THE PSEUDOVORTEX APPROXIMATION TO VORTICES AT HIGH DENSITY

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Abelian Higgs model on a compact Riemann surface Σ at critical coupling

[Why? Eg vortex gas, vortex lattices]

Higgs field φ $\longleftrightarrow$ $\varphi \in \Gamma(L)$ ← $\mathbb{C}$ line bundle over Σ

Gauge potential A ⟷ connexion A.

$$\boxed{n = \deg L \in \mathbb{Z}}$$

$d_A = d - iA$.

$$E(\varphi, A) = \frac{1}{2}\|d_A\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{2}\left\|\frac{1}{2}\tau - |\varphi|^2\right\|_{L^2}^2$$

Bogomolnyi argument $\left(\langle F_A, |\varphi|^2\omega_\Sigma\rangle_{L^2} = \|\partial_A\varphi\|^2 - \|\bar\partial_A\varphi\|^2 \right)$

$E \geq \tau\pi n$.

Equality $\iff$ $\bar\partial_A\varphi = 0$ — (V1) $\quad *F_A = \frac{1}{2}(\tau - |\varphi|^2)$ — (V2)

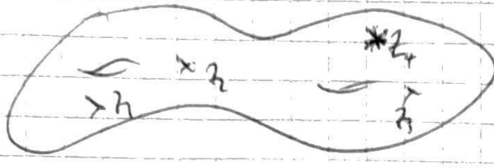
Bradlow bound (1990) $\int_\Sigma$ (V2) $\quad \pi n = \frac{1}{2}\tau|\Sigma| - \frac{1}{2}\|\varphi\|_{L^2}^2$

$$\Rightarrow \quad \varepsilon := \|\varphi\|_{L^2}^2 = \tau|\Sigma| - 2\pi n \geq 0$$

Bradlow / dissolving / high density limit: $\varepsilon \searrow 0$.

Existence Theorem (Bradlow 1990, Garcia Prada 1991)

~~Given a collection of~~ Assume $\varepsilon > 0$. Then given a collection of points $D = [z_1, z_2, \ldots, z_n] \in \Sigma^n/\ldots$ not necessarily distinct, there exists a unique (up to gauge) solution of (V1) (V2) whose φ vanishes precisely at $z_1, z_2, \ldots, z_n$ (with multiplicity).



$M_n \cong \Sigma^n / S_n$

Dissolved limit: $\varepsilon = 0 \implies \varphi = 0$

(V2): $*F_A = \frac{1}{2}\tau = \frac{2\pi n}{|\Sigma|}$ ← constant curvature connexion

$\hat{A} = \tilde{A} + \alpha$ $\qquad d\alpha = 0,\ \delta\alpha = 0$ i.e. α a harmonic 1-form

$M_n = \mathbb{R}^{2g}/\Lambda$

The Pseudovortex approximation ($0 < \varepsilon \ll 1$)
(Manton, Baptista 2003) $\left(\Sigma = S^2_{\text{round}}\right)$.

Take $A = \hat{A}$ — constant curvature connexion (unique on S^2)

$\varphi = \sqrt{\varepsilon}\,\hat{\varphi}$ $\qquad \hat{\varphi} \in H^0(L, \bar{\partial}_{\hat{A}})$ $\qquad \|\hat{\varphi}\|_{L^2} = 1$

(φ, A) solves (V1) exactly, (V2) "on average"

$\boxed{(*)\ \text{DESCRIPTION OF } H^0(L, \bar{\partial}_A)}$

How good is this approximation?

Fix a divisor D and choose $\hat{\varphi} \in H^0(L, \bar{\partial}_{\hat{A}})$ with $\|\hat{\varphi}\|_{L^2} = 1$, $\quad \hat{\varphi}^{-1}(0) = D$

The ~~the~~ real vortex with $\varphi^{-1}(0) = D$ is something of the form

$$\varphi = \sqrt{\varepsilon}\,\hat{\varphi}\, e^{\frac{1}{2}u} \qquad u : \Sigma \to \mathbb{R}.$$

This is holomorphic w.r.t. $A = \hat{A} + \frac{1}{2} * du$

$\Rightarrow (\varphi, A)$ solves (V1).

(V2) $\iff$ $\boxed{\Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon |\hat{\varphi}|^2 e^u = 0}$

Theorem (JMS, GL) $\exists\, C_n > 0$ (depending only on n) st., for all degree n divisors,

$$\|u\|_{C^0} \leq C_n \varepsilon$$

Vortices are uniformly close to pseudovortices for a given fixed n (if ε is small)

Question: what if we fix ε (>0, small) but allow $n \to \infty$?
Can we remove the n dependence from estimate above?

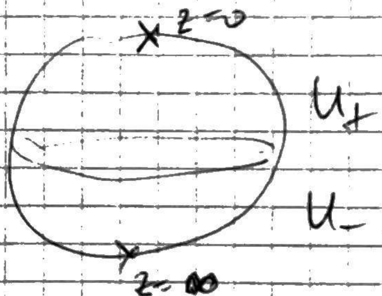
Answer: No!

