Two-vortex dynamics on flat tori

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Vortices

- $\begin{array}{cc} (L,h) & \text{degree } n \\ \downarrow & \\ (\Sigma, g_\Sigma) & \text{genus } g \end{array}$: section φ, connexion A

$$\begin{aligned} E(\varphi, A) &= \frac{1}{2}\|d_A\varphi\|^2_{L^2} + \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{2}\left\|\frac{1}{2}(\tau - |\varphi|^2)\right\|^2_{L^2} \\ &= \frac{1}{2}\left\|*F_A - \frac{1}{2}(\tau - |\varphi|^2)\right\|^2_{L^2} + \|\overline{\partial}_A\varphi\|^2_{L^2} + \frac{\tau}{2}\int_\Sigma F_A \\ &\geq \tau\pi n \end{aligned}$$

with equality iff

$$\overline{\partial}_A\varphi = 0 \qquad (V1), \qquad\qquad *F_A = \frac{1}{2}(\tau - |\varphi|^2) \qquad (V2).$$

Moduli space

- $M_n = \{\text{solutions of (V1), (V2)}\}/\text{gauge transformations}$
- Bradlow bound: $\int_\Sigma (V2)$:

$$\begin{aligned} 2\pi n &= \frac{\tau}{2}|\Sigma| - \frac{1}{2}\|\varphi\|_{L^2}^2 \\ \Rightarrow \|\varphi\|_{L^2}^2 &= \tau|\Sigma| - 4\pi n =: \varepsilon \geq 0 \end{aligned}$$

- Bradlow (1990), Garcia Prada (1991):

$$M_n = \begin{cases} \emptyset & \varepsilon < 0, \\ T^{2g} & \varepsilon = 0, \\ S^n\Sigma & \varepsilon > 0 \end{cases}$$

where $S^n\Sigma = \Sigma^n / S_n$

Geodesic approximation

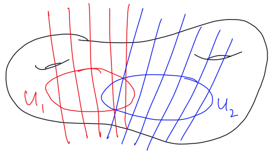
- Low energy vortex dynamics : geodesic motion in (M_n, g_{L^2})
- Curve of static vortex solutions $(\varphi(t), A(t))$, chosen so that

$$\delta \dot{A} + h(i\varphi, \dot{\varphi}) = 0.$$

- $g_{L^2}((\dot{\varphi}, \dot{A}), (\dot{\varphi}, \dot{A})) = \|\dot{\varphi}\|_{L^2}^2 + \|\dot{A}\|_{L^2}^2.$

Holomorphic structure on L

- A **holomorphic structure** on L is defined by a local trivialization with **holomorphic** transition functions $\tau_{ij} : U_i \cap U_j \to \mathbb{C}^\times$



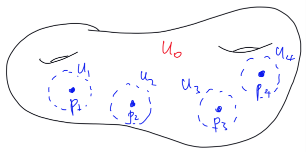
- $\overline{\partial}\varphi = 0$ now well-defined.
- Space of solutions $H^0(\Sigma, L)$, dim $n + 1 - g$ complex vector space.

Divisor $\Rightarrow$ holomorphic structure on L

- Divisor

$$D = n_1 p_1 + n_2 p_2 + \cdots + n_k p_k, \qquad n_1 + n_2 + \cdots + n_k = n$$

defines a holomorphic structure on L



$$\tau_{i0}(z) = z^{n_i}$$

- L_D comes with a holomorphic section

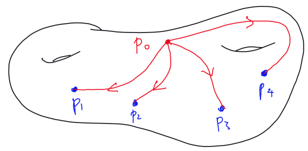
$$\widehat{\varphi}_D(z) = \begin{cases} 1 & z \in U_0 \\ z^{n_i} & z \in U_i \end{cases}$$

unique holomorphic section vanishing on D (up to $\widehat{\varphi}_D \mapsto c\widehat{\varphi}_D$)

