Ricci Magnetic Geodesic Motion of Vortices and Lumps

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Vortices

$$\mathcal{L} = \frac{1}{2}\left(D_\mu \varphi \overline{D^\mu \varphi} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4}(|\varphi|^2 - 1)^2 \right)$$

- Finite total energy $\implies |\varphi| \to 1$, $D\varphi \to 0$ as $r \to \infty$.
- At large r, $\varphi \sim e^{i\chi(\theta)}$, $A \sim -i\varphi^{-1} d\varphi \sim d\chi$
- Flux quantization: $B = F_{12}$

$$\int_{\mathbb{R}^2} B = \oint_{S^1_\infty} A = \chi(2\pi) - \chi(0) = 2\pi n.$$

- $n =$ number of zeroes of φ (with multiplicity). Energy peaks.

Bogomol'nyi argument

$$E = \frac{1}{2}\int_{\mathbb{R}^2} |D_i\varphi|^2 + F_{12}^2 + \frac{1}{4}(1 - |\varphi|^2)$$

- For a **static** field ($\partial_0 = 0$, $A_0 = 0$) with winding n,

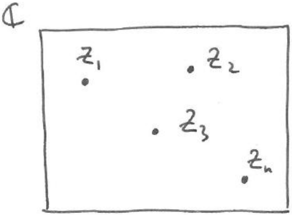
$$\begin{aligned} 0 &\leq \frac{1}{2}\int_{\mathbb{R}^2} |D_1\varphi + iD_2\varphi|^2 + (B - \frac{1}{2}(1 - |\varphi|^2))^2 \\ &= E - \frac{1}{2}\int_{\mathbb{R}^2} B + i(\partial_1(\overline{\varphi}D_2\varphi) - \partial_2(\overline{\varphi}D_1\varphi)) \\ &= E - \pi n \end{aligned}$$

- So $E \geq \pi n$, with equality iff

$$\begin{aligned} (BOG1) &\qquad D_1\varphi + iD_2\varphi = 0 \\ (BOG2) &\qquad B = \frac{1}{2}(1 - |\varphi|^2) \end{aligned}$$

Taubes's existence theorem

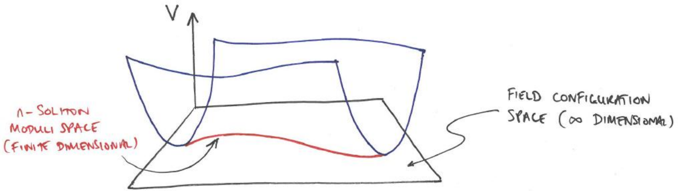
- Given any collection of points $Z_1, \ldots, Z_n$ in $\mathbb{C} \equiv \mathbb{R}^2$ there is a unique (up to gauge) n-vortex solution of the Bogomol'nyi equations with $\varphi = 0$ precisely at $Z_1, \ldots, Z_n$. Roughly, $Z_r =$ vortex positions.



$$\Longleftrightarrow \quad p(z) = (z - z_1) \cdots (z - z_n) = z^n + p_1 z^{n-1} + \cdots + p_n$$

- Moduli space of n-vortices: $M_n \equiv \mathbb{C}^n$
- Global coords $p_1, \ldots, p_n$
- Local coords $Z_1, \ldots, Z_n$ on $M_n \backslash \Delta$

Geodesic approximation



- **Restrict** dynamics to M_n

$$\begin{aligned} S &= \int (T - V)dt = \int (T - \pi n)dt \\ T &= \frac{1}{2}\int_{\mathbb{R}^2} |\partial_0 \varphi|^2 + (\partial_0 A_1)^2 + (\partial_0 A_2)^2 \end{aligned}$$

- Geodesic motion w.r.t. metric induced on M_n by T. Denote this metric γ, the L^2 metric

Strachan-Samols formula for the metric

- Expand $\log|\varphi|^2$ in a neighbourhood of Z_r

$$\log|\varphi|^2 = 2\log|z - Z_r| + a_r + \frac{1}{2}b_r(z - Z_r) + \frac{1}{2}\overline{b}_r(\bar{z} - \overline{Z}_r) + \cdots$$

