Skyrme Crystals

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Joint work with Derek Harland and Paul Leask

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General question

- When is a soliton on a torus

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- Clearly an artifact of b.c.s!
- φ should minimize energy E w.r.t. all variations of field **and period lattice** Λ

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$$f : \mathbb{T}^3 \to \mathbb{R}^k/\Lambda, \qquad f(\mathbf{x}) = x_1\mathbf{X}_1 + x_2\mathbf{X}_2 + \cdots + x_k\mathbf{X}_k$$

Now mfd is fixed, but **metric** depends on Λ

$$g_\Lambda = f^* g_{Euc} = g_{ij}dx_i dx_j, \qquad g_{ij} = \mathbf{X}_i \cdot \mathbf{X}_j \text{ const}$$

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- Criticality $\leftrightarrow$ stress tensor
- Manoeuvre works provided $E(\varphi, g)$ is **geometrically natural**

$$E(\varphi \circ f, g) = E(\varphi, (f^{-1})^* g)$$

Skyrme model

$$E : Maps(\mathbb{T}^3, SU(2)) \times SPD_3 \to \mathbb{R}$$

Two minimization problems:

- Fix g. Does $E(\cdot, g) : Maps \to \mathbb{R}$ attain a min in each homotopy class?

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- Fix φ? Does $E(\varphi, \cdot) : SPD_3 \to \mathbb{R}$ attain a min? YES! And it's a global min, and there are no other critical points!

The Skyrme energy

$$E(\varphi, g) = \int_{\mathbb{T}^3} \left(-\frac{1}{2} \mathrm{tr}(L_i L_j) g^{ij} - \frac{1}{16} \mathrm{tr}([L_i, L_j][L_k, L_l]) g^{ik} g^{jl} + V(\varphi) \right) \sqrt{|g|} d^3 x$$

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- **Constants:**
  - H, Ω: symmetric positive **semidefinite** matrices
  - $C \geq 0$
- Nondegeneracy assumption: φ is C^1 and immersive somewhere (automatic if $B \neq 0$).

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$$E_4 = \frac{1}{4}\int_{\mathbb{T}^3} |\varphi^*\omega|_g^2 \mathrm{vol}_g, \qquad \omega \in \Omega^2(G)\otimes\mathfrak{g}$$

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- Clearly $\sqrt{|g|}X_\varphi^g = X_\varphi$
- Now $X_\varphi^g = \sharp_g *_g \varphi^*\omega$, so

$$|\varphi^*\omega|_g^2 = |X_\varphi^g|_g^2 = \frac{1}{|g|}g(X_\varphi, X_\varphi)$$

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- Hence

$$E_4(g) = \frac{g_{ij}}{\sqrt{|g|}}\Omega_{ij}$$
$$\Omega_{ij} = \frac{1}{4}\int_{T^3} h(X_i, X_j) d^3x$$

Existence of minimizing metrics

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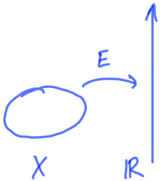

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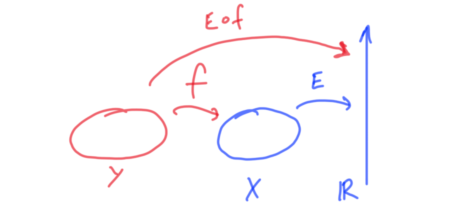

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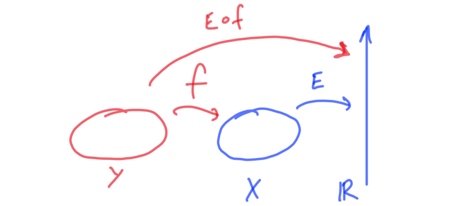

- $f : (0,\infty)^3 \times O(3) \to SPD_3$, $f(\boldsymbol{\lambda}, \mathscr{O}) = \mathscr{O} D_{\boldsymbol{\lambda}} \mathscr{O}^T$
- We will show $E \circ f : (0,\infty)^3 \times O(3) \to \mathbb{R}$ attains a min

