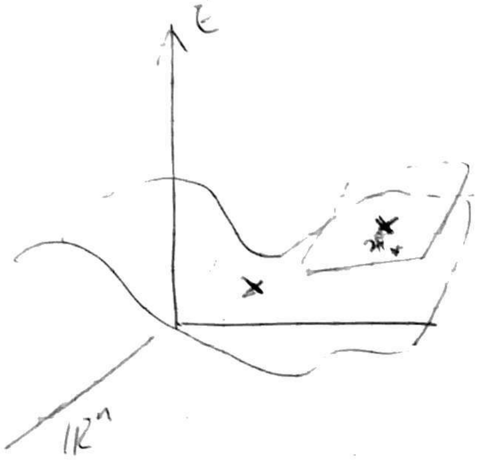
SHAPE MODES OF CRITICALLY COUPLED VORTICES 18/6/26 1

Joint work with Nora Gavrea, Derek Harland (Leeds)



$E: \mathbb{R}^n \to \mathbb{R}$

Static solutions $\left.\frac{\partial E}{\partial x_i}\right|_{x_*} = 0$

2nd variation

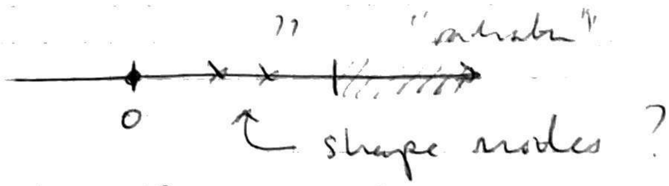
$$H_{ij} = \left.\frac{\partial^2 E}{\partial x_i \partial x_j}\right|_{x_*}$$

Spectrum: stability, vibrational dynamics, quantization

Field theory: $\mathbb{R}^n \longrightarrow C^\infty(\Sigma, X) = \mathcal{C}$

matrix $H \longrightarrow$ self adjoint linear operator

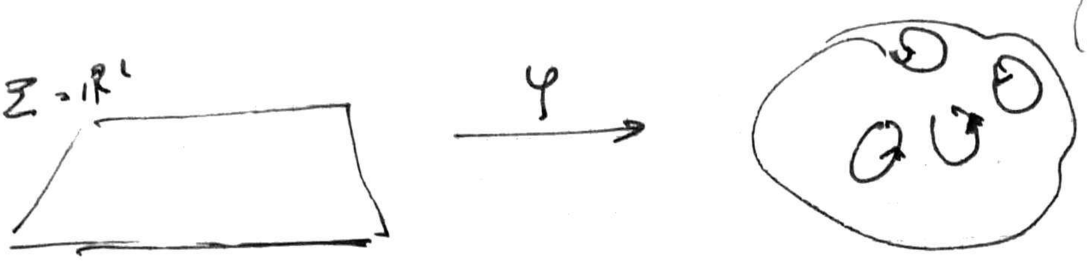
$$J : T_{\varphi_*}\mathcal{C} \to T_{\varphi_*}\mathcal{C}.$$



In general: difficult to
1. prove they exist
2. construct them

This talk: large class of gauged σ-models $\ni$ AHM
for which 1, 2 reduces to SCALAR SCHRÖDINGER PROBLEM

$$\Delta\psi + K(x)\psi = \lambda\psi \qquad \psi: \Sigma \to \mathbb{R}.$$



$(X, h, J \Rightarrow \omega)$

Circle action by holomorphic isometries

$e^{i\theta}: X \to X$

$e^{i\theta_1} \circ e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

Killing vector field Z on X.

$d_A\varphi \; \cancel{} = d\varphi - A\, Z(\varphi)$

curvature of A $\quad F_A = dA$.

