1$

Theorem (Haskins,JMS)

For all tangent *sections* L^2 *orthogonal to* $\ker J_\psi$,

$$\langle Y, J_\psi Y\rangle \geq c(\psi)\|Y\|_1^2.$$

The constant $c(\psi) > 0$ *and depends continuously on* ψ.

- Define tangent projection $P(Z) = Z - (\psi \cdot Z)\psi$.

  Then $(\alpha = \psi \cdot Z)$,

$$\begin{aligned}
\langle Z, LZ\rangle &= \langle P(Z) + \alpha\psi, JP(Z) + L(\alpha\psi)\rangle \\
&= \langle P(Z), JP(Z)\rangle + \|d\alpha\|^2 \geq \langle P(Z), JP(Z)\rangle.
\end{aligned}$$

Coercivity of the Hessian(s)

- Essence of Stuart's method: use quasi-conservation of $\|Y_t\|_0^2 + \langle Y, LY\rangle$ to bound growth of $\|Y_t\|_0^2 + \|Y\|_1^2$
- Key ingredient: show that $\langle Y, LY\rangle$ controls $\|Y\|_1$

Theorem (Haskins,JMS)

For all tangent *sections* L^2 *orthogonal to* $\ker J_\psi$,

$$\langle Y, J_\psi Y\rangle \geq c(\psi)\|Y\|_1^2.$$

The constant $c(\psi) > 0$ *and depends continuously on* ψ.

- Define tangent projection $P(Z) = Z - (\psi \cdot Z)\psi$.
  Then ($\alpha = \psi \cdot Z$),

$$\begin{aligned}\langle Z, LZ\rangle &= \langle P(Z) + \alpha\psi, JP(Z) + L(\alpha\psi)\rangle \\ &= \langle P(Z), JP(Z)\rangle + \|d\alpha\|^2 \geq \langle P(Z), JP(Z)\rangle.\end{aligned}$$

Coercivity of the Hessian(s)

- Essence of Stuart's method: use quasi-conservation of $\|Y_t\|_0^2 + \langle Y, LY\rangle$ to bound growth of $\|Y_t\|_0^2 + \|Y\|_1^2$
- Key ingredient: show that $\langle Y, LY\rangle$ controls $\|Y\|_1$

Theorem (Haskins,JMS)

For all tangent *sections* L^2 *orthogonal to* $\ker J_\psi$*,*

$$\langle Y, J_\psi Y\rangle \geq c(\psi)\|Y\|_1^2.$$

The constant $c(\psi) > 0$ *and depends continuously on* ψ*.*

- Define tangent projection $P(Z) = Z - (\psi \cdot Z)\psi$.
  Then ($\alpha = \psi \cdot Z$),

$$\begin{aligned} \langle Z, LZ\rangle &= \langle P(Z) + \alpha\psi, JP(Z) + L(\alpha\psi)\rangle \\ &= \langle P(Z), JP(Z)\rangle + \|\mathrm{d}\alpha\|^2 \geq \langle P(Z), JP(Z)\rangle. \end{aligned}$$

Coercivity of the Hessian(s)

- If $Y \perp_{L^2} \ker J$, so is $P(Y)$, so above Theorem implies

$$\langle Y, LY\rangle \geq c(\psi)\|P(Y)\|_1^2 \geq c(\psi)(\|Y\|_1^2 - \varepsilon^2\|Y\|_2^4)$$

- Similarly, can show that, if $Y \perp_{L^2} \ker J$

$$\langle LY, LLY\rangle \geq c(\psi)\left\{\|Y\|_3^2 - \varepsilon^2(\|Y\|_3^2 + \|Y\|_3^4)\right\}$$

- Also need Garding inequality (standard elliptic estimate)

$$\|LY\|_k^2 \geq c(\psi)\{\|Y\|_{k+2}^2 - \|Y\|_0^2\}$$

(True for any 2nd order elliptic operator)

Coercivity of the Hessian(s)

- If $Y \perp_{L^2} \ker J$, so is $P(Y)$, so above Theorem implies

$$\langle Y, LY\rangle \geq c(\psi)\|P(Y)\|_1^2 \geq c(\psi)(\|Y\|_1^2 - \varepsilon^2\|Y\|_2^4)$$

