Near-BPS Skyrmions and Restricted Harmonic Maps

Martin Speight
University of Leeds, UK

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Near BPS Skyrme models

$$\varphi : \mathbb{R}^3 \to S^3$$

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$$\varphi : M \longrightarrow N$$

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$$E = E_2 + E_4$$

$$\begin{aligned} E_2 &= \frac{1}{2}\int_M |\mathrm{d}\varphi|^2 \\ E_4 &= \frac{1}{2}\int_M |\mathrm{d}\varphi \wedge \mathrm{d}\varphi|^2 \end{aligned}$$

- Usual model: classical binding energies much too large

Near BPS Skyrme models

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$$\begin{aligned} E &= E_0 + E_2 + E_4 \\ E_0 &= \frac{1}{2}\int_M U(\phi)^2 \\ E_2 &= \frac{1}{2}\int_M |\mathrm{d}\varphi|^2 \\ E_4 &= \frac{1}{2}\int_M |\mathrm{d}\varphi \wedge \mathrm{d}\varphi|^2 \end{aligned}$$

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Near BPS Skyrme models

$$\varphi : M \to N$$

$$\begin{aligned}
E &= E_0 + E_2 + E_4 + E_6 + \cdots \\
E_0 &= \frac{1}{2}\int_M U(\phi)^2 \\
E_2 &= \frac{1}{2}\int_M |\mathrm{d}\varphi|^2 \\
E_4 &= \frac{1}{2}\int_M |\mathrm{d}\varphi \wedge \mathrm{d}\varphi|^2 \\
E_6 &= \frac{1}{2}\int_M |\varphi^* \mathrm{vol}_N|^2
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- Adam, Sanchez-Guillen, Wereszczynski: BPS model

Near BPS Skyrme models

$$\varphi : M \to N, \qquad E = E_6 + E_0$$

$$\begin{aligned} 0 &\leq \frac{1}{2}\|\varphi^*\mathrm{vol}_N - *U\circ\varphi\|_{L^2}^2 = E - \langle\varphi^*\mathrm{vol}_N, *U\circ\phi\rangle_{L^2} \\ E &\geq \int_M \varphi^*(U\mathrm{vol}_N) = \textit{constant}\ B \end{aligned}$$

- Equality iff

$$\varphi^*\mathrm{vol}_N = (U\circ\varphi)\mathrm{vol}_M \qquad (BOG)$$

- Solutions are volume-preserving maps $M \to N' = N\backslash U^{-1}(0)$ where $\mathrm{vol}_{N'} = \mathrm{vol}_N/U$,

$$\varphi^*\mathrm{vol}_{N'} = \mathrm{vol}_M$$

Near BPS Skyrme models

$$\varphi^* \mathrm{vol}_{N'} = \mathrm{vol}_M$$

- Given solution φ and volume-preserving map $\psi : M \to M$ (meaning $\psi^* \mathrm{vol}_M = \mathrm{vol}_M$), $\varphi \circ \psi$ also solves (BOG)
- ASW use this: given charge 1 BPS solution (hedgehog), obtain charge B solutions $\varphi_{hedgehog} \circ \psi_B$ where

$$\psi_B : \mathbb{R}^3 \backslash \mathbb{R}_z \to \mathbb{R}^3 \backslash \mathbb{R}_z, \qquad \psi_B(r, \theta, \phi) = (B^{-1/3} r, \theta, B\phi)$$

- BPS Skyrmions come in infinite dimensional families: orbits of φ under $\mathrm{SDiff}(M, g)$.

Near-BPS Skyrme models

$$E = E_0 + E_6$$

- BPS model is sick: field equation doesn't uniquely determine time evolution even locally
- Need to include E_2. Take $\varepsilon > 0$ small. **Near** BPS Skyrme model
- Use BPS Skyrmions as starting approx in near BPS theory. But which ones? They come in infinite dimensional families!
- φ_{BPS} should **minimize** E_2 among all maps in its SDiff orbit
- Suggests natural variant of harmonic map problem

Near-BPS Skyrme models

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Restricted harmonic maps

- $\mathrm{SDiff}(M,g) =$ volume preserving diffeos of M of **compact support**
  - $supp(\psi) = \overline{\{x \in M : \psi(x) \neq x\}}$