$\Sigma = S^2_{\mathrm{round}}$, $R = 1$

Theorem (JMS, GC, DH) For fixed $\varepsilon > 0$, $\exists$ sequences $D_n, \hat{D}_n$ of degree n divisors st.

$$\|u_{D_n}\|_{C^0} \longrightarrow \infty \qquad \|u_{\hat{D}_n}\|_{C^0} \leq C\varepsilon.$$

(*) $H^0(L, \bar{\partial}_A)$. Choose holomorphic, local trivialization

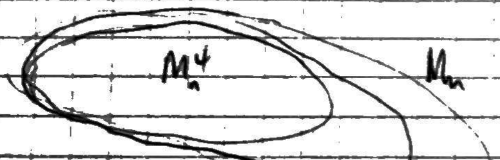


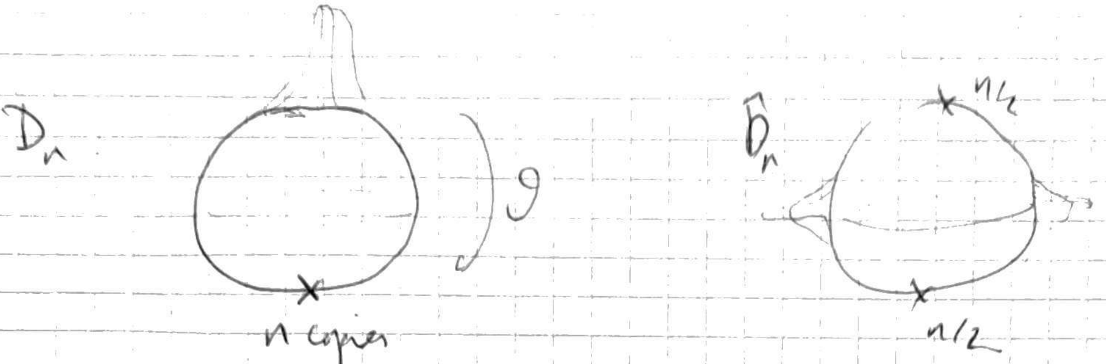
$$\varphi_+(z) = z^n \varphi_-(z)$$

$$\varphi_+(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

$$\langle \varphi, \psi \rangle_{L^2} = \int_{S^2} \frac{\overline{\varphi_+(z)}\, \psi_+(z)}{(1+|z|^2)^n}$$

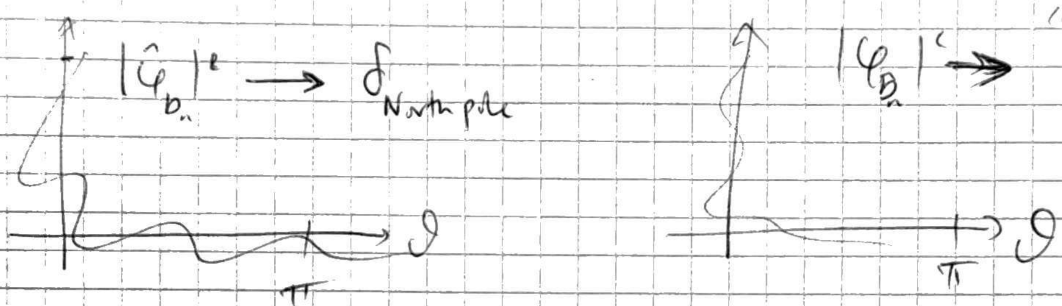
$\Sigma = S^2_{\mathrm{rnd},1}$





$$\Delta u - \frac{\varepsilon}{4\pi} + \varepsilon |\hat{\varphi}_D|^2 e^u = 0$$

$|\hat{\varphi}_{D_n}|^2 \longrightarrow \delta_{\text{North pole}}$ $\qquad$ $|\varphi_{D_n}|^2 \Rightarrow$ "δ_{equator}"



$$\Delta u - \frac{\varepsilon}{4\pi} + 0 = 0$$

unbounded

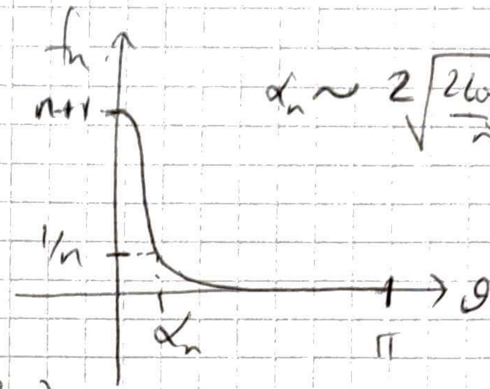
D_n : singularity at North pole $\Longrightarrow$ log divergence