- Similarly, can show that, if $Y \perp_{L^2} \ker J$

$$\langle LY, LLY\rangle \geq c(\psi)\left\{\|Y\|_3^2 - \varepsilon^2(\|Y\|_3^2 + \|Y\|_3^4)\right\}$$

- Also need Garding inequality (standard elliptic estimate)

$$\|LY\|_k^2 \geq c(\psi)\{\|Y\|_{k+2}^2 - \|Y\|_0^2\}$$

(True for any 2nd order elliptic operator)

Coercivity of the Hessian(s)

- If $Y \perp_{L^2} \ker J$, so is $P(Y)$, so above Theorem implies

$$\langle Y, LY\rangle \geq c(\psi)\|P(Y)\|_1^2 \geq c(\psi)(\|Y\|_1^2 - \varepsilon^2\|Y\|_2^4)$$

- Similarly, can show that, if $Y \perp_{L^2} \ker J$

$$\langle LY, LLY\rangle \geq c(\psi)\left\{\|Y\|_3^2 - \varepsilon^2(\|Y\|_3^2 + \|Y\|_3^4)\right\}$$

- Also need Garding inequality (standard elliptic estimate)

$$\|LY\|_k^2 \geq c(\psi)\{\|Y|_{k+2}^2 - \|Y\|_0^2\}$$

(True for any 2nd order elliptic operator)

Energy estimates

- Take a solution of the ODE/PDE system, and consider the quantity

$$Q_1(t) = \frac{1}{2}\|Y_t\|_0^2 + \frac{1}{2}\langle Y, LY\rangle$$

- This is "quasi-conserved":

$$\begin{aligned}
\frac{dQ_1}{dt} &= \langle Y_t, -LY + k + \varepsilon j'\rangle + \langle Y_t, LY\rangle + \varepsilon\langle Y, L_\tau Y\rangle \\
&= \frac{d}{dt}\langle Y, k\rangle + \varepsilon\{-\langle Y, k_\tau\rangle + \langle Y_t, j'\rangle + \langle Y, L_\tau Y\rangle\} \\
Q_1(t) &\leq c + c(q)(|q_{\tau\tau}| + |q_\tau|^2)\|Y(t)\|_0 \\
&\quad + \varepsilon\int_0^t c(|q|, |q_\tau|, |q_{\tau\tau}|, |q_{\tau\tau\tau}|, \|Y\|_3, \|Y_t\|_2)
\end{aligned}$$

Energy estimates

- Take a solution of the ODE/PDE system, and consider the quantity

$$Q_1(t) = \frac{1}{2}\|Y_t\|_0^2 + \frac{1}{2}\langle Y, LY\rangle$$

- This is "quasi-conserved":

$$\begin{aligned}
\frac{dQ_1}{dt} &= \langle Y_t, -LY + k + \varepsilon j'\rangle + \langle Y_t, LY\rangle + \varepsilon\langle Y, L_\tau Y\rangle \\
&= \frac{d}{dt}\langle Y, k\rangle + \varepsilon\{-\langle Y, k_\tau\rangle + \langle Y_t, j'\rangle + \langle Y, L_\tau Y\rangle\} \\
Q_1(t) &\leq c + c(q)(|q_{\tau\tau}| + |q_\tau|^2)\|Y(t)\|_0 \\
&\quad + \varepsilon\int_0^t c(|q|, |q_\tau|, |q_{\tau\tau}|, |q_{\tau\tau\tau}|, \|Y\|_3, \|Y_t\|_2)
\end{aligned}$$