- **Definition:** $\varphi : (M,g) \to (N,h)$ is **restricted** harmonic if $E_2(\varphi)$ is finite and φ locally extremizes the restriction of E_2 to its $\mathrm{SDiff}(M,g)$ orbit
  i.e. for all smooth curves ψ_t in $\mathrm{SDiff}(M,g)$ through $\psi_0 = \mathrm{Id}_M$,

$$\frac{d}{dt}E_2(\varphi \circ \psi_t)\Big|_{t=0} = 0$$

- Clearly harmonic $\Rightarrow$ restricted harmonic, but converse is false (e.g. **all** closed curves $S^1 \to (N,h)$ are restricted harmonic!)
- Will also want local stability criterion (need φ to **minimize** E_2 not just **extremize**)

Theorem

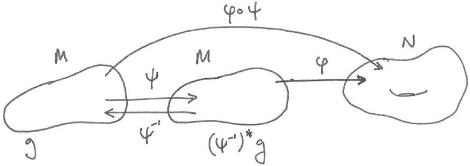
A smooth map $\varphi : (M, g) \to (N, h)$ of finite E_2 is restricted harmonic iff $\operatorname{div}\varphi^* h$ is exact $\quad$ ($= \mathrm{d}f$ for some $f : M \to \mathbb{R}$)

- $(\operatorname{div} T)(X) = \sum_i (\nabla_{e_i} T)(e_i, X)$.
  - On $\mathbb{R}^m$, $(\operatorname{div} T)_k = \sum_i \partial_i T_{ik}$
  - Useful fact: $\operatorname{div}(fT) = \iota_{\nabla f} T + f \operatorname{div} T$ (so $\operatorname{div}(fg) = \mathrm{d}f$)
- $\mathrm{d}(\operatorname{div}\varphi^* h) = 0$ third order nonlinear PDE for φ, plus finite collection of integral constraints (parametrized by $H^{m-1}_{compact}(M)$)
- Proof uses a nice trick

Geometric naturalness (naturality?)

- **Fact:** $E_2(\varphi, g)$ is **geometrically natural**:
  For all maps $\varphi : M \to N$, diffeos $\psi : M \to M$ and Riemannian metrics g on M,

  $$E_2(\varphi \circ \psi, g) = E_2(\varphi, (\psi^{-1})^* g)$$

  $\Rightarrow$ Variations of φ through diffeos of M (with g fixed) are equivalent to variations of g via pullback (with φ fixed)!
- Leads us to consider variations of g

- Given any curve of metrics g_t through $g_0 = g$ tangent to $\varepsilon = \partial_t g_t|_{t=0}$,

$$\frac{d}{dt} E(\varphi, g_t)\Big|_{t=0} = \frac{1}{2}\langle S(\varphi), \varepsilon\rangle_{L^2}$$

- Uniquely determines a symmetric $(0,2)$ tensor $S(\varphi)$, the **stress tensor**
- For E_2,

$$S(\varphi) = \frac{1}{2}|\mathrm{d}\varphi|^2 g - \varphi^* h$$

(Partial) proof of first variation theorem

- M compact, φ restricted harmonic $\Rightarrow$ $\quad \mathrm{d}(\operatorname{div} \varphi^* h) = 0$
- Choose any 2-form ν on M. Then $X = \sharp\delta\nu$ is a divergenceless vector field on M. Hence its flow ψ_t is a curve in $\mathrm{SDiff}(M, g)$ through Id_M

$$0 = \frac{d}{dt} E_2(\varphi \circ \psi_t, g)\Big|_{t=0} = \frac{d}{dt} E_2(\varphi, \psi_{-t}^* g)\Big|_{t=0} = \frac{1}{2}\langle S(\varphi), -\mathscr{L}_X g\rangle_{L^2}$$

$$= \int_M (\operatorname{div} S)(X) = \langle \operatorname{div} S, \flat X\rangle_{L^2} = \langle \operatorname{div} S, \delta\nu\rangle_{L^2} = \langle \mathrm{d}(\operatorname{div} S), \nu\rangle_{L^2}$$

- But $S = fg - \varphi^* h$ so $\operatorname{div} S = \mathrm{d}f - \operatorname{div}(\varphi^* h)$ $\square$

