Quantum dynamics of a CP1 lump on the two-sphere

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Joint work with Steffen Krusch (Kent)

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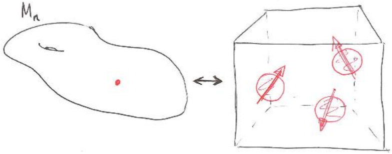

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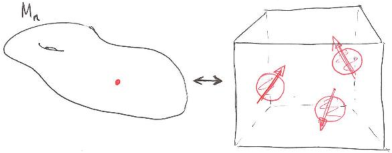

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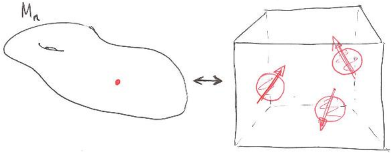

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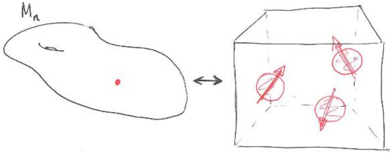

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- Examples: Yangs-Mills-Higgs (monopoles), abelian Higgs (vortices), $O(3)$ sigma model ($\mathbb{C}P^1$ lumps)

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- Do these corrections change the quantum dynamics **qualitatively**? E.g. H_0 may have only cts spectrum, H_{BO} only discrete. Or H_{BO} may have extra bound states. Or the degeneracies of energy levels may change.

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  - kähler, but not hyperkähler ($\kappa \neq 0$)
  - can compute $\mathscr{C}$ (numerically) using ideas from diff geom

$\mathbb{C}P^1$ lumps on S^2

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i.e. iff φ **holomorphic**

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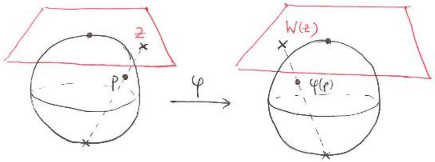

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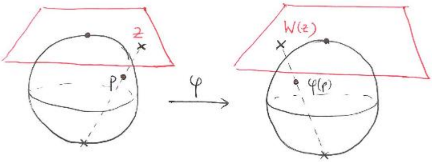

- So $M_n = \mathrm{Rat}_n \subset \mathbb{C}P^{2n+1}$

$$W(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_n z^n} \leftrightarrow [a_0, \ldots, a_n, b_0, \ldots, b_n]$$

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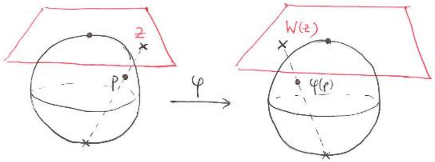

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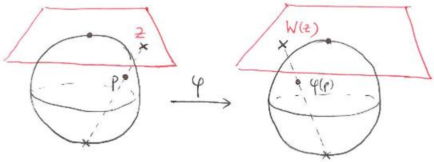

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- $PSL(2, \mathbb{C}) \cong PU(2) \times \mathbb{R}^3 \cong SO(3) \times \mathbb{R}^3$

$$\begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} = UH = U(\sqrt{1 + \lambda^2}\mathbb{I}_2 + \vec{\lambda} \cdot \vec{\tau})$$

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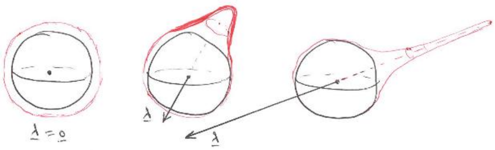

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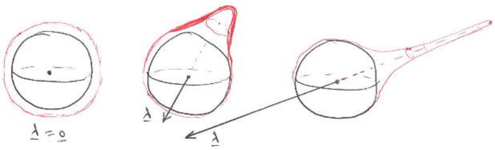

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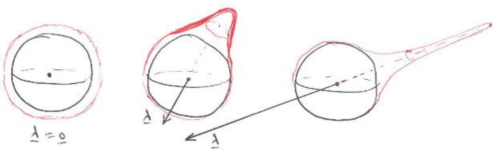