Energy estimates

- Take a solution of the ODE/PDE system, and consider the quantity

$$Q_1(t) = \frac{1}{2}\|Y_t\|_0^2 + \frac{1}{2}\langle Y, LY\rangle$$

- This is “quasi-conserved”:

$$\begin{aligned}
\frac{dQ_1}{dt} &= \langle Y_t, -LY + k + \varepsilon j'\rangle + \langle Y_t, LY\rangle + \varepsilon\langle Y, L_\tau Y\rangle \\
&= \frac{d}{dt}\langle Y, k\rangle + \varepsilon\{-\langle Y, k_\tau\rangle + \langle Y_t, j'\rangle + \langle Y, L_\tau Y\rangle\}
\end{aligned}$$

$$\begin{aligned}
Q_1(t) &\leq c + c(q)(|q_{\tau\tau}| + |q_\tau|^2)\|Y(t)\|_0 \\
&\quad + \varepsilon\int_0^t c(|q|, |q_\tau|, |q_{\tau\tau}|, |q_{\tau\tau\tau}|, \|Y\|_3, \|Y_t\|_2)
\end{aligned}$$

Energy estimates

- Take a solution of the ODE/PDE system, and consider the quantity

$$Q_1(t) = \frac{1}{2}\|Y_t\|_0^2 + \frac{1}{2}\langle Y, LY\rangle$$

- This is “quasi-conserved”:

$$\begin{aligned}
\frac{dQ_1}{dt} &= \langle Y_t, -LY + k + \varepsilon j'\rangle + \langle Y_t, LY\rangle + \varepsilon\langle Y, L_\tau Y\rangle \\
&= \frac{d}{dt}\langle Y, k\rangle + \varepsilon\{-\langle Y, k_\tau\rangle + \langle Y_t, j'\rangle + \langle Y, L_\tau Y\rangle\} \\
Q_1(t) &\leq c + c(q)(|q_{\tau\tau}| + |q_\tau|^2)\|Y(t)\|_0 \\
&\quad + \varepsilon\int_0^t c(|q|, |q_\tau|, |q_{\tau\tau}|, |q_{\tau\tau\tau}|, \|Y\|_3, \|Y_t\|_2)
\end{aligned}$$

Energy estimates

- Higher energy,

$$Q_2(t) = \frac{1}{2}\|(LY)_t\|_0^2 + \frac{1}{2}\langle LY, LLY\rangle$$

- LY satisfies a PDE with same analytic structure. One finds

$$\begin{aligned} Q_2(t) &\leq c + c(q)(|q_{\tau\tau}| + |q_\tau|^2)\|Y(t)\|_3 \\ &\quad + \varepsilon\int_0^t c(|q|, |q_\tau|, |q_{\tau\tau}|, |q_{\tau\tau\tau}|, \|Y\|_3, \|Y_t\|_2) \end{aligned}$$

- Key point: dominant term in growth of **quadratic** form Q_2 (which controls $\|Y\|_3^2$) is **linear** in $\|Y\|_3$

Energy estimates

- Higher energy,

$$Q_2(t) = \frac{1}{2}\|(LY)_t\|_0^2 + \frac{1}{2}\langle LY, LLY\rangle$$

- LY satisfies a PDE with same analytic structure. One finds

$$\begin{aligned} Q_2(t) \quad &\leq \quad c + c(q)(|q_{\tau\tau}| + |q_\tau|^2)\|Y(t)\|_3 \\ &\quad + \varepsilon \int_0^t c(|q|, |q_\tau|, |q_{\tau\tau}|, |q_{\tau\tau\tau}|, \|Y\|_3, \|Y_t\|_2) \end{aligned}$$

- Key point: dominant term in growth of **quadratic** form Q_2 (which controls $\|Y\|_3^2$) is **linear** in $\|Y\|_3$

Energy estimates

- Higher energy,

$$Q_2(t) = \frac{1}{2}\|(LY)_t\|_0^2 + \frac{1}{2}\langle LY, LLY\rangle$$

- LY satisfies a PDE with same analytic structure. One finds

$$\begin{aligned} Q_2(t) \quad \leq \quad & c + c(q)(|q_{\tau\tau}| + |q_\tau|^2)\|Y(t)\|_3 \\ & +\varepsilon \int_0^t c(|q|, |q_\tau|, |q_{\tau\tau}|, |q_{\tau\tau\tau}|, \|Y\|_3, \|Y_t\|_2) \end{aligned}$$

- Key point: dominant term in growth of **quadratic** form Q_2 (which controls $\|Y\|_3^2$) is **linear** in $\|Y\|_3$

A priori bound

Can repeatedly apply short time existence theorem, whilever $|q| + |q_\tau| + \|Y\|_3 + \|Y_t\|_2$ remains bounded. In this way, produce maximally extended solution with $Y(0) = Y_t(0) = 0$.

