Skyrme Crystals

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Joint work with Derek Harland and Paul Leask

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Skyrme model

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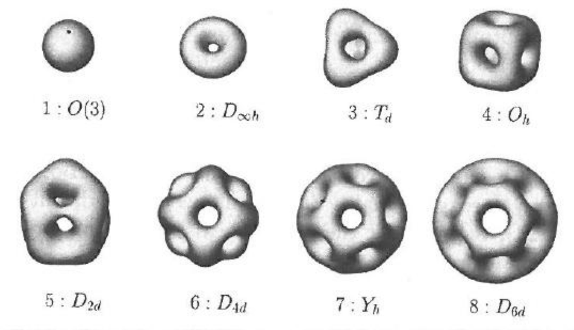
- Faddeev bound: $E(\varphi) \geq E_0|B|$, unattainable
- Degree B minimizer $\leftrightarrow$ nucleus of atomic weight B

Skyrme model

- Numerics



Battye and Sutcliffe

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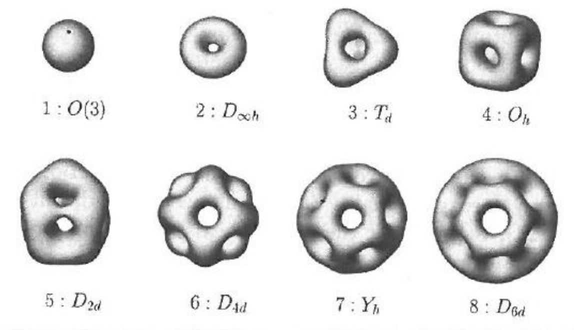

Battye and Sutcliffe

- E/BE_0 monotonically decreases e.g. 1.232 ($B = 1$), 1.096 ($B = 8$).

Skyrme model

- Suggests Skyrmions may be able to form a **crystal**

$$\varphi : \mathbb{R}^3/\Lambda \to G, \qquad \Lambda = \{n_1\mathbf{X}_1 + n_2\mathbf{X}_2 + n_3\mathbf{X}_3 \ : \ \mathbf{n} \in \mathbb{Z}^3\}$$

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- $V = 0$: Castillejo et al, Kugler et al, chose $\Lambda = L\mathbb{Z}^3$, found $B = 4$ minimizer for each $L > 0$, minimized over L. Found φ with $E/BE_0 = 1.036$.

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- But is this really a crystal? Given **any** Λ, B, there exists a degree B minimizer $\varphi : \mathbb{R}^3/\Lambda \to G$ (Auckly, Kapitanski).

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For most Λ, lifted map $\mathbb{R}^3 \to G$ clearly isn't a genuine solution: artifact of bc's.

General question

- Given a minimizer $\varphi : \mathbb{R}^k/\Lambda \to N$ of some energy functional $E(\varphi)$, when is the lifted map $\mathbb{R}^k \to N$ a genuine crystal?

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- Should be critical (in fact stable) with respect to variations of Λ too.

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- Identify them all with $M = \mathbb{R}^3/\mathbb{Z}^3$, the cubic torus. Now mfd is fixed, but **metric** depends on Λ

$$g_\Lambda = g_{ij} dx_i dx_j, \qquad g_{ij} = \mathbf{X}_i \cdot \mathbf{X}_j \text{ const}$$

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- So now

$$E(\varphi, g) = \int_{T^3} (|d\varphi|_g^2 + \frac{1}{4}|\varphi^*\omega|_g^2 + V(\varphi))\text{vol}_g$$

  and we want to minimize w.r.t. both $\varphi \in C^2_B(T^3, G)$ **and** $g \in SPD_3$ (space of symmetric positive definite 3×3 matrices)

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- Does a min exist? Dunno, but...
- Fix g: $E(\varphi)$ certainly attains a min in each homotopy class (at least in H^1 - low regularity) – Auckly, Kapitanski
- What if we fix φ? Does $E(g)$ attain a min?