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$$(E \circ f)(\boldsymbol{\lambda}, \mathscr{O}) = \mathrm{tr}(\mathscr{O}^{-1} H \mathscr{O} D_{\boldsymbol{\lambda}}^{-1}) + \mathrm{tr}(\mathscr{O}^{-1} \Omega \mathscr{O} D_{\boldsymbol{\lambda}}) + \frac{C}{\lambda_1 \lambda_2 \lambda_3}$$

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- Consider the smooth functions $O(3) \to (0, \infty)$

$$\mathscr{O} \mapsto (\mathscr{O}^{-1} H \mathscr{O})_{aa}, \qquad \mathscr{O} \mapsto (\mathscr{O}^{-1} \Omega \mathscr{O})_{aa}$$

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- Exists $\alpha > 0$ s.t. for all $(\boldsymbol{\lambda}, \mathscr{O})$,

$$(E \circ f)(\boldsymbol{\lambda}, \mathscr{O}) \geq \alpha \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \lambda_1 + \lambda_2 + \lambda_3 \right). \quad (*)$$

Existence

- Consider now a sequence $(\boldsymbol{\lambda}_n, \mathscr{O}_n)$ s.t.

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- Continuity of E implies $(E \circ f)(\boldsymbol{\lambda}_*, \mathscr{O}_*) = E_*$, i.e. $E \circ f$ attains a min

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- Metric on SPD_3?

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- Geodesic through $\Sigma(0)$: $\Sigma(t) = A\exp(t\xi)A^T$ where $AA^T = \Sigma(0)$

Uniqueness

$$\begin{aligned} E_4(\Sigma) &= \mathrm{tr}(\Omega\Sigma) \\ E_4(\Sigma(t)) &= \mathrm{tr}(\Omega A \exp(t\xi) A^T) \\ &= \mathrm{tr}(\Omega_A \exp(t\xi)), \qquad \Omega_A = A^T \Omega A \\ \frac{d^2}{dt^2} E_4(\Sigma(t))\bigg|_{t=0} &= \mathrm{tr}(\Omega_A \xi^2) > 0 \end{aligned}$$

- So E_4 is strictly convex.

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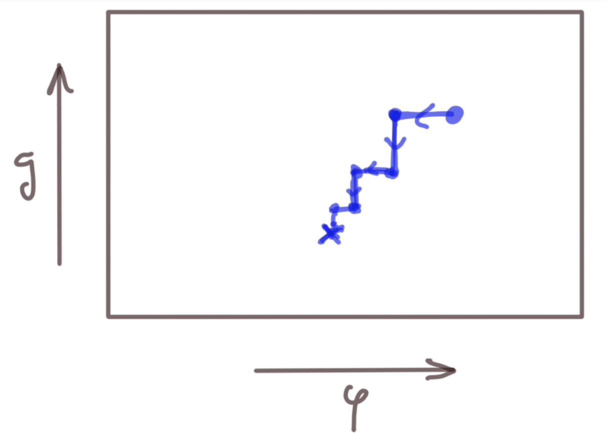
- $\det : SPD_3 \to \mathbb{R}$ is convex
- Hence $E_0 = \det \circ \iota$ is convex
- So $E = E_2 + E_4 + E_0$ is strictly convex. Hence it has at most one critical point. (Assume Σ_*, Σ_{**} both cps, apply Rolle's Theorem to $(E \circ \gamma)'$ where γ is the geodesic between them.)

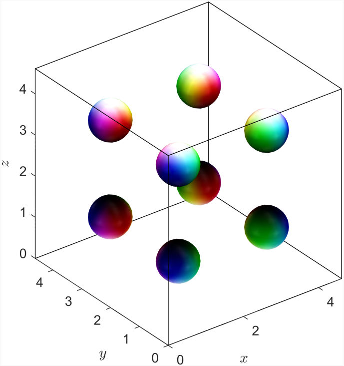
The numerical problem

$$\ddot{x} = -\operatorname{grad} E(x)$$



The Kugler-Shtrikman crystal (massless model)

$$E = E_2 + E_4$$



$$(x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$$
$$(\varphi_0, \varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_0, \varphi_2, \varphi_3, \varphi_1)$$

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$$E_t = E_0 + t\int_{\mathbb{T}^3}(1-\varphi_0)\sqrt{|g|}d^3x$$