- Let $q(t) = q_0(\tau) + \varepsilon^2 \tilde{q}(t)$ where $q_0(\tau)$ solves exact geodesic flow.
- There exists $T_* > 0$ such that $|q_0(\tau)| \leq 1$ for all $\tau \in [0, T_*]$.
- $M(t) = \max_{0 \leq s \leq t} \{|q(s)| + |\tilde{q}_t(s)| + \|Y(s)\|_3^2 + \|Y_t(s)\|_2^2\}$
- Whilever solution exists and $t < T_*/\varepsilon$,

$$\begin{aligned} |q| &\leq 1 + \varepsilon^2 c(M) \\ |\tilde{q}_t| &\leq \varepsilon c(M) \\ |q_\tau| &\leq c + \varepsilon |\tilde{q}_t| \leq c + \varepsilon M \\ |q_{\tau\tau}| &\leq c + \varepsilon c(M) \\ |q_{\tau\tau\tau}| &\leq c(M) \end{aligned}$$

A priori bound

Can repeatedly apply short time existence theorem, whilever $|q|+|q_\tau|+\|Y\|_3+\|Y_t\|_2$ remains bounded. In this way, produce maximally extended solution with $Y(0)=Y_t(0)=0$.

- Let $q(t)=q_0(\tau)+\varepsilon^2\widetilde{q}(t)$ where $q_0(\tau)$ solves exact geodesic flow.
- There exists $T_*>0$ such that $|q_0(\tau)|\leq 1$ for all $\tau\in[0,T_*]$.
- $M(t)=\max\limits_{0\leq s\leq t}\{|q(s)|+|\widetilde{q}_t(s)|+\|Y(s)\|_3^2+\|Y_t(s)\|_2^2\}$
- Whilever solution exists and $t<T_*/\varepsilon$,

$$
\begin{aligned}
|q| &\leq 1+\varepsilon^2 c(M)\\
|\widetilde{q}_t| &\leq \varepsilon c(M)\\
|q_\tau| &\leq c+\varepsilon|\widetilde{q}_t|\leq c+\varepsilon M\\
|q_{\tau\tau}| &\leq c+\varepsilon c(M)\\
|q_{\tau\tau\tau}| &\leq c(M)
\end{aligned}
$$

A priori bound

Can repeatedly apply short time existence theorem, whilever $|q| + |q_\tau| + \|Y\|_3 + \|Y_t\|_2$ remains bounded. In this way, produce maximally extended solution with $Y(0) = Y_t(0) = 0$.

- Let $q(t) = q_0(\tau) + \varepsilon^2 \tilde{q}(t)$
  where $q_0(\tau)$ solves exact geodesic flow.
- There exists $T_* > 0$ such that $|q_0(\tau)| \leq 1$ for all $\tau \in [0, T_*]$.
- $M(t) = \max_{0 \leq s \leq t} \{|q(s)| + |\tilde{q}_t(s)| + \|Y(s)\|_3^2 + \|Y_t(s)\|_2^2\}$
- Whilever solution exists and $t < T_*/\varepsilon$,

$$
\begin{aligned}
|q| &\leq 1 + \varepsilon^2 c(M) \\
|\tilde{q}_t| &\leq \varepsilon c(M) \\
|q_\tau| &\leq c + \varepsilon|\tilde{q}_t| \leq c + \varepsilon M \\
|q_{\tau\tau}| &\leq c + \varepsilon c(M) \\
|q_{\tau\tau\tau}| &\leq c(M)
\end{aligned}
$$

A priori bound

Can repeatedly apply short time existence theorem, whilever $|q| + |q_\tau| + \|Y\|_3 + \|Y_t\|_2$ remains bounded. In this way, produce maximally extended solution with $Y(0) = Y_t(0) = 0$.