Existence and uniqueness of minimizing metrics

- Want to think of E, for a **fixed** $\varphi : T^3 = \mathbb{R}^3/\mathbb{Z}^3 \to G$ as a function of the metric g on T^3:

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- **Theorem** Let $\varphi : T^3 \to G$ be C^1 and somewhere immersive. Then E_φ attains a global minimum at some $g_* \in SPD_3$ and has no other critical points.
- Proof: First note that

$$E_\varphi(g) = \sqrt{|g|}\,\mathrm{tr}(Hg^{-1}) + \frac{1}{\sqrt{|g|}}\,\mathrm{tr}(\Omega g) + C\sqrt{|g|}$$

where $H, \Omega \in SPD_3$ and $C \in [0, \infty)$ are fixed.

Existence and uniqueness of minimizing metrics

$$E_2(g) = \int_{T^3} \varphi^* h(\partial_i, \partial_j) g^{ij} \sqrt{|g|} d^3x = \sqrt{|g|} g_{ij}^{-1} H_{ij}$$

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$$E_4(g) = \frac{1}{4} \|\varphi^* \omega\|^2_{L^2(g)} = \frac{1}{4} \int_{T^3} \frac{1}{|g|} g(X_\varphi, X_\varphi) \mathrm{vol}_g = \frac{g_{ij}}{\sqrt{|g|}} \Omega_{ij}$$

$$\Omega_{ij} = \frac{1}{4} \int_{T^3} h(X_i, X_j) d^3 x$$

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- Define $\mathscr{G} = g/\sqrt{|g|}$. Then

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- Surjection $f : (0,\infty)^3 \times O(3) \to SPD_3$

$$(\boldsymbol{\lambda}, \mathscr{O}) \mapsto \mathscr{O}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\mathscr{O}^T = \mathscr{O} D_{\boldsymbol{\lambda}} \mathscr{O}^T$$

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- We will show $E \circ f : (0,\infty)^3 \times O(3) \to \mathbb{R}$ attains a min

Existence

$$\begin{aligned}(E \circ f)(\boldsymbol{\lambda}, \mathscr{O}) &= \mathrm{tr}(H(\mathscr{O} D_{\boldsymbol{\lambda}} \mathscr{O}^{-1})^{-1}) + \mathrm{tr}(\Omega \mathscr{O} D_{\boldsymbol{\lambda}} \mathscr{O}^{-1}) + \frac{C}{\lambda_1 \lambda_2 \lambda_3} \\ &= \mathrm{tr}(\mathscr{O}^{-1} H \mathscr{O} D_{\boldsymbol{\lambda}}^{-1}) + \mathrm{tr}(\mathscr{O}^{-1} \Omega \mathscr{O} D_{\boldsymbol{\lambda}}) + \frac{C}{\lambda_1 \lambda_2 \lambda_3}\end{aligned}$$

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- Consider the smooth functions $O(3) \to (0, \infty)$

$$\mathscr{O} \mapsto (\mathscr{O}^{-1} H \mathscr{O})_{aa}, \qquad \mathscr{O} \mapsto (\mathscr{O}^{-1} \Omega \mathscr{O})_{aa}$$

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- Exists $\alpha > 0$ s.t. for all $(\boldsymbol{\lambda}, \mathscr{O})$,

  $$(E \circ f)(\boldsymbol{\lambda}, \mathscr{O}) \geq \alpha \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \lambda_1 + \lambda_2 + \lambda_3 \right). \quad (*)$$

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- Continuity of E implies $(E \circ f)(\boldsymbol{\lambda}_*, \mathscr{O}_*) = E_*$, i.e. $E \circ f$ attains a min
- Let $\mathscr{G}_* = \mathscr{O}_* D_{\boldsymbol{\lambda}_*} \mathscr{O}_*^{-1} \in SPD_3$. E_φ attains a min at $g_* = \mathscr{G}_* / \det(\mathscr{G}_*)$.

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- Metric on SPD_3?

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- Geodesic through $\mathscr{G}(0)$: $\mathscr{G}(t) = A\exp(t\xi)A^T$ where $AA^T = \mathscr{G}(0)$

Uniqueness

$$\begin{aligned}
E_4(\mathscr{G}) &= \mathrm{tr}(\Omega\mathscr{G}) \\
E_4(\mathscr{G}(t)) &= \mathrm{tr}(\Omega A \exp(t\xi) A^T) \\
&= \mathrm{tr}(\Omega_A \exp(t\xi)), \qquad \Omega_A = A^T \Omega A \\
\frac{d^2}{dt^2} E_4(\mathscr{G}(t))\bigg|_{t=0} &= \mathrm{tr}(\Omega_A \xi^2) > 0
\end{aligned}$$

- So E_4 is strictly convex.