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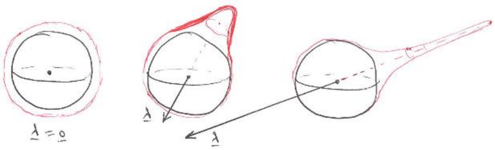

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- Natural metric on Rat_1 assigns squared length

$$\int_{S^2} \varphi_t \cdot \varphi_t$$

to the tangent vector to any curve $\varphi(t)$ in Rat_1

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- It's kähler, and invariant under both $SO(3)$ actions:

$$\gamma = A_1\, d\vec{\lambda}\cdot d\vec{\lambda} + A_2(\vec{\lambda}\cdot d\vec{\lambda})^2 + A_3\,\vec{\sigma}\cdot\vec{\sigma} + A_4(\vec{\lambda}\cdot\vec{\sigma})^2 + A_1\vec{\lambda}\cdot(\vec{\sigma}\times d\vec{\lambda})$$

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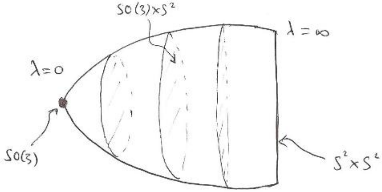
- Ricci positive with unbounded scalar curvature $\kappa(\lambda)$



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where $\partial_a = \frac{\partial}{\partial \lambda_a}$, $\theta_a =$ usual left-invariant vector fields on $SO(3)$, $B_i(\lambda)$ explicitly known

The laplacian on Rat_1

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The laplacian on Rat_1

- Define angular momentum operators

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- Wavefunction $\psi : [0,\infty)\times S^2\times SO(3)\to\mathbb{C}$

The laplacian on Rat_1

- Define angular momentum operators

$$\vec{J} = -i\vec{\theta}, \qquad \vec{K} = -i\vec{\xi}, \qquad \vec{L} = -i\vec{\lambda}\times\vec{\partial}, \qquad \vec{T} = \vec{J}+\vec{L} = -i\vec{X}$$

Recall Δ commutes with $\vec{K}$ and $\vec{T}$.

- $\vec{J}\cdot\vec{J} = \vec{K}\cdot\vec{K}$, so Δ commutes with $\vec{J}\cdot\vec{J}$
- Laplacian in physics language

$$\Delta f = -B_1 \frac{\partial}{\partial\lambda}\left(B_2 \frac{\partial f}{\partial\lambda}\right) + B_3 \vec{L}\cdot\vec{L} f + B_4\{\vec{J}\cdot\vec{J} + \vec{T}\cdot\vec{T} + B_5(\vec{\lambda}\cdot\vec{T})^2\} f$$

- Wavefunction $\psi : [0,\infty)\times S^2 \times SO(3) \to \mathbb{C}$
- Expand S^2 dependence in spherical harmonics Y_{ll_3}

$$\vec{L}\cdot\vec{L} Y_{ll_3} = l(l+1) Y_{ll_3}, \qquad L_3 Y_{ll_3} = l_3 Y_{ll_3}$$

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- Expand $SO(3)$ dependence in $\pi^{(j)}_{j_3,k_3}$

$$\psi = \sum_{j\in\mathbb{N}} \sum_{j_3=-j}^{j} \sum_{k_3=-j}^{j} \sum_{l\in\mathbb{N}} \sum_{l_3=-l}^{l} A^{jl}_{j_3 k_3 l_3}(\lambda)\pi^{(j)}_{j_3 k_3} Y_{l l_3}$$

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- Does **not** preserve l, l_3 or j_3. Standard angular momentum algebra: there exists a basis $|t,j,l,t_3\rangle$ for the space

$$\mathscr{V}_{jt} = \bigoplus_{j_3=-j}^{j}\ \bigoplus_{l=|j-t|}^{|j+t|}\ \bigoplus_{l_3=-l}^{l} \pi^{(j)}_{j_3 0} Y_{l l_3}$$

such that

$$\begin{aligned}
\vec{T}\cdot\vec{T}|t,j,l,t_3\rangle &= t(t+1)|t,j,l,t_3\rangle \\
\vec{J}\cdot\vec{J}|t,j,l,t_3\rangle &= j(j+1)|t,j,l,t_3\rangle \\
\vec{L}\cdot\vec{L}|t,j,l,t_3\rangle &= l(l+1)|t,j,l,t_3\rangle \\
T_3|t,j,l,t_3\rangle &= t_3|t,j,l,t_3\rangle
\end{aligned}$$