Abel-Jacobi map

- Holomorphic one-forms on Σ: $\nu_1, \nu_2, \ldots, \nu_g$

$$AJ(\{p_1, p_2, \ldots, p_n\}) = \begin{pmatrix} \int_{p_0}^{p_1} \nu_1 + \int_{p_0}^{p_1} \nu_1 + \cdots + \int_{p_0}^{p_n} \nu_1 \\ \int_{p_0}^{p_1} \nu_2 + \int_{p_0}^{p_1} \nu_2 + \cdots + \int_{p_0}^{p_n} \nu_2 \\ \vdots \\ \int_{p_0}^{p_1} \nu_g + \int_{p_0}^{p_1} \nu_g + \cdots + \int_{p_0}^{p_n} \nu_g \end{pmatrix}$$



- $AJ : S^n\Sigma \to \mathbb{C}^g/\Lambda_{Abel} = T^{2g}_{Abel}$

The fibration of M_n

- Amazing fact: $D \sim D'$ (i.e. they equip L with the same holomorphic structure) iff $AJ(D) = AJ(D')$.

$$\begin{array}{ccc} \mathbb{P}(H^0(\Sigma, L_{[D]})) & \hookrightarrow & M_n \\ & & \downarrow \quad AJ \\ & & T^{2g}_{Abel} \end{array}$$

- M_n has structure of a $\mathbb{C}P^{n-g}$ bundle over T^{2g}

The dissolving limit

- Fix a fibre $[D] \subset M_n$. Has unique (up to gauge) constant curvature connexion $\widehat{A}$ s.t. $\overline{\partial}_{\widehat{A}} = \overline{\partial}_{[D]}$.
- For each $D \in [D]$, let $\widehat{\varphi}_D$ be "the" holo section vanishing on D with $\|\widehat{\varphi}_D\|_{L^2} = 1$.
- Pseudovortex for divisor D: $(\sqrt{\varepsilon}\widehat{\varphi}_D, \widehat{A})$

$$\overline{\partial}_{\widehat{A}}(\sqrt{\varepsilon}\widehat{\varphi}_D = 0,)$$

- Rink (2013) after Baptista and Manton (2003) conjectured:
  (1) For small $\varepsilon > 0$, Pseudovortex for divisor D is a good approx to actual vortex for divisor D
  (2) L^2 metric on fibre $[D]$ is well approximated by L^2 metric on space of pseudovortices $= \varepsilon g_{FS}$

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The dissolving limit

- **Theorem** (Chaudhuri, Harland, JMS 2024) There exists $C > 0$ (depending on $[D]$ and g_Σ) such that, for all $D \in [D]$

$$\|\varepsilon^{-1/2}|\varphi_D| - |\widehat{\varphi}_D|\|_{C^0} + \|F_{A_D} - F_{\widehat{A}}\|_{C^0} \leq C\varepsilon.$$

  Furthermore

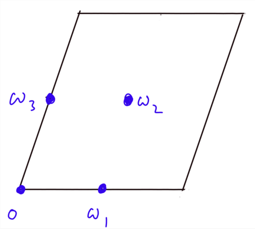
$$\|\varepsilon^{-1} g_{L^2}|_{[D]} - g_{FS}\|_{C^1} \leq C\varepsilon$$

- Vortices in a fixed fibre converge uniformly to pseudovortices, and the induced L^2 metric on each fibre converges in C^1 to the Fubini-Study metric.