Weakly conformal maps are restricted harmonic

- A map $\varphi : (M, g) \to (N, h)$ is **weakly conformal** if $\varphi^* h = fg$ for some smooth function $f : M \to \mathbb{R}$
- **Corollary:** Let φ have finite E_2 and be weakly conformal. Then φ is restricted harmonic
  Proof: $\mathrm{div}\, \varphi^* h = \mathrm{div}\,(fg) = \mathrm{d}f$ $\square$
- E.g. inverse stereographic projection $\mathbb{R}^3 \to S^3$ is conformal, hence restricted harmonic. Also a charge 1 BPS Skyrmion, for $U(\varphi) = (1 - \varphi_0)^3$

Suspension maps $\mathbb{R}^3 \to S^3$

- Fix maps $\mathscr{R} : S^2 \to S^2$ of degree B and $f : [0,\infty) \to (0,\pi]$
  The **suspension** of $\mathscr{R}$ by f is

$$\varphi : \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4, \qquad \varphi(rn) = (\cos f(r), \sin f(r)\mathscr{R}(n))$$

  where $r \geq 0$ and $n \in S^2 \subset \mathbb{R}^3$. Degree B Skyrme field
- Such a map has $\varphi^* h = f'(r)^2 dr^2 + \sin^2 f(r) \mathscr{R}^* g_{S^2}$
- **Hedgehog ansatz:** $\mathscr{R} = \mathrm{Id}_{S^2}$

$$\begin{aligned} \varphi^* h &= a(r) g + b(r) dr^2 \qquad \text{since } g = dr^2 + r^2 g_{S^2} \\ \Rightarrow \quad \mathrm{div}\,(\varphi^* h) &= da + c(r) dr \end{aligned}$$

  always exact.
- All fields in hedgehog ansatz are restricted harmonic

Suspension maps $\mathbb{R}^3 \to S^3$

$$\varphi(rn) = (\cos f(r), \sin f(r)\mathscr{R}(n)), \qquad \varphi^* h = f'(r)^2 dr^2 + \sin^2 f(r)\mathscr{R}^* g_{S^2}$$

- **Rational map ansatz:** $\mathscr{R}$ =holomorphic, hence weakly conformal, $\mathscr{R}^* g_{S^2} = \lambda(n) g_{S^2}$

$$\begin{aligned} \varphi^* h &= a(r,n)g + b(r)\lambda(n)dr^2 \\ \Rightarrow \quad \mathrm{div}\,(\varphi^* h) &= \mathrm{d}a + \lambda(n)c(r)dr \end{aligned}$$

  Never closed unless λ =constant, i.e., $\mathscr{R}$ an isometry (hedgehog ansatz).
- Rational map ansatz gives no new restricted harmonic maps

Suspension maps $\mathbb{R}^3 \to S^3$

$$\varphi(rn) = (\cos f(r), \sin f(r)\mathscr{R}(n)), \qquad \varphi^* h = f'(r)^2 dr^2 + \sin^2 f(r)\mathscr{R}^* g_{S^2}$$

- **ASW ansatz:** $\mathscr{R}(\theta, \phi) = (\theta, B\phi)$

$$\begin{aligned} \varphi^* h &= f'(r)^2 dr^2 + \sin^2 f(r)(d\theta^2 + B^2 \sin^2\theta d\phi^2) \\ \Rightarrow \quad \mathrm{ddiv}\,(\varphi^* h) &= (B^2 - 1)(\cdots) \neq 0 \quad \text{if } B \neq \pm 1 \end{aligned}$$

- Again, no restricted harmonic maps except hedgehogs ($B = 1$)

- Can construct RH maps $\mathbb{R}^3 \to S^3$ in each htpy class by taking $\mathscr{R} = \mathrm{Id}_{S^2}$ and $f(0) = B\pi$, $f(\infty) = 0$, but these are rather fake

- $M = \mathbb{R}$: trivial as $\mathrm{SDiff}(\mathbb{R}, dx^2) = \{\mathrm{Id}_M\}$
- $M = S^1$: also trivial as $\mathrm{SDiff}(S^1, dx^2) = \{\text{translations}\}$
- Baby skyrme fields: $M = R^2$, $N = S^2$
  Any rotationally equivariant map $\varphi(rn) = (\cos f(r), \sin f(r) n^B)$ is restricted harmonic

