The geometry of the moduli space of BPS vortex-antivortex pairs

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joint with
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Plan

- Gauged sigma model on $\mathbb{R}^2$ with S^2 target: two species of vortex
- North vortices and South antivortices can coexist in stable equilibrium
- Moduli space $=$ {pairs **disjoint** effective divisors}: noncompact in a nontrivial way
- Natural Riemannian metric g_{L^2}. Complete?
- Focus on $(1,1)$ case, depends only on vortex-antivortex separation $\varepsilon > 0$
- (Almost) explicit formula for g_{L^2}, careful numerics
- Conjectured asymptotics:
  - $\varepsilon \to 0$ "self similarity"
  - $\varepsilon \to \infty$ point vortex model

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- Choose $\mathbf{e} = (0, 0, 1)$.

Flux quantization, vortices

- As $r \to \infty$, $\mathbf{e} \cdot \mathbf{n} \to 0$ and $D\mathbf{n} \to 0$

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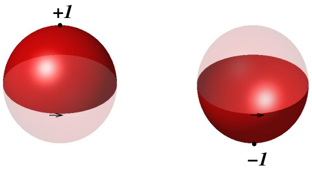

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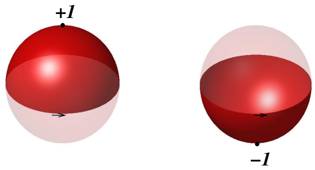

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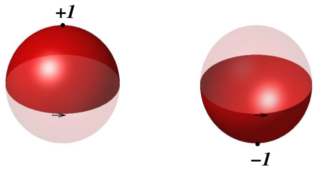

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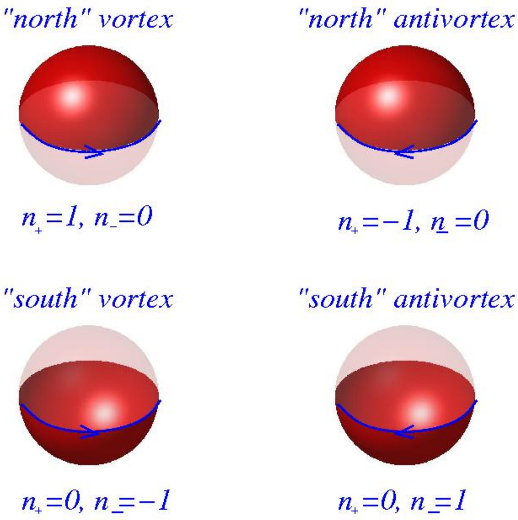
(Anti)vortices



Bogomol'nyi argument

- Let $Q = (\mathbf{e} \cdot \mathbf{n})A$. Note

$$(\mathbf{n} \times D\mathbf{n}) \cdot D\mathbf{n} = \mathbf{n}^*\omega + \mathrm{d}Q - (\mathbf{e} \cdot \mathbf{n})B$$

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with equality iff

$$\begin{aligned} D_1\mathbf{n}+\mathbf{n}\times D_2\mathbf{n} &= 0, && (BOG1) \\ *B &= \mathbf{e}\cdot\mathbf{n} && (BOG2) \end{aligned}$$

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$$\nabla^2 h - 2\tanh\frac{h}{2} = 0$$

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$$\nabla^2 h - 2\tanh\frac{h}{2} = \; 4\pi\left(\sum_{r=1}^{n_+}\delta(z - z_r^+) - \sum_{r=1}^{n_-}\delta(z - z_r^-)\right)$$

- vortices at z_r^+, antivortices at z_r^-

The Taubes equation

- **Theorem** (Yang, 1999): For each pair of disjoint effective divisors $[z_1^+, \ldots, z_{n_+}^+], [z_1^-, \ldots, z_{n_-}^-]$ there exists a unique solution of (TAUBES), and hence a unique (up to gauge) solution of (BOG1), (BOG2).

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- Moduli space of vortices: $\mathrm{M}_{n_+,n_-} \equiv (\mathbb{C}^{n_+} \times \mathbb{C}^{n_-}) \backslash \Delta_{n_+,n_-}$

Solving the (1,1) Taubes equation (numerically)

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta(z-\varepsilon) - \delta(z+\varepsilon)\right)$$

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- Solve with b.c. $\widehat{h}(\infty) = 0$