The dissolving limit

- Strengthens Garcia-Lara, JMS 2023 in 2 ways:
  - Generalizes to arbitrary genus (from $g = 0$)
  - C^1 convergence of metric (from C^0 convergence)
  - Implies pointwise convergence of **geodesics**.
- But geodesics in **fibre** are only dynamically interesting if the fibre is totally geodesic! For generic Σ, g_Σ, n, none are!
- Two simple cases with all fibres totally geodesic:
  - $\Sigma = S^2$ (**any** g_Σ): $T^{2g}_{Abel} = \{pt\}$, the whole M_n is a single fibre!
  - $\Sigma = T^2 = \mathbb{C}/\Lambda$, $n = 2$: $T^{2g}_{Abel} \equiv \Sigma$, fibres $\equiv \mathbb{C}P^1 \equiv S^2$

$M_2(T^2)$

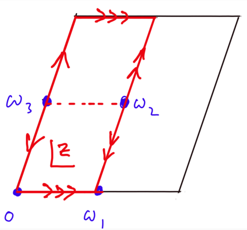


- $g = 1$: only one holo one-form, $\nu = dz$
- $AJ(\{z_1, z_2\}) = \int_0^{z_1} dz + \int_0^{z_2} dz = z_1 + z_2 \in \mathbb{C}/\Lambda$
- Isometry $T_a : M_2 \to M_2$, $D = \{z_1, z_2\} \mapsto \{z_1 + a, z_2 + a\}$ maps fibre above $AJ(D)$ to fibre above $AJ(D) + na$
- Hence all fibres are isometric

$M_2(T^2)$

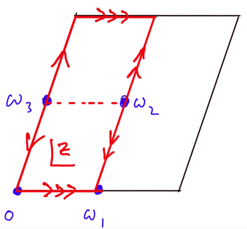
- Isometry $R : M_2 \to M_2$, $\{z_1, z_2\} \mapsto \{-z_1, -z_2\}$
- $AJ^{-1}(0) = \{\{z_1, z_2\} \,:\, z_1 + z_2 = 0\}$ = fixed point set of R.
- Hence $AJ^{-1}(0)$ is totally geodesic!
- Isometries map totally geodesic submfds to totally geodesic submfds, so all fibres are totally geodesic.

$AJ^{-1}(0)$ = pillowcase



- Recall, in limit $\varepsilon \to 0$, the metric on this pillowcase converges in C^1 to round metric on S^2!
- Can we write down the isometry $f : AJ^{-1}(0) \to S^2_{round}$?
  - It's holomorphic (w.r.t. coord z)
  - It has branch points at the half periods
  - It extends to an even degree 2 holomorphic map $\Sigma \to S^2$
  - It must be of the form $f : z \mapsto M(\wp(z))$

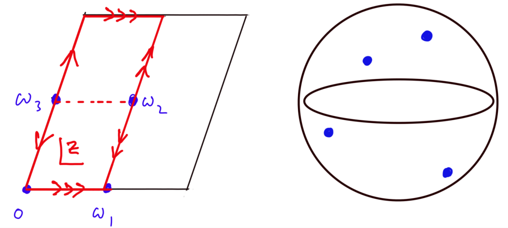
$AJ^{-1}(0) =$ pillowcase



- $T_{\omega_i} : z \mapsto z + \omega_i$ is an isometry of $AJ^{-1}(0)$.
- These generate $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.
- Only isometric action of K_4 on S^2 is generated by rotations by π about a pair of orthogonal axes.
- WLOG can assume

$$T_{\omega_1} \leftrightarrow R_3(\pi), \qquad T_{\omega_3} \leftrightarrow R_2(\pi).$$

$AJ^{-1}(0) =$ pillowcase

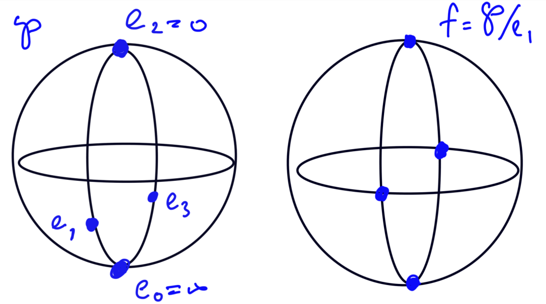


$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \qquad e_i = \wp(\omega_i)$$

- Hence M must map the critical values of $\wp$ to 4 points constituting a K_4 orbit in S^2.
- Uniquely determines M up to conjugation by $SU(2)$.