  Defines coefficients $b_r(Z_1, \ldots, Z_n)$, $r = 1, 2, \ldots, n$

- Metric: $$\gamma = \pi \sum_{r,s=1}^{n} \left(\delta_{rs} + 2\frac{\partial \overline{b}_s}{\partial Z_r}\right) \mathrm{d}Z_r \mathrm{d}\overline{Z}_s$$
- Hermitian, since T real: $$\frac{\partial \overline{b}_s}{\partial Z_r} = \frac{\partial b_r}{\partial \overline{Z}_r} \qquad (KC)$$
- Kähler form

$$\omega = \frac{i\pi}{2} \sum_{r,s=1}^{n} \left(\delta_{rs} + 2\frac{\partial \overline{b}_s}{\partial Z_r}\right) \mathrm{d}Z_r \wedge \mathrm{d}\overline{Z}_s$$

  Closed by (KC). M_n is a Kähler manifold.

M_2

M_2°

$$M_2 \cong \mathbb{C}_{com} \times M_2^\circ$$

Chern-Simons Vortices

$$\mathcal{L} = \frac{1}{2}\Big(D_\mu\varphi\overline{D^\mu\varphi} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \kappa\,\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \partial_\mu N\partial^\mu N$$
$$-\frac{1}{4}(|\varphi|^2 - 1 - 2\kappa N)^2 + |\varphi|^2 N^2\Big)$$

- Finite energy: $|\varphi| \to 1$, $N \to 0$, [or $\varphi \to 0$, $N \to -(2\kappa)^{-1}$] as $r \to \infty$
- Flux quantization unchanged
- $\kappa = 0$: usual AHM embeds with $N = 0$

Bogomol'nyi argument (Lee-Lee-Min)

- Consider all **stationary** fields ($\partial_0\varphi = \partial_0 A_\mu = 0$) satisfying Gauss's law (E-L eqn from varying A_0)

$$\nabla^2 A_0 - |\varphi|^2 A_0 - \kappa B = 0$$

$$\begin{aligned}
E &= \frac{1}{2}\int_{\mathbb{R}^2} |-iA_0\varphi|^2 + \overline{D_i\varphi}D_i\varphi + \partial_i A_0 \partial_i A_0 + B^2 + V(\varphi, N) \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\Big(|(D_1 + iD_2)\varphi|^2 + \big(B + \frac{1}{2}(|\varphi|^2 - 1 - 2\kappa N)\big)^2 \\
&\quad + |\varphi|^2 (A_0 - N)^2 - 2N(\nabla^2 A_0 - |\varphi|^2 A_0 - \kappa B) \\
&\quad + (\nabla A_0 - \nabla N)^2 + B\Big)\, d^2\mathbf{x} \\
&\geq \pi n \qquad \text{with equality iff}
\end{aligned}$$

$$(D_1 + iD_2)\varphi = 0 \qquad B + \frac{1}{2}(|\varphi|^2 - 1 - 2\kappa N) = 0 \qquad A_0 = N$$

- Formal index theorem argument suggests $M_n \equiv \mathbb{C}^n$ again (Lee-Min-Rim)

Kim-Lee approximation (small κ, speed)

- Curve $\alpha(t)$ in $M_n \equiv M_n|_{\kappa=0}$

$$
\begin{aligned}
L &= \frac{1}{2}\gamma_{L^2}(\dot\alpha,\dot\alpha) + \mathcal{A}_1(\dot\alpha) + \mathcal{A}_2(\dot\alpha) + O(\kappa^3, \kappa^2 v, \kappa v^2, v^3) \\
\mathcal{A}_1 &= i\frac{\pi\kappa}{2}\sum_r (b_r dz_r - \overline{b_r} d\overline{z}_r) \\
\mathcal{A}_2 &= i\frac{\pi\kappa}{8}\sum_r (H_r dz_r - \overline{H_r} d\overline{z}_r) \\
H_r &= -b_r + \sum_{s\neq r}\left\{(z_r - z_s)\frac{\partial b_r}{\partial z_s} + (\overline{z}_r - \overline{z}_s)\frac{\partial b_r}{\partial \overline{z}_s}\right\}.
\end{aligned}
$$

- **Magnetic** geodesic motion on M_n, $\mathcal{B} = d(\mathcal{A}_1 + \mathcal{A}_2)$
- Collie and Tong's (amazing) claim: $\mathcal{B} = \kappa\rho$!
  (Recall: $\rho(X, Y) = Ric(JX, Y)$, a closed two-form on any Kähler mfd)
- Argument is extremely indirect.

