Wave-map flow on a compact Riemann surface

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Wave-map flow

- $\varphi : \mathbb{R} \times \Sigma \to N \subset \mathbb{R}^k$,

$$(\Box\varphi)(t,x) \perp T_{\varphi(t,x)}N \qquad \text{for all } (t,x) \in M$$

  where $\Box = \partial_t^2 - \Delta_\Sigma$
- $N = S^2 \subset \mathbb{R}^3$

$$\begin{aligned} \Box\varphi - (\varphi \cdot \Box\varphi)\varphi &= 0 \\ \Box\varphi + (|\varphi_t|^2 - |d\varphi|^2)\varphi &= 0 \end{aligned}$$

  semilinear wave equation.
- Henceforth $\dim_{\mathbb{R}} \Sigma = 2$

Static solutions

- $\varphi : \Sigma \to N$ static wave map $\Leftrightarrow$ harmonic, i.e. critical pt of

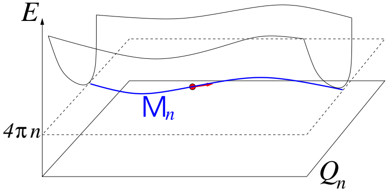
$$E(\varphi) = \frac{1}{2}\int_\Sigma |\mathrm{d}\varphi|^2$$

- Holomorphic maps $\Sigma \to S^2$ minimize E in their htpy class

$$M_n = Hol_n(\Sigma, S^2) = \begin{cases} \mathrm{Rat}_n^* & \Sigma = \mathbb{C} & \dim_{\mathbb{C}} = 2n \\ \mathrm{Rat}_n & \Sigma = S^2 & \dim_{\mathbb{C}} = 2n+1 \\ ??? & \Sigma = \Sigma_g & \dim_{\mathbb{C}} = 2n+1-g \end{cases}$$

- Noncompact, lump bubbling

The conjecture (Ward, after Manton)



- For $(\varphi(0), \varphi_t(0)) \in T_{\varphi(0)} M_n$, $|\varphi_t(0)|$ small, wave map $\varphi(t)$ is well approximated by the **geodesic** in M_n (with respect to the L^2 metric).
- Motivates study of geodesic flow in M_n (Ward, Leese, JMS, Sadun, Cova, Baptista, McGlade, Romão)

Theorem (JMS)

Let Σ be a compact Riemann surface of genus g and $n \geq g$.
For fixed $\varphi_0 \in \mathsf{M}_n$ and $\varphi_1 \in T_{\varphi_0}\mathsf{M}_n$ consider the one parameter family of wave-map IVPs

$$\varphi(0) = \varphi_0, \qquad \varphi_t(0) = \varepsilon\varphi_1,$$

parametrized by $\varepsilon > 0$.There exist constants $\tau_* > 0$ and $\varepsilon_* > 0$ such that for all $\varepsilon \in (0, \varepsilon_*]$, the problem has a unique solution for $t \in [0, \tau_*/\varepsilon]$.Furthermore, the time re-scaled solution

$$\varphi_\varepsilon : [0, \tau_*] \times \Sigma \to S^2, \qquad \varphi_\varepsilon(\tau, x) = \varphi(\tau/\varepsilon, x)$$

converges uniformly in C^1 to $\psi : [0, \tau_*] \times \Sigma \to S^2$, the geodesic in M_n with the same initial data, as $\varepsilon \to 0$.

$$\delta_\varepsilon = \sup_{(\tau,x)\in[0,\tau_*]\times\Sigma} \{|\varphi_\varepsilon - \psi|, |d\varphi_\varepsilon - d\psi|, |\dot{\varphi}_\varepsilon - \dot{\psi}|\} \to 0$$

Proof strategy

- Proof uses Stuart's projection/energy estimate strategy
- He dealt with vortices on $\mathbb{C}$ and monopoles on $\mathbb{R}^3$
- What's easier here: Σ is compact, no gauge symmetry
- What's harder here: M_n is (badly) noncompact, target S^2 nonlinear

Projection to M_n

- Slow time $\tau = \varepsilon t$, $\dot{} = d/d\tau$
- Given curve $\psi(\tau) \in M_n$ and wave-map φ, define error section Y

$$\varphi = \psi + \varepsilon^2 Y$$

- φ wave-map iff

$$\begin{aligned} Y_{tt} + J_{\psi(\tau)} Y &= k + \varepsilon j \\ k &= -(\psi_{\tau\tau} + |\psi_\tau|^2 \psi) \\ j &= \mathit{polynomial}(\varepsilon, \psi, \mathrm{d}\psi, \psi_\tau, Y, \mathrm{d}Y, Y_t) \\ J_\psi Y &= -\Delta Y - |\mathrm{d}\psi|^2 Y - 2(\mathrm{d}\psi, \mathrm{d}Y)\psi \end{aligned}$$

