Supercurrent coupling in the abstract Faddeev-Skyrme model

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The model

$$E(\varphi, C) = \frac{1}{8}\|\mathrm{d}\varphi\|^2 + \frac{1}{2}\|\mathrm{d}C + \frac{1}{2}\varphi^*\omega\|^2 + \frac{1}{2}\|C\|^2$$

- $\varphi : M \to N$
  (M, g) Riemannian, (N, h, J) Kähler, $\omega(X, Y) = h(JX, Y)$
- $C \in \Omega^1(M)$
- $\|\cdot\| = L^2$ norm: $\|C\|^2 = \int_M |C|^2 = \int_M C \wedge *C$ etc.
- e.g. $M = \mathbb{R}^3$, $N = S^2$ ($JX = \varphi \times X$)
  Looks like Faddeev-Skyrme model (coincide when $C = 0$)
- "Supercurrent coupled" FS model

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Why? Multicomponent GL theory

- $\psi_1, \ldots, \psi_k : M \to \mathbb{C}, \qquad A \in \Omega^1(M)$

$$E_{GL} = \frac{1}{2}\sum_{a=1}^{k} \|\mathrm{d}_A\psi_a\|^2 + \frac{1}{2}\|\mathrm{d}A\|^2 + \int_M U(\psi)$$

- Sigma model limit: assume U strongly confines ψ to sphere

$$|\psi_1|^2 + \cdots + |\psi_k|^2 = 1$$

  e.g. $U = \lambda(1 - |\psi|^2)^2$, $\lambda \to \infty$

- Fact (Hindmarsh, Babaev-Faddeev-Niemi,...): $E_{GL} = E(\varphi, C)$ where

$$\begin{aligned} \varphi &= [\psi_1, \cdots, \psi_k] : M \to \mathbb{C}P^{k-1} \\ C &= \frac{i}{2}\sum_a (\overline{\psi}_a \mathrm{d}_A\psi_a - \psi_a \overline{\mathrm{d}_A\psi_a}) = \text{total supercurrent} \end{aligned}$$

- $k = 2$, $M = \mathbb{R}^3$: FS model has knot solitons.
  SCFS model? If so, expect they persist to 2-cpt GL theory.

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$$\begin{aligned}
&= \frac{1}{4}\left\{\frac{1}{2}\|\mathrm{d}\varphi\|^2 + \frac{1}{2}\|\varphi^*\omega\|^2\right\} + \frac{1}{2}\left\{\|\mathrm{d}C\|^2 + \|C\|^2\right\} + \frac{1}{2}\langle \mathrm{d}C, \varphi^*\omega\rangle \\
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$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)
- SCFS model: $\inf\{E(\varphi, C) \,:\, Q(\varphi) = n\} = 0$ for all $n \in \mathbb{Z}$

  Proof: $\exists$ φ with $\varphi = (0, 0, 1)$ outside $\overline{B}$.

  $\exists$ C s.t. $\varphi^*\omega = -2\mathrm{d}C$ ($H^2(M) = 0$).

  Can assume $C = 0$ outside $\overline{B}$ ($H^1(M\backslash\overline{B}) = 0$).

  Let $D_\lambda : M \to M$, $D_\lambda(x) = \lambda x$, $\lambda > 0$.

$$E(\varphi \circ D_\lambda, D_\lambda^* C) = \frac{1}{2\lambda}\|\mathrm{d}\varphi\|^2 + 0 + \frac{1}{2\lambda}\|C\|^2$$

- SCFS can't have (nontrivial) global minimizers. Local minimizers?

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  Proof: Choose A s.t. $\mathrm{d}A = \varphi^*\omega$.

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A &= A_{\mathrm{harmonic}} + \mathrm{d}A_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\
E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle \mathrm{d}A, \mathrm{d}A\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\
16\pi^2 Q &= \int A \wedge \mathrm{d}A \leq \|A\|\|\mathrm{d}A\| \leq \|A\|\sqrt{2E_2(\varphi)} \\
E_2(\varphi) &\geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi).
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$$\flat : TM \to T^*M, \quad \flat X = g(X,\cdot), \qquad \sharp = \flat^{-1} : T^*M \to TM$$

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for all X, Y

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$$\begin{aligned}\delta(\mathrm{d}C + \frac{1}{2}\varphi^*\omega) + C &= 0,\\ \tau(\varphi) - 2J\mathrm{d}\varphi\sharp C &= 0.\end{aligned}$$

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Embedded Hopf map $S^3_R \longrightarrow S^2$