Kim-Lee flow on M_2 is ill-defined!

- COM/relative coords

$$Z = \frac{1}{2}(z_1 + z_2), \qquad \zeta = \sigma e^{i\theta} = \frac{1}{2}(z_1 - z_2)/2$$

- Translation/reflexion symmetry $\implies$

$$b_1(\zeta) = b(\sigma)e^{-i\theta} = -b_2(\zeta), \qquad b \text{ real}$$

- $\mathcal{B} = f(\sigma)d\sigma \wedge \sigma d\theta$ where

$$f(\sigma) = \frac{\pi\kappa}{\sigma}\frac{d}{d\sigma}\left(-2\sigma b(\sigma) + \frac{1}{2}\sigma^2 b'(\sigma)\right)$$

Defines $\mathcal{B}$ on all M_2 except coincidence set, $\sigma = 0$

Kim-Lee flow on M_2 is ill-defined!

- Small σ asymptotics:

$$\begin{aligned} b(\sigma) &= \frac{1}{\sigma} - \frac{1}{2}\sigma + O(\sigma^2) \\ \implies \qquad f(\sigma) &= \frac{3}{2}\pi\kappa + O(\sigma^2) \end{aligned}$$

- But $\zeta = \sigma e^{i\theta}$ is not a global coordinate on M_2 (since $\zeta \equiv -\zeta$).

$$p(z) = (z - z_1)(z - z_2) = z^2 - 2Zz + (Z^2 - \zeta^2)z$$

  so good global coords are Z, w where $w = \zeta^2$.

- $\mathcal{B} = \frac{3}{2}\pi\kappa \left(\frac{1}{|w|} + O(1) \right) \frac{i}{8} dw \wedge d\overline{w}$
- $\mathcal{B}$ blows up on $\Delta \subset M_2$.
- $\mathcal{B} \neq \kappa\rho$

Ricci magnetic geodesic flow

- Kim-Lee flow on M_n is **not** RMG flow
- In fact, it's not a globally well-defined flow at all (undefined when vortices coincide)
- RMG flow certainly is globally defined, so maybe Collie-Tong are right (despite being "wrong"...)?
- RMG flow makes sense on any Kähler manifold

Ricci magnetic geodesic flow

$$\nabla^{\alpha}_{d/dt}\dot{\alpha} = \kappa \sharp \iota_{\dot{\alpha}}\rho$$

- Obvious properties:
  - reduces to geodesic flow when $\kappa = 0$
  - conserves speed $\|\dot{\alpha}(t)\|^2 = \gamma(\dot{\alpha}, \dot{\alpha})$
  - $\alpha(t)$ is RMG_κ iff $\alpha(ct)$ is $\mathrm{RMG}_{c\kappa}$
  - can assume $\kappa = 1$, or $\|\dot{\alpha}\| = 1$
- On a **surface**, $\rho = K\omega$
  - so α is RMG_κ iff it has constant speed and

$$\text{signed curvature} = \langle \frac{\nabla_{d/dt}\dot{\alpha}}{\|\dot{\alpha}\|^2}, J\frac{\dot{\alpha}}{\|\dot{\alpha}\|} \rangle = \frac{\kappa}{\|\dot{\alpha}\|} K$$

Ricci magnetic geodesic flow

$$\nabla^{\alpha}_{d/dt}\dot{\alpha} = \kappa\, \sharp\iota_{\dot{\alpha}}\rho$$

- RMG curves are **not** preserved by time reversal, or by general isometries
- RMG curves **are** preserved by **holomorphic** local isometries
- **Corollary:** Let G be a group of holomorphic isometries of M and M^G be its fixed point set. General nonsense implies M^G is a complex submanifold of M. Then any RMG curve in M with initial data tangent to M^G remains on M^G for all time.
- **Warning!** M^G is itself a Kähler mfd (w.r.t $\iota^*\gamma$) so has its own RMG flow. These two RMG flows (extrinsic and intrinsic) do **not** coincide in general!