The laplacian on Rat_1

- $[\Delta, T_3] = 0$ so Δ preserves T_3 eigenspace $\mathscr{V}_{jtt_3}$, $-t \leq t_3 \leq t$.
- Spectral problem for $H_0 = \frac{1}{2}\Delta$ reduces to infinite sequence of vector Sturm-Liouville problems for maps $\psi : [0, \infty) \to \mathscr{V}_{jt0}$, vector space of dimension $2\min\{j, t\} + 1$, spanned by

$$|t, j, l, 0\rangle \qquad |j - t| \leq l \leq j + t$$

- Boundary conditions at $\lambda = 0$, $\lambda = \infty$: standard SL classification applies. Worst case: LCN
- Spectrum of $H_0 = \frac{1}{2}\Delta$ computed numerically

Spectrum of $H_0 = \frac{1}{2}\Delta$

energy	degeneracy	$\{j,t\}^P$
0.00	1	$\{0,0\}^+$
1.06	6	$\{0,1\}^-$
1.46	9	$\{1,1\}^-$
2.30	1	$\{0,0\}^+$
2.72	9	$\{1,1\}^-$
2.76	10	$\{0,2\}^+$
3.05	9	$\{1,1\}^+$
3.18	30	$\{1,2\}^+$
3.91	25	$\{2,2\}^-$
4.30	6	$\{0,1\}^-$
4.93	9	$\{1,1\}^-$
5.01	30	$\{1,2\}^+$
5.11	14	$\{0,3\}^-$
5.33	30	$\{1,2\}^-$

Born-Oppenheimer corrections

$$H_{BO} = H_0 + \frac{1}{4}\kappa(\lambda) + \mathscr{C}(\lambda) + \cdots$$

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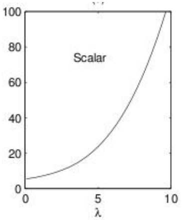

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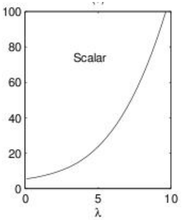

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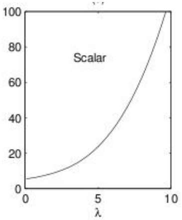

$\kappa \sim \lambda^4/(\log\lambda)^3$

- Changes BCs at ∞ (but not radically)
- Casimir energy $\mathscr{C}(\lambda)$ much murkier

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$$\mathscr{E} = \frac{1}{2}\sum_i \omega_i$$

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- $SO(2)$ symmetry: reduces to infinite sequence of **scalar** SL problems on interval $\theta \in [0, \pi]$. Numerics $\to \omega_i^2(\lambda)$.

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- ζ-function regularization $\longrightarrow$ finite renormalized Casimir energies $\mathscr{C}_*^0$, $\mathscr{C}_*^\infty$ for these two spectra

Casimir energy

- Cut off divergent infinite sum

$$\mathscr{C}_k(\lambda) = \frac{1}{2}\sum_{i=1}^{k}(\omega_i(\lambda) - \omega_i(\infty)), \qquad k = 10, 24, 42, \ldots, 4\ell + 2, \ldots$$

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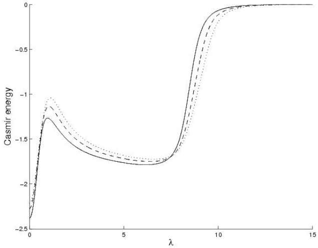

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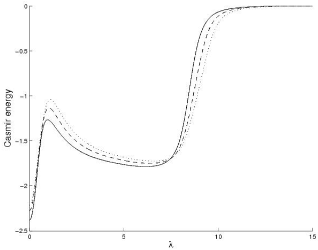

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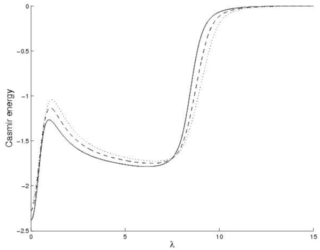