Stability: second variation of E_2

- $Hess_\varphi : \Gamma_0(TM) \times \Gamma_0(TM) \to \mathbb{R}$ is the symmetric bilinear form

$$Hess_\varphi(X, Y) = \left. \frac{\partial^2 E_2(\varphi \circ \psi_{s,t})}{\partial s \partial t} \right|_{s=t=0}$$

where $\psi_{s,t}$ is an arbitrary two-param variation of $\psi_{0,0} = \mathrm{Id}_M$ in $\mathrm{SDiff}(M, g)$, with

$$X = \partial_s \psi_{s,0}|_{s=0}, \qquad Y = \partial_t \psi_{0,t}|_{t=0}$$

- φ is **stable** if $Hess_\varphi(X, X) \geq 0$ for all X, **unstable** otherwise

Theorem

$Hess_\varphi(X, Y) = \frac{1}{2} \langle \mathscr{L}_X \varphi^* h, \mathscr{L}_Y g \rangle_{L^2}$

- Doesn't *look* symmetric, but it is

Stability

- **Corollary:** Let φ have finite E_2 and be weakly conformal. Then φ is stable restricted harmonic
- Proof: $\mathscr{L}_X(\varphi^* h) = \mathscr{L}_X(f^2 g) = f^2 \mathscr{L}_X g + X[f^2]g$,so

$$Hess_\varphi(X,X) = \frac{1}{2}\|f\mathscr{L}_X g\|_{L^2}^2 + \frac{1}{2}\langle X[f^2]g, \mathscr{L}_X g\rangle_{L^2} = \frac{1}{2}\|f\mathscr{L}_X g\|_{L^2}^2$$

since $\mathscr{L}_X g$ is pointwise orthogonal to g. $\square$

Concluding remarks

- In order to be a sensible approximant of a near-BPS Skyrmion in the model $E_0 + E_6 + \varepsilon E_2$, φ_{BPS} should be restricted harmonic (and have $Hess \geq 0$)
- φ is restricted harmonic iff $\mathrm{div}\,(\varphi^* h)$ is an exact one-form on M
- Whole analysis easily generalizes to $E_0 + E_6 + \varepsilon F$, where $F(\varphi, g)$ is any geometrically natural functional, e.g. $E_2 + E_4$.
  φ is restricted F-critical iff $\mathrm{div}\,(\varphi^* S_F)$ is exact
- Constructing restricted harmonic maps $\mathbb{R}^3 \to S^3$ is hard. The fields used by ASW, Marleau et al are not restricted harmonic

Concluding remarks

- Finite dimensional analogue: given $\varphi : \mathbb{R}^k \to N$, minimize

$$E_\varphi : SL(k, \mathbb{R}) \to \mathbb{R}, \qquad E_\varphi(A) = E_2(\varphi \circ A) = \frac{1}{2}\mathrm{tr}(A^T M A)$$

where $M_{ij} = \int_{\mathbb{R}^k} \mathscr{D}_{ij} d^k x =$ "average strain matrix"

- **Fact:** $E_\varphi(A) \geq \frac{k}{2}(\det M)^{1/k}$, with equality if and only if $A^T M A = \mu \mathbb{I}_k$, and equality is always attained
  - E.g. for $\varphi_B^{ASW} = \varphi_{hedgehog} \circ \psi_B$,

$$A_B = \begin{pmatrix} \lambda_B & & \\ & \lambda_B & \\ & & \lambda_B^{-2} \end{pmatrix}$$

  where $\lambda_B \to 0$ monotonically as $B \to \infty$. Squashed BPS Skyrmions
  - $E_2(\varphi_B^{ASW}) \sim B^{7/3}$ whereas $E_2(\varphi_B^{ASW} \circ A_B) \sim B^{5/3}$

Concluding remarks

- As a natural geometric variational problem, restricted harmonic maps have lots of interesting open questions:
  - cohomological (local/global RHMs)
  - rigidity theory (dimension of the moduli space of RHMs)
  - spectral approach to stability...