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- $G = SU(2)$, $K = \{\mathrm{diag}(\lambda, \bar{\lambda}) : \lambda \in U(1)\}$, $\varphi : x \mapsto xK$

  Left-invariant vector fields θ_a, one-forms σ_a

  At $x = e$, $\theta_a = \frac{i}{2}\tau_a$.Radius $R \Rightarrow |\theta_a| = \frac{R}{2}$

$$\text{Try } C = \mu\sigma_3 \quad \text{[then (2) holds automatically]}$$

$$\varphi^*\omega = -\sigma_1 \wedge \sigma_2$$

$$\delta(\varphi^*\omega) = -*\mathrm{d}*\sigma_1 \wedge \sigma_2 = -\frac{4}{R^2}\sigma_3$$

$$(1) \Leftrightarrow \mu = \frac{2}{4+R^2} \quad [B = -\frac{4}{R(4+R^2)}\vartheta_3]$$

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Second variation

$\varphi_{s,t} : M \longrightarrow N$ $\qquad$ $C_{s,t} \in \Omega^1(M)$

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- Can compute $\mathscr{H}$ explicitly
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(In)stability of the embedded Hopf map

$$G = SU(2), \quad K = S(U(1) \times U(1)), \qquad \varphi(x) = xK, \quad C = \frac{2\sigma_3}{4 + R^2}$$

- Can compute $\mathscr{H}$ explicitly
- $X = f_1 \mathrm{d}\varphi\theta_1 + f_2 \mathrm{d}\varphi\theta_2$, $Y = f_3\sigma_1 + f_4\sigma_2 + f_5\sigma_3$, $f_1, \ldots, f_5 \in C^\infty(G)$
- $\mathscr{J}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{4}{R^2}\begin{pmatrix} -(\theta_1^2 + \theta_2^2 + \theta_3^2) & -2\theta_3 \\ 2\theta_3 & -(\theta_1^2 + \theta_2^2 + \theta_3^2) \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$
- Similar expressions for $\mathscr{L}, \mathscr{A}, \mathscr{B}, \mathscr{C}, \delta\mathrm{d}$ as matrices of diffops
- Peter-Weyl theorem: matrix elements of unitary irreps of G form basis for $L^2(G)$. Expand each $f_a : G \to \mathbb{R}$ in this basis.
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(In)stability of the embedded Hopf map

$$\mathscr{J} = \begin{bmatrix} 3\,R^{-2} & 0 & \frac{-4\,i}{R^2} & 0 \\ 0 & 3\,R^{-2} & 0 & \frac{4\,i}{R^2} \\ \frac{4\,i}{R^2} & 0 & 3\,R^{-2} & 0 \\ 0 & \frac{-4\,i}{R^2} & 0 & 3\,R^{-2} \end{bmatrix}, \quad \mathscr{L} = \begin{bmatrix} 8\,R^{-4} & 0 & \frac{-4\,i}{R^4} & 0 \\ 0 & 8\,R^{-4} & 0 & \frac{4\,i}{R^4} \\ \frac{4\,i}{R^4} & 0 & 8\,R^{-4} & 0 \\ 0 & \frac{-4\,i}{R^4} & 0 & 8\,R^{-4} \end{bmatrix}, \ldots$$

- Maple finds negative eigenvalue of $\mathscr{H}$

$$\lambda(R) = \frac{p(R) - \sqrt{p(R)^2 + 3840R^4 + 1536R^6 + 208R^8 + 16R^{10}}}{8R^4(4+R^2)}$$

where $p(R) = 48 + 144R^2 + 51R^4 + 4R^6$

- Total multiplicity 4. Eigenvector “mixed”
- No other negative eigenvalues $0 \leq s \leq 10$. Index probably 4.

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Concluding remarks

- Embedded hopf "soliton" unstable on S^3_R for all R
- On M^3, supercurrent coupling destroys VK bound (even if keep topology)
- Not so on M^2
- Can find other exact embedded critical points of SCFS.
  E.g. Hopf map
  $\mathbb{C}^k \supset S^{2k-1} \to \mathbb{C}P^{k-1}, \quad (z_1, \ldots, z_k) \mapsto [z_1, \ldots, z_k]$
  critical for $C = \mu \flat iz$
- $(\varphi, 0)$ critical iff φ harmonic and coclosed.
  E.g. $\mathrm{Id} : N \to N$, $\pi : N \times K \to N$, $\pi_k : \mathrm{Flag}(\mathbb{C}^n) \to Gr_{k,n}$
- $(\mathrm{Id}, 0)$ is **stable**. Suggests better luck on M^2.

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