- $J_\psi =$ Jacobi operator for harmonic map ψ
  - Self-adjoint operator on tangent sections ($\psi \cdot Y = 0$)
  - Elliptic
  - $\ker J_\psi = T_\psi M_n$
  - Coercive: for all tangent sections $\perp_{L^2}$ to $\ker J_\psi$,

$$\langle Y, J_\psi Y \rangle_{L^2} \geq c(\psi) \| Y \|^2_{H^1}$$

what's H^1?

Sobolev space interlude

- $H^k = \{f : \Sigma \to \mathbb{R} \, : \, f, \nabla f, \ldots, \nabla^k f \in L^2\}$
- Banach space wrt $\|f\|_k^2 = \|f\|^2 + \|\nabla f\|^2 + \cdots + \|\nabla^k f\|^2$
- Banach **algebra** if $k \geq 2$: if $f, g \in H^k$ then $fg \in H^k$ and $\|fg\|_k \leq C\|f\|_k\|g\|_k$
- $\|f\|_{C^0} \leq C\|f\|_2$

back to the plot

Projection to M_n

- Problem: Y is **not** a tangent section,

$$\psi \cdot Y = -\frac{1}{2}\varepsilon^2 |Y|^2 \quad (*)$$

- J_ψ is **not** self-adjoint on $\psi^{-1}\underline{\mathbb{R}^3}$

$$J_\psi Y = -\Delta Y - |\mathrm{d}\psi|^2 Y + A_\psi Y$$

- Improved Jacobi operator: $L_\psi = J_\psi + A_\psi^\dagger$
  - Self-adjoint operator on $\psi^{-1}\underline{\mathbb{R}^3}$
  - Elliptic
  - $L_\psi = J_\psi$ on tangent sections
  - Nearly coercive: for all $(*)$-sections $\perp_{L^2}$ to $\ker J_\psi$

$$\langle Y, L_\psi Y\rangle_{L^2} \geq c(\psi)\|Y\|_1^2 - \varepsilon^2 \widetilde{c}(\psi)\|Y\|_2^3$$

Projection to M_n

- On $(*)$ sections,

$$A_\psi^\dagger Y = \varepsilon^2 \{ |Y|^2 \Delta\psi + (\mathrm{d}|Y|^2, \mathrm{d}\psi) \} =: \varepsilon \hat{j}$$

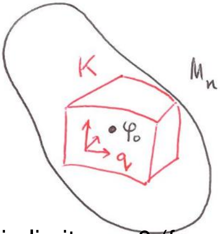
  so φ wave-map iff

$$Y_{tt} + L_\psi Y = k + \varepsilon j', \qquad j' := j + \hat{j}$$

- Dynamics for $\psi(\tau)$: demand $Y(t) \perp_{L^2}$ to $\ker J_{\psi(\tau)}$

$$\begin{aligned}
\langle Y, \frac{\partial\psi}{\partial q^\mu} \rangle &= 0 \\
\langle Y_{tt}, \frac{\partial\psi}{\partial q^\mu} \rangle &= O(\varepsilon) \\
\langle -L_\psi Y + k, \frac{\partial\psi}{\partial q^\mu} \rangle &= O(\varepsilon) \\
\langle \psi_{\tau\tau}, \frac{\partial\psi}{\partial q^\mu} \rangle &= O(\varepsilon)
\end{aligned}$$

Geodesic flow in limit $\varepsilon \to 0$ (formally)



Step 1: local existence

$$\begin{aligned} \ddot{q} + G(q)\dot{q}\dot{q} &= \varepsilon h(q, \dot{q}, Y, \mathrm{d}Y, Y_t) \\ Y_{tt} + L_q Y &= k + \varepsilon j'(q, \dot{q}, Y, \mathrm{d}Y, Y_t) \qquad (CS) \end{aligned}$$

- Given initial data $q(0) \in K$, $\dot{q}(0), Y(0), Y_t(0)$ with $|\dot{q}(0)| + \|Y(0)\|_3 + \|Y_t(0)\|_2 \leq \Gamma$, system (CS) has a unique solution for $t \in [0, T(\Gamma)]$.The solution remains in the ball of radius $c\Gamma$ in $K \times \mathbb{R}^k \times H^3 \times H^2$.If the initial data are tangent to the $(*)$ and $\perp_{L^2}$ constraints, the solution preserves them.
- Prove using Picard's method.
- Can apply iteratively: solution exists whilever q remains in K and $\dot{q}$, $\|Y\|_3$, $\|Y_t\|_2$ remain bounded.