Hyperbolic vortices

- Metric on M_n not known for vortices on $\mathbb{R}^2$
- Nice fact: Bogomol'nyi eqns are integrable if we put the model on $\mathbb{H}^2$

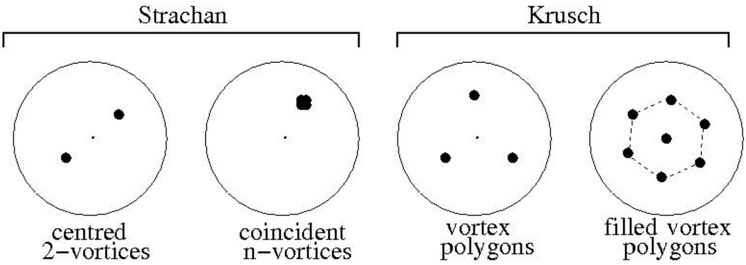
$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} \; : \; y > 0\}, \qquad g = \frac{8}{y^2}\,(dx^2 + dy^2)$$

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} \; : \; |x + iy| < 1\}, \qquad g = \frac{8(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

- Allows one (in principle) to compute metric on M_n exactly

Hyperbolic vortices

- In practice, only metric on certain 2-dim submanifolds of M_n known exactly



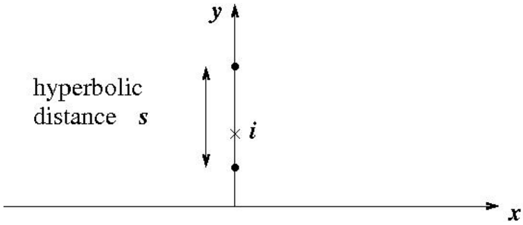
$$\gamma_n^{co} = \frac{\pi n(n+2)}{2} g_{\mathbb{H}^2}$$
$$\gamma_{n,n-1}^{poly} = \frac{3}{2}\pi n g_{\mathbb{H}^2}$$

Metric on $M_2 = (\mathbb{H}^2 \times \mathbb{H}^2)/S_2$

- $G = PL(2, \mathbb{R})$ acts isometrically on $\mathbb{H}^2$, hence on M_2

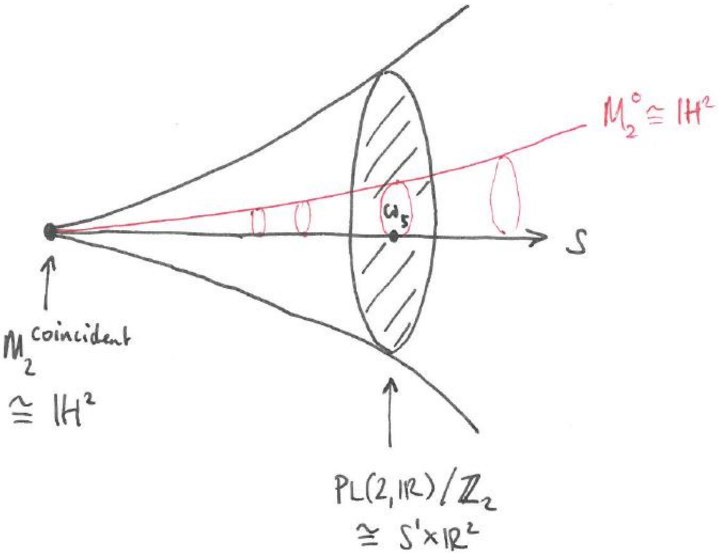
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

- Every G orbit contains a unique point $w_s = [(ie^{s/2}, ie^{-s/2})]$, $s \geq 0$



- Generic isotropy group $K = \{\mathbb{I}_2, Q\}$, $Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Metric on $M_2 = (\mathbb{H}^2 \times \mathbb{H}^2)/S_2$



Metric on $M_2 = (\mathbb{H}^2 \times \mathbb{H}^2)/S_2$

- γ determined by it values on $V_s = T_{w_s} M_2 = \langle \partial/\partial s \rangle \oplus \mathfrak{g}$
- $\mathfrak{g}$ = traceless real 2×2 matrices, basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Most general $Ad(K)$ invariant inner product on V_s

$$\gamma_s = A_1(s)ds^2 + A_2(s)\sigma_1^2 + A_3(s)\sigma_2^2 + A_4(s)\sigma_3^2 + A_5(s)ds\sigma_2 + A_6(s)\sigma_1\sigma_3$$

where σ_i = left-invariant one forms dual to e_i

- Almost complex structure

$$Je_1 = \cosh(s/2)e_3, \qquad Je_2 = -4\sinh(s/2)\frac{\partial}{\partial s}$$

- $\gamma(JX, JY) = \gamma(X, Y) \Longrightarrow$

$$A_3 \equiv 16\sinh^2(s/2)A_1, \quad A_4 \equiv \frac{A_2}{\cosh^2(s/2)}, \quad A_5 \equiv A_6 \equiv 0$$