- Let $q(t) = q_0(\tau) + \varepsilon^2 \widetilde{q}(t)$
  where $q_0(\tau)$ solves exact geodesic flow.
- There exists $T_* > 0$ such that $|q_0(\tau)| \leq 1$ for all $\tau \in [0, T_*]$.
- $M(t) = \max_{0 \leq s \leq t} \{|q(s)| + |\widetilde{q}_t(s)| + \|Y(s)\|_3^2 + \|Y_t(s)\|_2^2\}$
- Whilever solution exists and $t < T_*/\varepsilon$,

$$\begin{aligned}
|q| &\leq 1 + \varepsilon^2 c(M) \\
|\widetilde{q}_t| &\leq \varepsilon c(M) \\
|q_\tau| &\leq c + \varepsilon|\widetilde{q}_t| \leq c + \varepsilon M \\
|q_{\tau\tau}| &\leq c + \varepsilon c(M) \\
|q_{\tau\tau\tau}| &\leq c(M)
\end{aligned}$$

A priori bound

Can repeatedly apply short time existence theorem, whilever $|q| + |q_\tau| + \|Y\|_3 + \|Y_t\|_2$ remains bounded. In this way, produce maximally extended solution with $Y(0) = Y_t(0) = 0$.

- Let $q(t) = q_0(\tau) + \varepsilon^2 \widetilde{q}(t)$
  where $q_0(\tau)$ solves exact geodesic flow.
- There exists $T_* > 0$ such that $|q_0(\tau)| \leq 1$ for all $\tau \in [0, T_*]$.
- $M(t) = \max\limits_{0 \leq s \leq t} \{|q(s)| + |\widetilde{q}_t(s)| + \|Y(s)\|_3^2 + \|Y_t(s)\|_2^2\}$
- Whilever solution exists and $t < T_*/\varepsilon$,

$$
\begin{aligned}
|q| &\leq 1 + \varepsilon^2 c(M) \\
|\widetilde{q}_t| &\leq \varepsilon c(M) \\
|q_\tau| &\leq c + \varepsilon|\widetilde{q}_t| \leq c + \varepsilon M \\
|q_{\tau\tau}| &\leq c + \varepsilon c(M) \\
|q_{\tau\tau\tau}| &\leq c(M)
\end{aligned}
$$

A priori bound

- By coercivity, Garding, energy estimates:

$$
\begin{aligned}
\|Y_t(t)\|_2^2 + \|Y(t)\|_3^2 &\leq c(|q|)\{\|LY_t\|_0^2 + \|Y_t\|_0^2 \\
&\quad + \langle LY, LLY\rangle + \varepsilon^2 c(\|Y\|_3)\} \\
&\leq c(1+\varepsilon^2 c(M))\{Q_1(t) + Q_2(t) + \varepsilon^2 c(M)\} \\
&\leq (c+\varepsilon^2 c(M))\{c + (c+\varepsilon c(M))\|Y(t)\|_3 \\
&\quad + \int_0^t c(M(s))ds\} \\
&\leq c + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))
\end{aligned}
$$

- Hence, whilever solution exists (and $t < T_*/\varepsilon$),

$$M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))$$

A priori bound

- By coercivity, Garding, energy estimates:

$$
\begin{aligned}
\|Y_t(t)\|_2^2 + \|Y(t)\|_3^2 &\leq c(|q|)\{\|LY_t\|_0^2 + \|Y_t\|_0^2 \\
&\quad + \langle LY, LLY\rangle + \varepsilon^2 c(\|Y\|_3)\} \\
&\leq c(1+\varepsilon^2 c(M))\{Q_1(t) + Q_2(t) + \varepsilon^2 c(M)\} \\
&\leq (c+\varepsilon^2 c(M))\{c + (c+\varepsilon c(M))\|Y(t)\|_3 \\
&\quad + \int_0^t c(M(s))ds\} \\
&\leq c + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))
\end{aligned}
$$