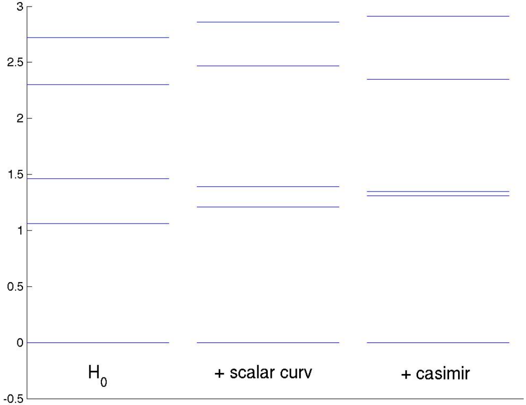
- Approximation: $\mathscr{C}(\lambda) = (\mathscr{C}_*^0 - \mathscr{C}_*^\infty)\mathscr{C}_{10}(\lambda)$
- $\mathscr{C}(\lambda)$ just a bounded potential: doesn't change BCs

Comparison of spectra

energy	degeneracy	$\{j,t\}^P$	energy	degeneracy	$\{j,t\}^P$	energy	degeneracy	$\{j,t\}^P$
0.00	1	$\{0,0\}^+$	1.79	1	$\{0,0\}^+$	0.58	1	$\{0,0\}^+$
1.06	6	$\{0,1\}^-$	3.00	6	$\{0,1\}^-$	1.89	6	$\{0,1\}^-$
1.46	9	$\{1,1\}^-$	3.18	9	$\{1,1\}^-$	1.93	9	$\{1,1\}^-$
2.30	1	$\{0,0\}^+$	4.26	1	$\{0,0\}^+$	2.93	1	$\{0,0\}^+$
2.72	9	$\{1,1\}^-$	4.65	9	$\{1,1\}^-$	3.49	9	$\{1,1\}^-$
2.76	10	$\{0,2\}^+$	4.68*	9	$\{1,1\}^+$	3.51*	9	$\{1,1\}^+$
3.05	9	$\{1,1\}^+$	4.84*	10	$\{0,2\}^+$	3.76*	10	$\{0,2\}^+$
3.18	30	$\{1,2\}^+$	5.07	30	$\{1,2\}^+$	3.95	30	$\{1,2\}^+$
3.91	25	$\{2,2\}^-$	5.56	25	$\{2,2\}^-$	4.25	25	$\{2,2\}^-$
4.30	6	$\{0,1\}^-$	6.36	6	$\{0,1\}^-$	5.07	6	$\{0,1\}^-$
4.93	9	$\{1,1\}^-$	6.64	9	$\{1,1\}^-$	5.39	9	$\{1,1\}^-$
5.01	30	$\{1,2\}^+$	6.96	30	$\{1,2\}^+$	5.81	30	$\{1,2\}^+$
5.11	14	$\{0,3\}^-$	7.01*	30	$\{1,2\}^-$	5.91*	30	$\{1,2\}^-$
5.33	30	$\{1,2\}^-$	7.30*	14	$\{0,3\}^-$	6.22*	14	$\{0,3\}^-$
5.42	25	$\{2,2\}^-$	7.44	25	$\{2,2\}^-$	6.24	25	$\{2,2\}^-$
5.52	42	$\{1,3\}^-$	7.57	42	$\{1,3\}^-$	6.44*	25	$\{2,2\}^+$
6.06	25	$\{2,2\}^+$	7.66	25	$\{2,2\}^+$	6.48*	42	$\{1,3\}^-$
6.30	70	$\{2,3\}^+$	8.10	70	$\{2,3\}^+$	6.96	70	$\{2,3\}^+$
6.46	1	$\{0,0\}^+$	8.55	1	$\{0,0\}^+$	7.22	1	$\{0,0\}^+$

- Low energy spectra rather similar

Comparison of spectra



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- BO corrections can be computed (almost) explicitly
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- Corrections don’t change low energy spectrum drastically
- Extension: add supersymmetry
  - Corrections vanish (?)
  - $\psi \in \Omega^{(0,p)}(M_n)$, $H_0 = \frac{1}{2}\Delta_{(0,p)}$