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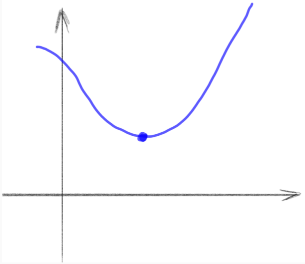
- No reason to expect **degenerate** critical points to survive perturbation

Degenerate critical points are unstable

f



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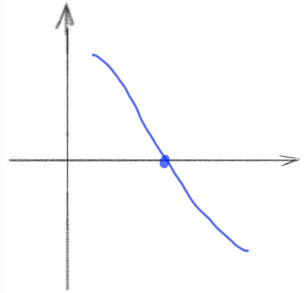

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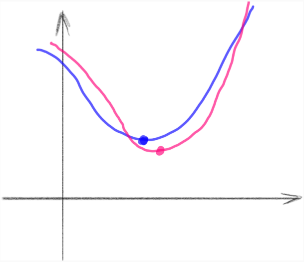

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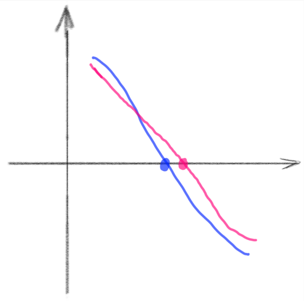

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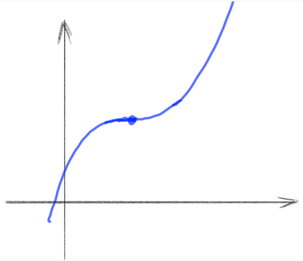

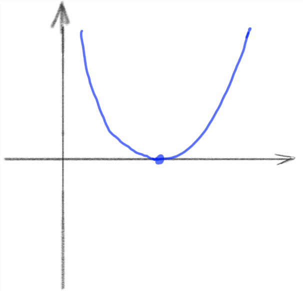

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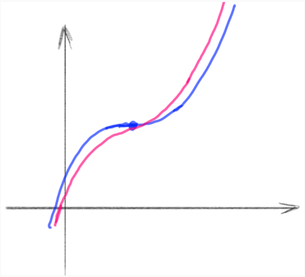

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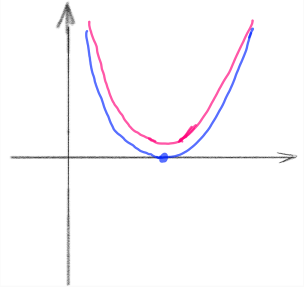

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$$E_t(gx) = E_t(x)$$

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- Nondegenerate $\Rightarrow$ isolated.

Symmetry analysis

- Apply this to

$$E_t = E_2 + E_4 + t\int_{\mathbb{T}^3}(1-\varphi_0)\mathrm{vol}_g$$

- $X = C^2(\mathbb{T}^3, SU(2)) \times SPD_3$
- $G = SO(3) \times Aut(T^3)$

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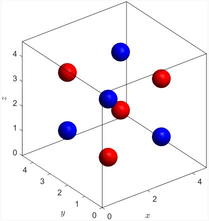
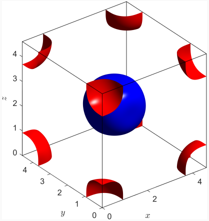
- Reduces to a problem in representation theory of subgroups of O_h

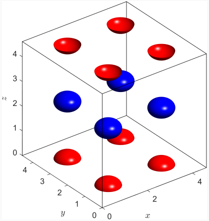


φ_{KS}

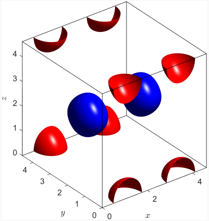


$R_\alpha \varphi_{KS}$

$\varphi_0 = 0.9$



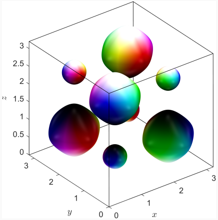
$R_{sheet} \varphi_{KS}$



$R_{chain} \varphi_{KS}$

$\varphi_0 = -0.9$

Skyrme crystals at pion mass $t = 1$



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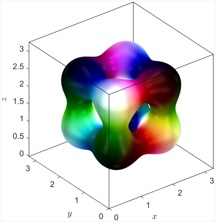

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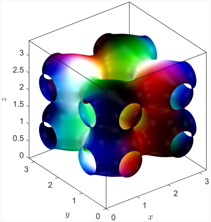