S^1 action Hamiltonian: $\quad \mu : X \to \mathbb{R}$

s.t. $d\mu = \iota_Z \omega$

$(\; Z = -J_X \nabla\mu \;)$

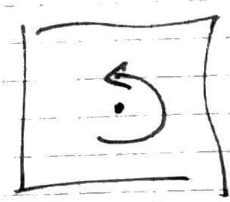
$$E(\varphi, A) = \frac{1}{2}\|d_A\varphi\|^2_{L^2} + \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{2}\|\mu\circ\varphi\|^2_{L^2}$$

Finite energy: $\quad \varphi|_{\partial_\infty\Sigma}$ stays in a gauge orbit. $\qquad \varphi \to \mu^{-1}(0) \subset X$ as $|x| \to \infty$.

$\Rightarrow$ winding number $n \in \mathbb{Z}$

$\Rightarrow$ Flux quantization (Stokes's theorem)

Examples: $\quad X = \mathbb{C}$



$\mu = \frac{1}{2}(\tau - |\varphi|^2)$

$X = S^2$

$\mu = \varphi\cdot \underline{n} - \tau$

"Bogomol'nyi" bound

$$0 \le \frac{1}{2}\|d_A\varphi(\partial_1) + J_X\, d_A\varphi(\partial_2)\|_{L^2}^2 + \frac{1}{2}\|{*F_A} - \mu(\varphi)\|_{L^2}^2$$
$$= E(\varphi, A) - \int_\Sigma \varphi^{A*}\omega - \int_\Sigma \mu(\varphi) F_A$$

$$\begin{aligned}(\varphi^{A*}\omega)(X,Y) &= \omega(d_A\varphi(X), d_A\varphi(Y)) \\ &= \omega(d\varphi(X) - A(X)\,Z(\varphi),\ d\varphi(Y) - A(Y)\,Z(\varphi)) \\ &= (\varphi^*\omega)(X,Y) - (A \wedge \varphi^*\iota_Z\omega)(X,Y) \\ &= (\varphi^*\omega - A \wedge \varphi^* d\mu)(X,Y)\end{aligned}$$

$$\begin{aligned}\Rightarrow \varphi^{A*}\omega + \mu(\varphi) F_A &= \varphi^*\omega + \mu(\varphi)\, dA + d(\mu(\varphi)) \wedge A \\ &= \varphi^*\omega + d(\mu(\varphi) A)\end{aligned}$$

$$\Rightarrow E(\varphi, A) \ge \int_\Sigma \varphi^*\omega + \int_{\partial\Sigma} \underset{0}{\underbrace{\mu(\varphi)}} A$$

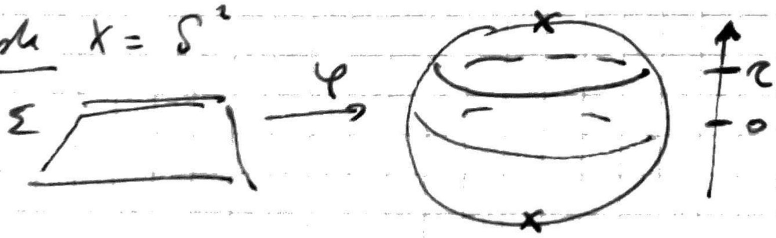
↳ a homotopy invariant of φ.

with equality $\Leftrightarrow$ $J_X \circ d_A\varphi = d_A\varphi \circ J_\Sigma$
$*F_A = \mu(\varphi)$

$$\mathrm{Bog}(\varphi, A) := \begin{pmatrix} \frac{1}{\sqrt{2}}\left(d_A\varphi + J_X \circ *_\Sigma d_A\varphi\right) \\ *F_A - \mu(\varphi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathrm{Bog}: C^\infty(\Sigma, X) \times \Omega^1(\Sigma) \to \Gamma(T^*\Sigma \otimes \varphi^{-1}TX) \times C^\infty(\Sigma, \mathbb{R})$$