Square torus

- $e_0 = \infty,\ e_1 = 6.875\cdots,\ e_2 = 0,\ e_3 = -e_1$
- $f(z) = \wp(z)/e_1$



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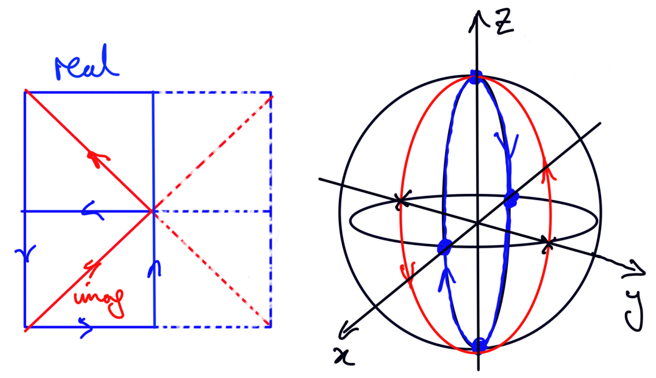

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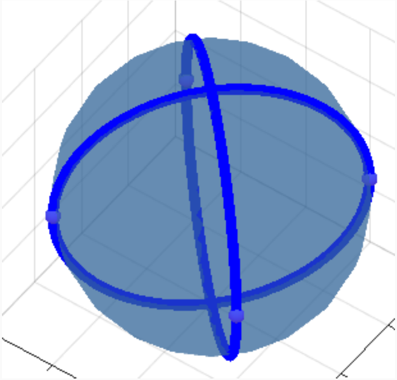
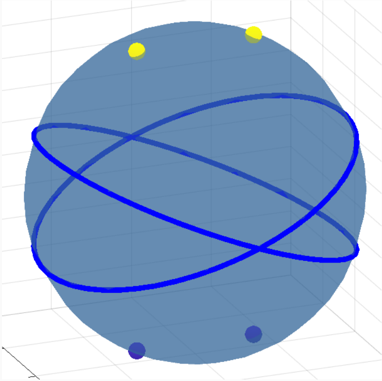


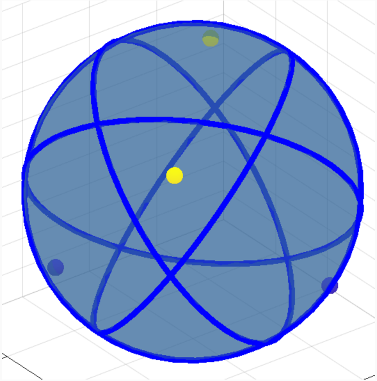
Rectangular torus

$$\omega_1 = 1.5, \quad \omega_3 = i$$

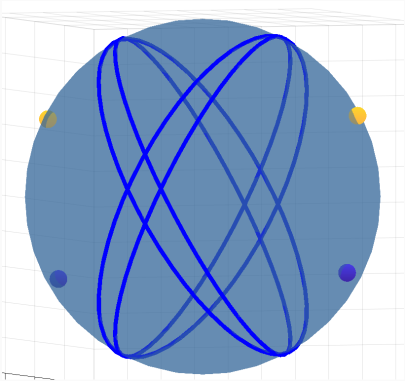


Equianharmonic torus

$$\omega_1 = 1, \quad \omega_3 = e^{i\pi/3}$$



Scalene torus



Summary

- Geodesics in $M_2(T^2)$ converge, in the dissolving limit, to geodesics in the pillowcase $AJ^{-1}(0)$ with the round metric.
- We can compute this metric explicitly (as a metric on the pillowcase).
- Geodesic flow depends qualitatively on Λ: if Λ is rectangular, all primitive non-scattering geodesics lift to $\Sigma \times \Sigma$
- If Λ not rectangular, there are "pursuit" geodesics. These only lift if you traverse them twice.
- Global structure of $M_2(T^2)$?