Metric on $M_2 = (\mathbb{H}^2 \times \mathbb{H}^2)/S_2$

- Kähler form $\omega(X, Y) = \gamma(JX, Y)$

$$\omega = 4\sinh(s/2)A_1 \, ds \wedge \sigma_2 + \frac{A_2}{\cosh(s/2)} \, \sigma_1 \wedge \sigma_3$$

- $\mathrm{d}\omega = 0 \Longrightarrow \frac{d}{ds}\left(\frac{A_2}{\cosh(s/2)}\right) - 8\sinh(s/2)A_1 = 0$
- **Proposition:** let γ be a G-invariant Kähler metric on M_2. Then, for some function $A_2(s) > 0$,

$$\gamma = A_1 ds^2 + A_2\sigma_1^2 + A_3\sigma_2^2 + A_4\sigma_3^2$$

where

$$\begin{aligned} A_1 &= \frac{1}{8\sinh(s/2)} \frac{d}{ds}\left(\frac{A_2(s)}{\cosh(s/2)}\right) \\ A_3 &= 2\sinh(s/2)\frac{d}{ds}\left(\frac{A_2(s)}{\cosh(s/2)}\right) \\ A_4 &= \frac{A(s)}{\cosh^2(s/2)} \end{aligned}$$

Metric on $M_2 = (\mathbb{H}^2 \times \mathbb{H}^2)/S_2$

- Strachan's formula for γ on M_2^0 determines A_0, hence A_2 up to an integration constant
- Regularity at $s = 0$ determines the constant

$$\frac{A_2(s)}{8\pi} = \cosh^2(s/2) + 1 + 2\sinh^2(s/2)\sqrt{\frac{\cosh^2(s/2)}{\sinh^4(s/2)} + 1}$$

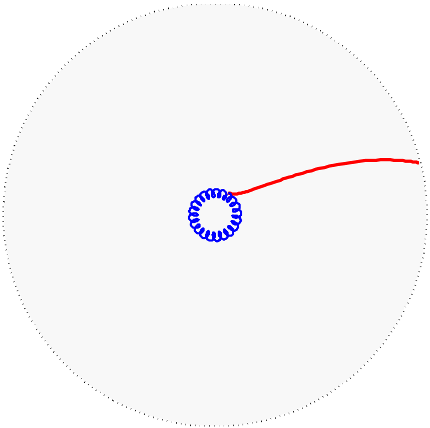
RMG flow on M_2^0

- Ricci form easy to compute (obeys same structure lemma as Kähler form)
- Consider the holomorphic isometry $Q : [(z_1, z_2)] \mapsto [(-1/z_2, -1/z_1)]$
- Fixed point set: $M_2^0 = \{[(\xi, -1/\xi)] \, : \, \xi \in \mathbb{H}^2\}$
- RMG curves initially tangent to M_2^0 stay on M_2^0 for all time. Two RMG flows

$$\begin{array}{llll} \text{Extrinsic:} & \mathcal{B} = \kappa\rho| & \sim & -\kappa e^{s/2} ds \wedge \sigma_2 \\ \text{Intrinsic:} & \mathcal{B} = \kappa K(s)\omega| & \sim & -\dfrac{\kappa}{2} e^{s/2} ds \wedge \sigma_2 \end{array}$$

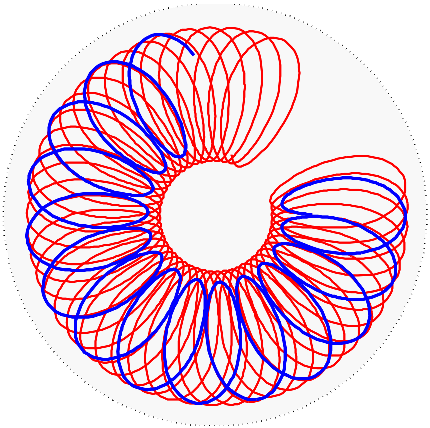
Compare flows with $\kappa_{intrinsic} = 2\kappa_{extrinsic}$