Example $X = S^2$



$k_+ - k_- = n$

$$\int_\Sigma \varphi^*\omega = 2\pi(1-\tau)k_+ + 2\pi(1+\tau)k_-$$

2nd variation $(\varphi_{s,t}, A_{s,t})$ two parameter smooth variation of $(\varphi, A) = (\varphi_{0,0}, A_{0,0})$ a critical pt of E

$$\partial_s (\varphi_{s,0}, A_{s,0})\Big|_{s=0} = (\hat{\varepsilon}, \hat{\alpha}) \in \underbrace{\Gamma(\varphi^{-1}TX) \oplus \Omega^1(\Sigma)}_{\mathcal{P}}$$

$$\partial_t (\varphi_{0,t}, A_{0,t})\Big|_{t=0} = (\varepsilon, \alpha) \in \mathcal{P}$$

$$\mathrm{Hess}_{(\varphi,A)}((\hat{\varepsilon},\hat{\alpha}),(\varepsilon,\alpha)) := \frac{\partial^2}{\partial s\,\partial t}\Big|_{s=t=0} E(\varphi_{s,t}, A_{s,t})$$

$$= \left\langle (\hat{\varepsilon},\hat{\alpha}), \mathcal{J}_{(\varphi,A)}(\varepsilon,\alpha) \right\rangle_{L^2}.$$

$\mathcal{J}_{\varphi,A} : \mathcal{P} \to \mathcal{P}$ — this is the operator whose spectrum we want

Now $E(\varphi_{s,t}, A_{s,t}) = \frac{1}{2} \| \mathrm{Bog}(\varphi_{s,t}, A_{s,t}) \|_{L^2}^2 +$ homotopy invariant.

$$\Rightarrow \mathrm{Hess}((\hat{\varepsilon},\hat{\alpha}),(\varepsilon,\alpha)) = \left\langle d\mathrm{Bog}_{\varphi,A}(\hat{\varepsilon},\hat{\alpha}), d\mathrm{Bog}_{\varphi,A}(\varepsilon,\alpha) \right\rangle_{L^2}$$

$$= \left\langle (\hat{\varepsilon},\hat{\alpha}), \mathcal{B}^{\dagger}\mathcal{B}(\varepsilon,\alpha) \right\rangle_{L^2}$$

where $\mathcal{B} = d\mathrm{Bog}_{(\varphi,A)} : \mathcal{P} \to \underbrace{\Gamma(T^*\Sigma \otimes \varphi^{-1}TX) \oplus C^\infty(\Sigma)}_{\mathcal{D}'}$

[Explicitly: $\mathcal{B}(\varepsilon,\alpha) = \begin{pmatrix} \frac{1}{\sqrt{2}}(\mathcal{L}(\varepsilon,\alpha) - {*}J_X \mathcal{L}(\varepsilon,\alpha)) \\ {*}d\alpha + h(Z, J_X\varepsilon) \end{pmatrix}$

where $\mathcal{L}(\varepsilon,\alpha) = \nabla^{\varphi}\varepsilon + A(\nabla^X_\varepsilon Z)\circ\varphi + \alpha\, Z\circ\varphi$]

$\mathcal{B}^{\dagger}\mathcal{B} =$ a bit of a brute.

2nd variation $(\varphi_{s,t}, A_{s,t})$ two parameter smooth variation of $(\varphi, A) = (\varphi_{0,0}, A_{0,0})$ a critical pt of E

$$\partial_s (\varphi_{s,0}, A_{s,0})\Big|_{s=0} = (\hat{\varepsilon}, \hat{\alpha}) \in \underbrace{T(\varphi^{-1}TX) \oplus \Omega^1(\Sigma)}_{P}$$

$$\partial_t (\varphi_{0,t}, A_{0,t})\Big|_{t=0} = (\varepsilon, \alpha) \in P$$

$$\mathrm{Hess}_{(\varphi,A)}((\hat{\varepsilon},\hat{\alpha}),(\varepsilon,\alpha)) := \frac{\partial^2}{\partial s\,\partial t}\Big|_{s=t=0} E(\varphi_{s,t}, A_{s,t})$$

$$= \langle(\hat{\varepsilon},\hat{\alpha}), \mathcal{J}_{(\varphi,A)}(\varepsilon,\alpha)\rangle_{L^2}.$$