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Uniqueness

$$E_2(\mathscr{G}) = \mathrm{tr}(H\mathscr{G}^{-1}) = (\widehat{E}_4 \circ \iota)(\mathscr{G})$$

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- $\det : SPD_3 \to \mathbb{R}$ is strictly convex
- Hence $E_0 = \det \circ \iota$ is strictly convex
- So $E = E_2 + E_4 + E_0$ is strictly convex. Hence it has at most one critical point. (Assume $\mathscr{G}_*$, $\mathscr{G}_{**}$ both cps, apply Rolle's Theorem to $(E \circ \gamma)'$ where γ is the geodesic between them.)

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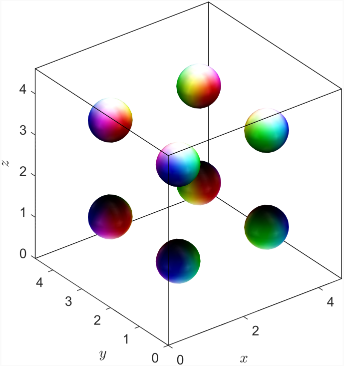
- Converges much faster than gradient flow.

The Kugler-Shtrikman crystal (massless model)

$$E = \|d\varphi\|^2 + \frac{1}{4}\|\varphi^*\omega\|^2$$



$$(x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$$
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  What happens to these critical points?
- No reason to expect **degenerate** critical points to survive perturbation

An instructive toy model

$$E_t : \mathbb{R}^2 \to \mathbb{R}, \qquad E_t(x, y) = x^2 + ty^2$$

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The case of the KS crystal

$$E_t : \underbrace{C^2(T^3, S^3) \times SPD_3}_{M} \to \mathbb{R}$$

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- Then $M^{\Gamma_R} = M$ and $(R, e) \cdot \varphi_{KS}$ is certainly **not** a nondegenerate cp of $E_0|$

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- For

$$R \in \left\{ \mathbb{I}_4, \underbrace{\begin{pmatrix} (0,1,1,1)/\sqrt{3} \\ * \end{pmatrix}}_{R_\alpha}, \underbrace{\begin{pmatrix} (0,0,0,1) \\ * \end{pmatrix}}_{R_{sheet}}, \underbrace{\begin{pmatrix} (0,0,1,1)/\sqrt{2} \\ * \end{pmatrix}}_{R_{chain}} \right\}$$

Γ_R is nontrivial, and M^{Γ_R} intersects the $SO(4)$ orbit of φ_{KS} transversely: implies $(R, e) \cdot \varphi_{KS}$ is an **isolated** cp of $E_0| : M^{\Gamma_R} \to \mathbb{R}$

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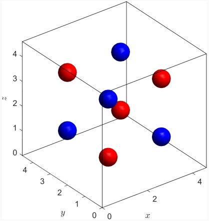
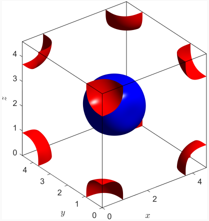
- PSC implies $\varphi(t)$ also a cp of E_t.

The KS crystals that (should) survive

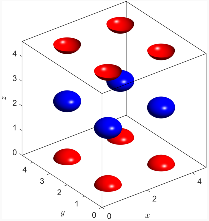


φ_{KS}



$R_\alpha \varphi_{KS}$

$\varphi_0 = 0.9$



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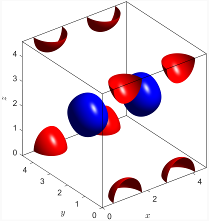
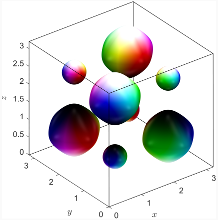


$R_{chain} \varphi_{KS}$

$\varphi_0 = -0.9$

Skyrme crystals at pion mass $t = 1$



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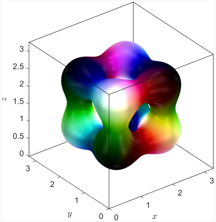