- Hence, whilever solution exists (and $t < T_*/\varepsilon$),

$$M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))$$

A priori bound

- By coercivity, Garding, energy estimates:

$$
\begin{aligned}
\|Y_t(t)\|_2^2 + \|Y(t)\|_3^2 &\leq c(|q|)\{\|LY_t\|_0^2 + \|Y_t\|_0^2 \\
&\quad + \langle LY, LLY\rangle + \varepsilon^2 c(\|Y\|_3)\} \\
&\leq c(1+\varepsilon^2 c(M))\{Q_1(t) + Q_2(t) + \varepsilon^2 c(M)\} \\
&\leq (c+\varepsilon^2 c(M))\{c + (c+\varepsilon c(M))\|Y(t)\|_3 \\
&\quad + \int_0^t c(M(s))ds\} \\
&\leq c + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))
\end{aligned}
$$

- Hence, whilever solution exists (and $t < T_*/\varepsilon$),

$$
M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))
$$

A priori bound

- By coercivity, Garding, energy estimates:

$$
\begin{aligned}
\|Y_t(t)\|_2^2 + \|Y(t)\|_3^2 &\leq c(|q|)\{\|LY_t\|_0^2 + \|Y_t\|_0^2 \\
&\quad + \langle LY, LLY\rangle + \varepsilon^2 c(\|Y\|_3)\} \\
&\leq c(1+\varepsilon^2 c(M))\{Q_1(t) + Q_2(t) + \varepsilon^2 c(M)\} \\
&\leq (c+\varepsilon^2 c(M))\{c + (c+\varepsilon c(M))\|Y(t)\|_3 \\
&\quad + \int_0^t c(M(s))ds\} \\
&\leq c + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))
\end{aligned}
$$

- Hence, whilever solution exists (and $t < T_*/\varepsilon$),

$$M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))$$

A priori bound

- By coercivity, Garding, energy estimates:

$$
\begin{aligned}
\|Y_t(t)\|_2^2 + \|Y(t)\|_3^2 &\leq c(|q|)\{\|LY_t\|_0^2 + \|Y_t\|_0^2 \\
&\quad + \langle LY, LLY\rangle + \varepsilon^2 c(\|Y\|_3)\} \\
&\leq c(1+\varepsilon^2 c(M))\{Q_1(t) + Q_2(t) + \varepsilon^2 c(M)\} \\
&\leq (c+\varepsilon^2 c(M))\{c + (c+\varepsilon c(M))\|Y(t)\|_3 \\
&\quad + \int_0^t c(M(s))ds\} \\
&\leq c + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))
\end{aligned}
$$

- Hence, whilever solution exists (and $t < T_*/\varepsilon$),

$$M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))$$

A priori bound

$$M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))$$

- Choose $M_* > 0$ large, and for each $\varepsilon > 0$ let t_ε be the first time when $M = M_*$.

  Claim there exists $\varepsilon_* > 0$ such that $\varepsilon t_\varepsilon$ is bounded away from 0 on $(0, \varepsilon_*]$.

  If not, there exists a sequence $\varepsilon \to 0$ so that $\varepsilon t_\varepsilon \to 0$, whence

$$M_* \leq c + cM_*^{\frac{1}{2}}$$

  a contradiction for M_* sufficiently large.

A priori bound

$$M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon t c(M(t))$$

- Choose $M_* > 0$ large, and for each $\varepsilon > 0$ let t_ε be the first time when $M = M_*$.
  Claim there exists $\varepsilon_* > 0$ such that $\varepsilon t_\varepsilon$ is bounded away from 0 on $(0, \varepsilon_*]$.
  If not, there exists a sequence $\varepsilon \to 0$ so that $\varepsilon t_\varepsilon \to 0$, whence

$$M_* \leq c + cM_*^{\frac{1}{2}}$$

  a contradiction for M_* sufficiently large.

A priori bound

$$M(t) \leq c + \varepsilon c(M(t)) + cM(t)^{\frac{1}{2}} + \varepsilon tc(M(t))$$

- Choose $M_* > 0$ large, and for each $\varepsilon > 0$ let t_ε be the first time when $M = M_*$.
  Claim there exists $\varepsilon_* > 0$ such that $\varepsilon t_\varepsilon$ is bounded away from 0 on $(0, \varepsilon_*]$.
  If not, there exists a sequence $\varepsilon \to 0$ so that $\varepsilon t_\varepsilon \to 0$, whence

$$M_* \leq c + cM_*^{\frac{1}{2}}$$

  a contradiction for M_* sufficiently large.