Step 2: energy estimates

$$
\begin{aligned}
E_1(t) &= \frac{1}{2}\|Y_t\|^2 + \frac{1}{2}\langle Y, LY\rangle \\
\frac{dE_1}{dt} &= \langle Y_t, -LY + k + \varepsilon j'\rangle + \langle Y_t, LY\rangle + \frac{1}{2}\varepsilon\langle Y, L_\tau Y\rangle \\
&= \langle Y_t, k\rangle + \varepsilon\{\cdots\} \\
&= \frac{d}{dt}\langle Y, k\rangle + \varepsilon\{\cdots\} \\
E_1(t) &\le C + C(|\ddot{q}| + |\dot{q}|^2)\|Y(t)\| + \varepsilon\int_0^t c(|\dot{q}|, |\ddot{q}|, |\dddot{q}|, \|Y\|_3, \|Y_t\|_2)
\end{aligned}
$$

Apply same argument to LY

$$
\begin{aligned}
E_2(t) &= \frac{1}{2}\|(LY)_t\|^2 + \frac{1}{2}\langle LY, LLY\rangle \\
E_2(t) &\le C + C(|\ddot{q}| + |\dot{q}|^2)\|Y(t)\|_2 + \varepsilon\int_0^t c(|\dot{q}|, |\ddot{q}|, |\dddot{q}|, \|Y\|_3, \|Y_t\|_2)
\end{aligned}
$$

Step 3: a priori bound

- Define deviation $\widetilde{q}$ of $q(\tau)$ from geodesic q_*

$$q = q_* + \varepsilon^2 \widetilde{q}$$

- Measure total error by

$$M(s) = \max_{0 \le t \le s} \{ \varepsilon^2 |\widetilde{q}(t)|^2 + |\widetilde{q}_t(t)|^2 + |\widetilde{q}_{tt}(t)|^2 + \| Y(t) \|_3^2 + \| Y_t(t) \|_2^2 \}$$

  Solution exists whilever $M(t)$ remains bounded

- q solves geodesic eqn $+O(\varepsilon) \Rightarrow$

$$\begin{array}{rrcll} & |\widetilde{q}_{tt}| & \le & \varepsilon c(M(t)) & \\ \Rightarrow & |\widetilde{q}_t| & \le & \varepsilon t c(M(t)) & (\widetilde{q}_t(0) = 0) \\ \Rightarrow & |\varepsilon \widetilde{q}| & \le & \varepsilon^2 t^2 c(M(t)) & (\widetilde{q}(0) = 0) \end{array}$$

Step 3: a priori bound

$$
\begin{aligned}
\|Y\|_3^2 + \|Y_t\|_2^2 &\le E_1(t) + E_2(t) + \varepsilon^2 c(M) \qquad \text{(coercivity)} \\
&\le C + CM^{\frac{1}{2}} + \varepsilon t c(M) + \varepsilon^2 c(M) \qquad \text{(energy estimate)} \\
\Rightarrow \quad M(t) &\le C + CM(t)^{\frac{1}{2}} + (\varepsilon t + \varepsilon^2 t^2 + \varepsilon^2) c(M(t))
\end{aligned}
$$

- Choose M_* large, define $t_\varepsilon = \sup\{t : M(t) \le M_*\}$
- Claim $\exists \varepsilon_* > 0$ s.t. $\varepsilon t_\varepsilon$ is bded away from 0 on $(0, \varepsilon_*]$
- Assume not: $\exists$ sequence $\varepsilon \to 0$ s.t. $\varepsilon t_\varepsilon \to 0$. Then

$$M_* \le C + CM_*^{\frac{1}{2}} \qquad \otimes$$

- M remains bded for time of order $1/\varepsilon$. Theorem follows.

Concluding remarks

- Loosely: the geodesic approximation "works" for times of order $1/\varepsilon$ when the initial velocities are of order ε
- Can't do much better: M_n incomplete!
- τ_* depends on how close φ_0 is to ∂M_n.
- Geodesic approx. certainly **fails** very close to blow-up (Rodnianski and Sterbenz)