Extrinsic vs intrinsic RMG flow on M_2^0



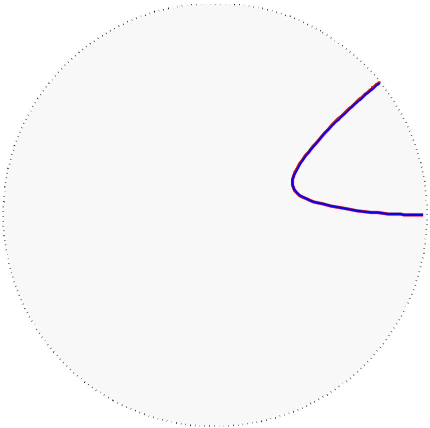
extrinsic, intrinsic

Extrinsic vs intrinsic RMG flow on M_2^0



extrinsic, intrinsic

Extrinsic vs intrinsic RMG flow on M_2^0



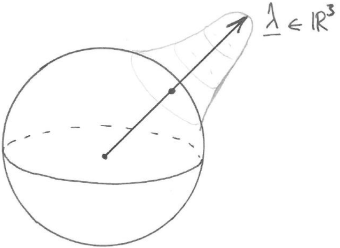
extrinsic, intrinsic

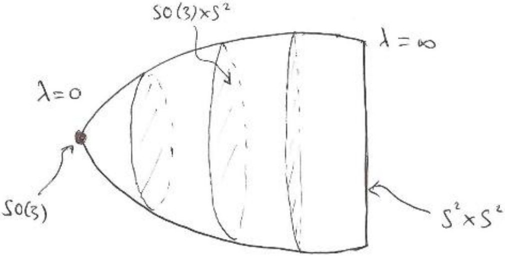
Completeness of RMG flow

- RMG flow constant speed: M geodesically complete implies M RMG complete
- Converse?
- $\alpha(t)$ is $\text{RMG}_{\kappa/c}$ iff $\alpha(ct)$ is RMG_{κ}
- Speed $\to \infty$ limit equivalent to $\kappa \to 0$ (geodesic) limit
- Naively suggests converse true
- Actually, it's FALSE!

Moduli space of charge 1 $O(3)$ sigma model lumps on S^2

- $M_1 = \mathrm{Rat}_1 = \{\frac{az+b}{cz+d}\ ad - bc \neq 0\} \equiv SO(3) \times \mathbb{R}^3$





- Kähler, invariant under $G = SO(3) \times SO(3)$
- Geodesically incomplete.

Moduli space of charge 1 $O(3)$ sigma model lumps on S^2

- G-invariance $\implies$ RMG flow conserves 6 angular momenta, K_i, I_i
- Also conserves energy $\|\dot{\alpha}\|^2$
- Define $q : T\mathrm{Rat}_1 \to \mathbb{R}^7$, $q(\dot{\alpha}) = (\|\dot{\alpha}\|^2, \mathbf{K}, \mathbf{I})$
- Every RMG curve confined to a level set of q
- **Theorem:** every level set of q is compact!
- **Corollary:** RMG flow on Rat_1 is complete

Summary

- RMG flow on $M_n(\mathbb{R}^2)$ proposed by Collie-Tong as low energy model of CS-Maxwell vortex dynamics
- Claimed it coincides with Kim-Lee flow
- **FALSE!** In fact Kim-Lee flow ill-defined on $\Delta \subset M_n$
- Intrinsic RMG flow on surfaces of revolution in $M_n(\mathbb{H}^2)$ studied by Krusch-JMS
- Claimed it coincides with extrinsic RMG flow
- **FALSE!** In fact they're qualitatively different
- Krusch-JMS conjectured that geodesic incompleteness implies RMG incompleteness
- **FALSE!** E.g. $(\mathrm{Rat}_1, \gamma_{L^2})$ is incomplete but RMG complete

Summary: open questions

- Does RMG flow really model CSM vortex dynamics?
  - numerics?
  - point vortex model (large separation)?
- When does RMG completeness imply geodesic (equiv. metric) completeness?
  - Uniformly bounded ρ?
  - Surfaces of bounded Gauss curvature?
- Quantization?
  - $\rho =$ curvature of canonical bundle. Suggests ψ a section thereof, and $H = \frac{1}{2}\Delta^{\nabla}$
  - What about κ? Quantized on compact M?