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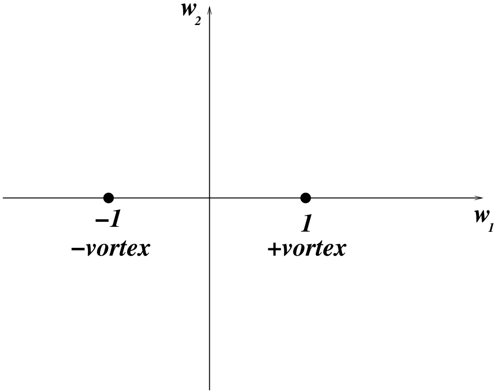

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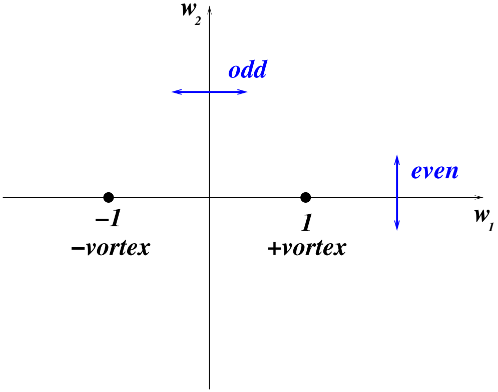

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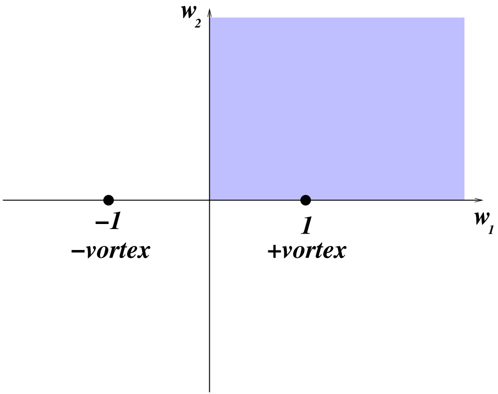

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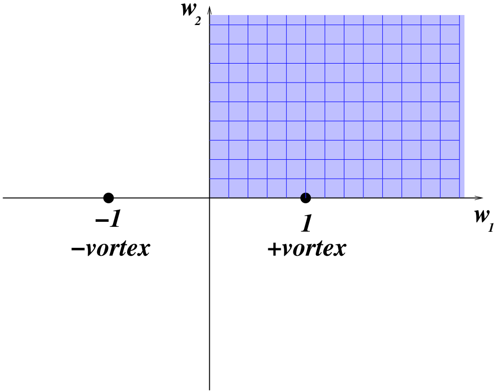

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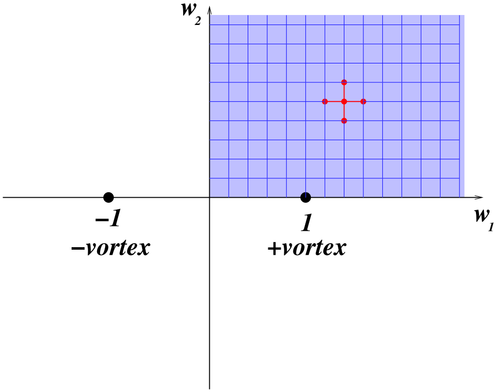

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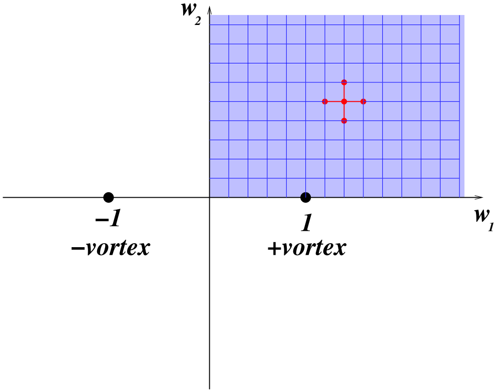
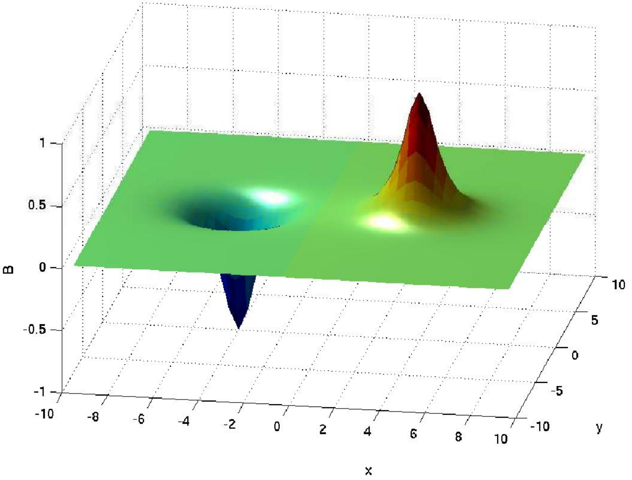