A priori bound

"Precise conjecture" follows:

- There exists $T_{**} = \inf\{\varepsilon t_\varepsilon \,:\, \varepsilon \in (0, \varepsilon_*]\} > 0$ such that, as $\varepsilon \to 0$,

$$q(\tau) \to q_0(\tau) \quad \text{uniformly on } [0, T_{**}]$$

- $\|Y\|_3$ remains bounded for $t \in [0, T_{**}/\varepsilon]$, and

$$\|Y\|_{C^0} \le c\|Y\|_2 \le c\|Y\|_3,$$

so $\varphi_\varepsilon(\tau)$ converges uniformly on $[0, T_{**}]$ to $\psi(q_0(\tau))$

A priori bound

"Precise conjecture" follows:

- There exists $T_{**} = \inf\{\varepsilon t_\varepsilon : \varepsilon \in (0, \varepsilon_*]\} > 0$ such that, as $\varepsilon \to 0$,

$$q(\tau) \to q_0(\tau) \quad \text{uniformly on } [0, T_{**}]$$

- $\|Y\|_3$ remains bounded for $t \in [0, T_{**}/\varepsilon]$, and

$$\|Y\|_{C^0} \leq c\|Y\|_2 \leq c\|Y\|_3,$$

so $\varphi_\varepsilon(\tau)$ converges uniformly on $[0, T_{**}]$ to $\psi(q_0(\tau))$

Concluding remarks

- Proved for $\Sigma = T^2$, but argument should immediately generalize to any compact RS
- Generalizing target space is much harder. Static model has right properties when N=compact Kähler, but what's analogue of $\varphi(t) = \psi(\tau) + \varepsilon^2 Y(t)$?
- We first tried (even for $N = S^2$!)

$$\varphi(t) = \exp_{\psi(\tau)} \varepsilon^2 Y(t)$$

  but this doesn't work (j has Y_{xx} terms, fatal for short time existence result and higher energy estimate)
- Presumably extrinsic set-up can be used. Hideous.

Concluding remarks

- Proved for $\Sigma = T^2$, but argument should immediately generalize to any compact RS
- Generalizing target space is much harder. Static model has right properties when N=compact Kähler, but what's analogue of $\varphi(t) = \psi(\tau) + \varepsilon^2 Y(t)$?
- We first tried (even for $N = S^2$!)

$$\varphi(t) = \exp_{\psi(\tau)} \varepsilon^2 Y(t)$$

  but this doesn't work (j has Y_{xx} terms, fatal for short time existence result and higher energy estimate)
- Presumably extrinsic set-up can be used. Hideous.

Concluding remarks

- Proved for $\Sigma = T^2$, but argument should immediately generalize to any compact RS
- Generalizing target space is much harder. Static model has right properties when N=compact Kähler, but what's analogue of $\varphi(t) = \psi(\tau) + \varepsilon^2 Y(t)$?
- We first tried (even for $N = S^2$!)

$$\varphi(t) = \exp_{\psi(\tau)} \varepsilon^2 Y(t)$$

but this doesn't work (j has Y_{xx} terms, fatal for short time existence result and higher energy estimate)

- Presumably extrinsic set-up can be used. Hideous.

Concluding remarks

- Proved for $\Sigma = T^2$, but argument should immediately generalize to any compact RS
- Generalizing target space is much harder. Static model has right properties when N=compact Kähler, but what's analogue of $\varphi(t) = \psi(\tau) + \varepsilon^2 Y(t)$?
- We first tried (even for $N = S^2$!)

  $$\varphi(t) = \exp_{\psi(\tau)} \varepsilon^2 Y(t)$$

  but this doesn't work (j has Y_{xx} terms, fatal for short time existence result and higher energy estimate)
- Presumably extrinsic set-up can be used. Hideous.