$\hat{D}_n$: singularity on equator $\Longrightarrow$ $\vee$ singularity

bounded

Proof: ~~Δu~~ $\frac{\varepsilon}{4\pi}$ take D_n = n copies of south pole

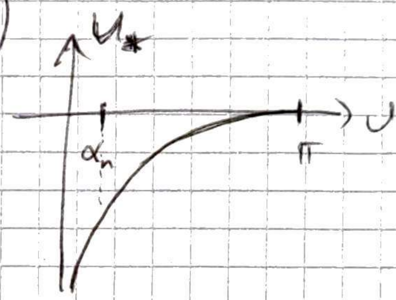
$$\Rightarrow |\hat{\varphi}|^2 = \frac{n+1}{4\pi} \cos^{2n} \vartheta/2 = \frac{1}{4\pi} f_n(\vartheta)$$

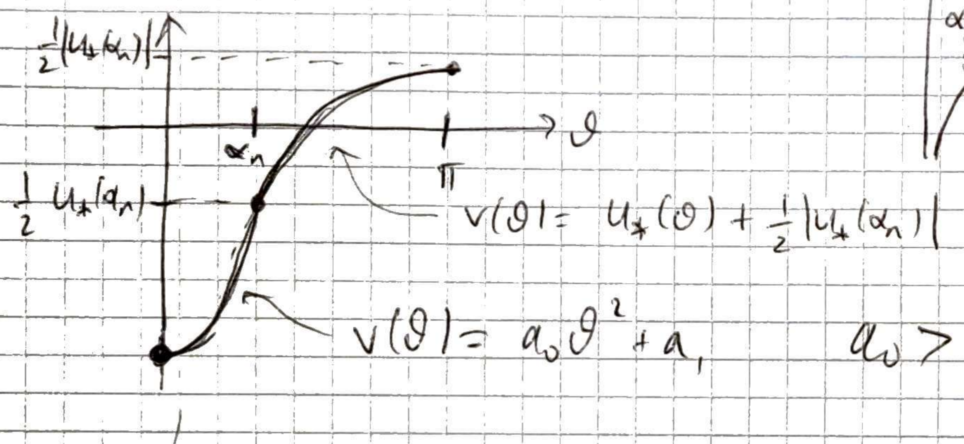


$\alpha_n \sim 2\sqrt{\frac{2\log n}{n}}$

$$\Delta u - \frac{\varepsilon}{4\pi} + \frac{\varepsilon}{4\pi} f_n e^u = 0$$

$$\Delta u_* = \frac{\varepsilon}{8\pi} \qquad u_*(\vartheta) = \frac{\varepsilon}{8\pi} \log(\sin^2 \vartheta/2)$$





$a_0 > \frac{\varepsilon n}{8\pi}$

Claim: v is a subsolution of $\circledast$ on $S^2 \setminus \{\vartheta = \alpha_n\}$

On $[\alpha_n, \pi]$: $\Delta v - \frac{\varepsilon}{4\pi} + \frac{\varepsilon}{4\pi} f_n e^v < -\frac{\varepsilon}{8\pi} + \frac{\varepsilon}{4\pi n} e^{\frac{1}{2}|u_*(\alpha_n)|}$

< 0 (n large)

On $[0, \alpha_n]$: $\Delta v = -\frac{d^2 v}{d\vartheta^2} - \cot\vartheta \frac{dv}{d\vartheta} < -2a_0$

$$\Rightarrow \Delta v - \frac{\varepsilon}{4\pi} + \frac{\varepsilon}{4\pi} f_n e^v < -2a_0 - \frac{\varepsilon}{4\pi} + \frac{\varepsilon}{4\pi}(n+1)$$

$$= -2a_0 + \frac{\varepsilon n}{4\pi} < 0$$

Claim: Either (1) $u \geq v$ for all $p \in S^2$ or

(2) $u(\alpha_n) < v(\alpha_n)$

(5)

Assume (1) is false. Then $u-v$ attains a negative min at some $p \in S^2$. If $p \in S^2 \setminus \{\theta = \alpha_n\}$ then $u-v$ is smooth at p so

$$\Delta(u-v)\Big|_p = -\operatorname{tr} \operatorname{Hess}(u-v)\Big|_p \leq 0$$

But

$$\Delta u - \frac{\varepsilon}{4\pi} + \frac{\varepsilon}{4\pi} f_n e^u = 0$$

$$\Delta v - \frac{\varepsilon}{4\pi} + \frac{\varepsilon}{4\pi} f_n e^v < 0$$

$$\Rightarrow \Delta(u-v)\Big|_p + \frac{\varepsilon}{4\pi} f_n \underbrace{\left(e^{u(p)} - e^{v(p)}\right)}_{<0 \text{ since } u(p)<v(p)!} > 0$$

$\otimes$.

Hence, if (1) is false, the negative min of $u-v$ must be attained at $\theta = \alpha_n$. But then $u(\alpha_n) < v(\alpha_n)$!

In either case, $\|u\|_{C^0} \geq \frac{1}{2}|u_1(\alpha_n)| \longrightarrow \infty$.

$\square$