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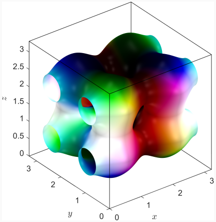
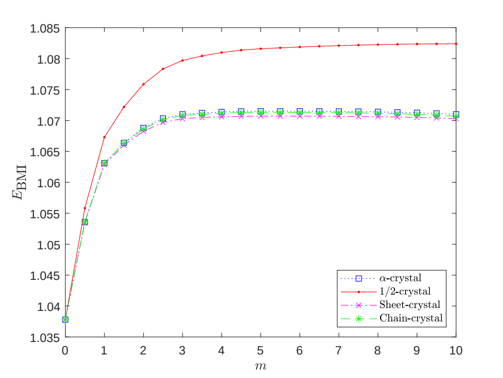


$R_{chain} \varphi_{KS}$

Energy ordering: sheet $<$ chain $<$ α $<$ KS



$$g_{sheet} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_1 & 0 \\ 0 & 0 & L_3 \end{pmatrix}$$

$L_3 > L_1$

$$g_{chain} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \end{pmatrix}$$

$L_2 > L_1$

trigonal, but **not** cubic!

Isospin inertia tensors

$$U_{KS} = \begin{pmatrix} 165.2 & 0 & 0 \\ 0 & 165.2 & 0 \\ 0 & 0 & 165.2 \end{pmatrix}, \quad U_{\alpha} = \begin{pmatrix} 135.5 & 0 & 0 \\ 0 & 135.5 & 0 \\ 0 & 0 & 167.3 \end{pmatrix},$$

$$U_{sheet} = \begin{pmatrix} 135.8 & 0 & 0 \\ 0 & 135.8 & 0 \\ 0 & 0 & 166.8 \end{pmatrix}, \quad U_{chain} = \begin{pmatrix} 135.6 & 0 & 0 \\ 0 & 135.7 & 0 \\ 0 & 0 & 167.2 \end{pmatrix}.$$

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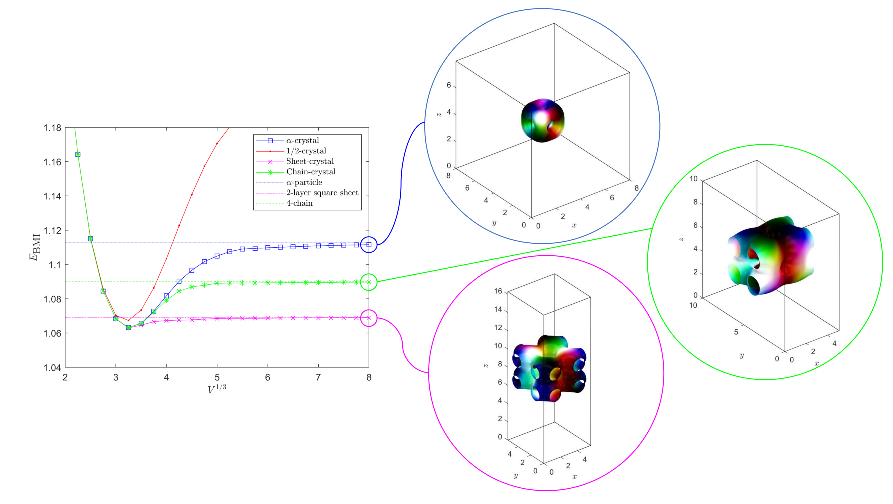

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- Many examples in condensed matter (cf work with Tom Winyard et al). True also for nuclear Skyrme model with massive pions
- Extreme case: baby Skyrme model $\varphi : M^2 \to S^2$

$$E(\varphi) = \int_M (\frac{1}{2}|d\varphi|^2 + \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi)).$$

Given **any** period lattice $\Lambda \subset \mathbb{R}^2$, can cook up a smooth potential $V : S^2 \to [0, \infty)$ s.t. $E(\varphi, g)$ has a global min at (φ_*, g_Λ) with φ_* degree 2 and holomorphic.

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- Existence result **at fixed volume** very generic

$$E(\varphi, g) = E_2(\varphi, g) + \text{positive, geom nat}$$

any dimension. Compactness argument works. E.g. ω-meson Skyrme model