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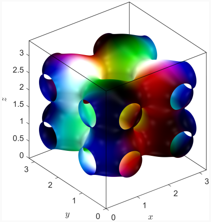

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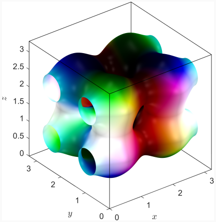
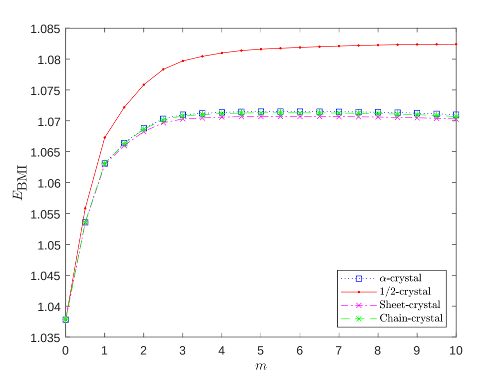


$R_{chain} \varphi_{KS}$

Energy ordering: sheet $<$ chain $< \alpha <$ KS



$$g_{sheet} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_1 & 0 \\ 0 & 0 & L_3 \end{pmatrix}$$

$L_3 > L_1$

$$g_{chain} = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_2 \end{pmatrix}$$

$L_2 > L_1$

trigonal, but **not** cubic!

Isospin inertia tensors

$$U_{KS} = \begin{pmatrix} 165.2 & 0 & 0 \\ 0 & 165.2 & 0 \\ 0 & 0 & 165.2 \end{pmatrix}, \quad U_{\alpha} = \begin{pmatrix} 135.5 & 0 & 0 \\ 0 & 135.5 & 0 \\ 0 & 0 & 167.3 \end{pmatrix},$$

$$U_{sheet} = \begin{pmatrix} 135.8 & 0 & 0 \\ 0 & 135.8 & 0 \\ 0 & 0 & 166.8 \end{pmatrix}, \quad U_{chain} = \begin{pmatrix} 135.6 & 0 & 0 \\ 0 & 135.7 & 0 \\ 0 & 0 & 167.2 \end{pmatrix}.$$

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- Can again solve numerically by ANF

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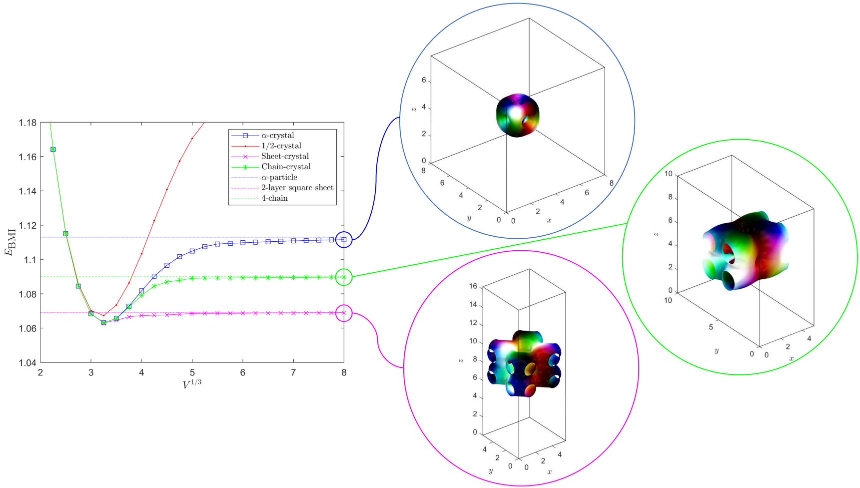

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- Many examples in condensed matter. True also for nuclear Skyrme model with massive pions
- Extreme case: baby Skyrme model $\varphi : M^2 \to S^2$

$$E(\varphi) = \int_M (\frac{1}{2}|d\varphi|^2 + \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi).$$

  Given **any** period lattice $\Lambda \subset \mathbb{R}^2$, can cook up a smooth potential $V : S^2 \to [0, \infty)$ s.t. $E(\varphi, g)$ has a global min at (φ_*, g_Λ) with φ_* degree 2 and holomorphic.

Concluding remarks

- So this crazy lattice **is** the period lattice of a baby Skyrmion crystal, at least for a (highly contrived) choice of V!