$\mathcal{J}_{\varphi,A} : P \to P$ this is the operator whose spectrum we want

Now $E(\varphi_{s,t}, A_{s,t}) = \frac{1}{2}\| \mathrm{Bog}(\varphi_{s,t}, A_{s,t})\|^2_{L^2} +$ homotopy invariant.

$$\Rightarrow \mathrm{Hess}((\hat{\varepsilon},\hat{\alpha}),(\varepsilon,\alpha)) = \langle d\mathrm{Bog}_{\varphi,A}(\hat{\varepsilon},\hat{\alpha}), d\mathrm{Bog}_{\varphi,A}(\varepsilon,\alpha)\rangle_{L^2}$$

$$= \langle(\hat{\varepsilon},\hat{\alpha}), \mathcal{B}^{+}\mathcal{B}(\varepsilon,\alpha)\rangle_{L^2}.$$

where $\mathcal{B} = d\mathrm{Bog}_{(\varphi,A)} : P \to \underbrace{T(T^*\Sigma \otimes \varphi^{-1}TX) \oplus C^\infty(\Sigma)}_{D'}$

[Explicitly: $\mathcal{B}(\varepsilon,\alpha) = \begin{pmatrix} \frac{1}{\sqrt{2}}(\mathcal{L}(\varepsilon,\alpha) + J_X \mathcal{L}(\varepsilon,\alpha)) \\ + d\alpha + h(Z, J_X\varepsilon) \end{pmatrix}$

where $\mathcal{L}(\varepsilon,\alpha) = \nabla^\varphi \varepsilon + A(\nabla^X_\varepsilon Z)\circ\varphi + \alpha\, Z\circ\varphi$]

$\mathcal{B}^{+}\mathcal{B} =$ a bit of a brute.

<u>Ab'ml GTS</u>

$$\mathcal{G} : C^\infty(\Sigma) \longrightarrow P$$
$$\chi \mapsto (\chi Z \circ \varphi,\ d\chi)$$

- $G_\Lambda := \mathcal{G}(C^\infty(\Sigma)) \subset \ker \mathbb{J}$

Assume $\mathbb{J}(\xi,\alpha) = \lambda(\xi,\alpha) \qquad \lambda \neq 0$

Then $\forall\, \chi \qquad \langle(\xi,\alpha), \mathcal{G}(\chi)\rangle_{L^2} = \frac{1}{\lambda}\langle \mathbb{J}(\xi,\alpha), \mathcal{G}(\chi)\rangle_{L^2}$

$$= \frac{1}{\lambda}\langle(\xi,\alpha), \mathbb{J}\,\mathcal{G}(\chi)\rangle_{L^2} = 0$$

i.e. $\mathcal{G}^+(\xi,\alpha) = 0. \quad = \delta\alpha + h(Z\circ\varphi, \xi)$

- So if we want eigensections of $\mathbb{J}$ outside $\ker \mathbb{J}$, we can demand $\mathcal{G}^+(\xi,\alpha) = 0$:

$$\tilde{\mathbb{B}} : \underbrace{\Gamma(\varphi^* TX) \oplus \Omega^1(\Sigma)}_{P} \longrightarrow \underbrace{\Gamma(T^*\Sigma \otimes \varphi^* TX) \oplus C^\infty(\Sigma) \oplus C^\infty(\Sigma)}_{D}$$

$$\tilde{\mathbb{B}}(\xi,\alpha) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\mathcal{L}(\xi,\alpha) - * J_X \mathcal{L}(\xi,\alpha)\right), \\ * d\alpha + h(Z, J_X \xi) \\ - * d * \alpha + h(Z, \xi) \end{pmatrix}$$

- $\tilde{\mathbb{J}} = \tilde{\mathbb{B}}^+ \tilde{\mathbb{B}}$ and $\mathbb{J} = \mathbb{B}^+ \mathbb{B}$ have the same spectrum.

$\tilde{\mathbb{B}}$ has a symmetry...

$$S_1 : P \to P \qquad (\xi,\alpha) \longmapsto (J_X \xi, *\alpha)$$

$$S_2 : D \to D \qquad (\eta, f_1, f_2) \longmapsto (J_X \eta, -f_2, f_1)$$

$$\tilde{\mathbb{B}} \circ S_1 = S_2 \circ \tilde{\mathbb{B}}$$

- $$\Rightarrow \quad \tilde{\mathbb{B}}^+ \circ S_2 = S_1 \circ \tilde{\mathbb{B}}$$

Proposition Let $\psi \in C^\infty(\Sigma)$ be an L^2 eigenstate of
$$G^\dagger G : C^\infty(\Sigma) \to C^\infty(\Sigma) \quad \text{with eigenvalue } \lambda.$$

Then $(\varepsilon, \alpha) = \delta_1^\vee G(\psi)$ is an L^2 eigensection of $\tilde{J}$ with eigenvalue λ.

Proof: $\delta_1^\vee G(\psi) \in \ker G^\dagger$

$$\Rightarrow \quad \tilde{J}\,\delta_1^\vee G(\psi) = (J + GG^\dagger)\,\delta_1^\vee G(\psi) = J(\delta_1^\vee G(\psi))$$

$$\| $$

$$\delta_1^\vee \tilde{J} G(\psi) = \delta_1^\vee (J G(\psi) + GG^\dagger G(\psi))$$

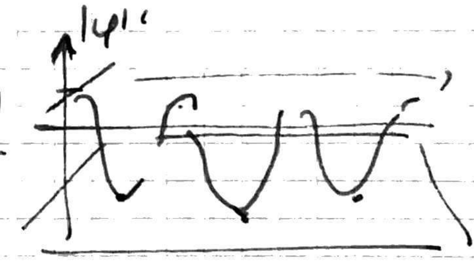
$$= \delta_1^\vee G(\lambda\psi) = \lambda\,\delta_1^\vee G(\psi) \qquad \square$$

Now $$G^\dagger G \psi = \Delta\psi + |Z\circ\varphi|^2 \psi$$

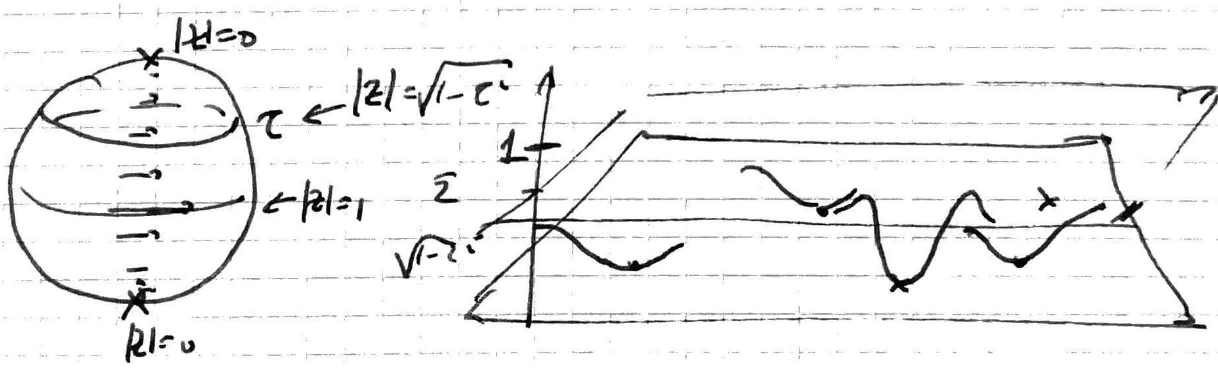
Schrödinger operator with potential $|Z\circ\varphi|^2 = |d\mu\circ\varphi|^2$

X = $\mathbb{C}$ $\quad Z = \frac{\partial}{\partial\theta} \quad |Z|^2 = |\varphi|^2$

— always have at least one bound state



X = S^2



If $k_- = 0$: always have a bound state.

If $\tau = 0$: always have a bound state

General X If we choose τ s.t. minimum mfd contains orbit of maximal $|Z|$: always have a bound state