- $F(\widehat{h_{ij}}) = 0$, solve with Newton-Raphson

(1,1) vortices



$\varepsilon = 4$

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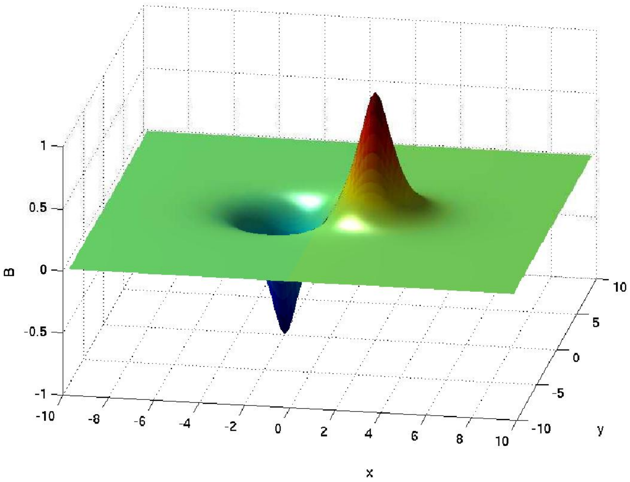
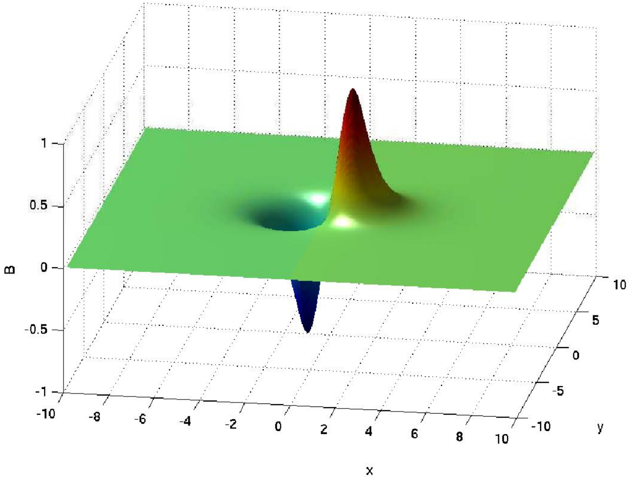


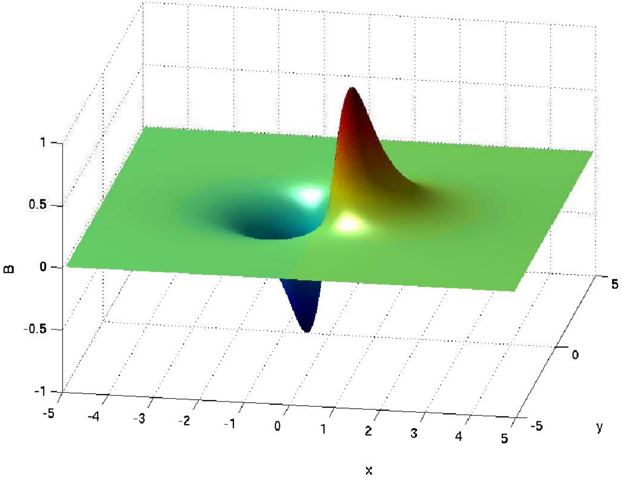
$\varepsilon = 2$

(1,1) vortices



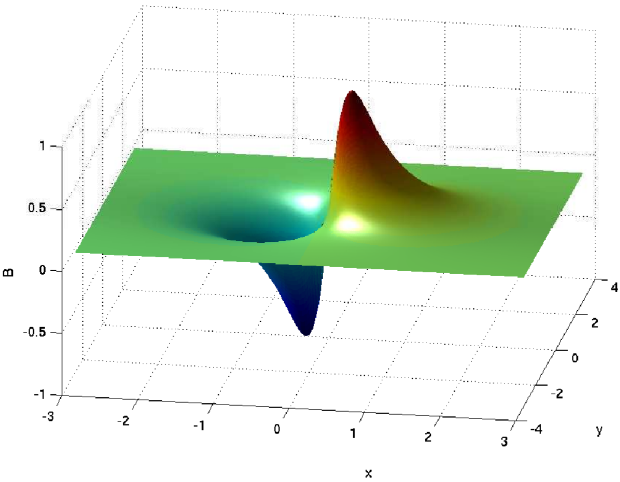
$\varepsilon = 1$

(1,1) vortices



$\varepsilon = 0.5$

(1,1) vortices



$\varepsilon = 0.3$

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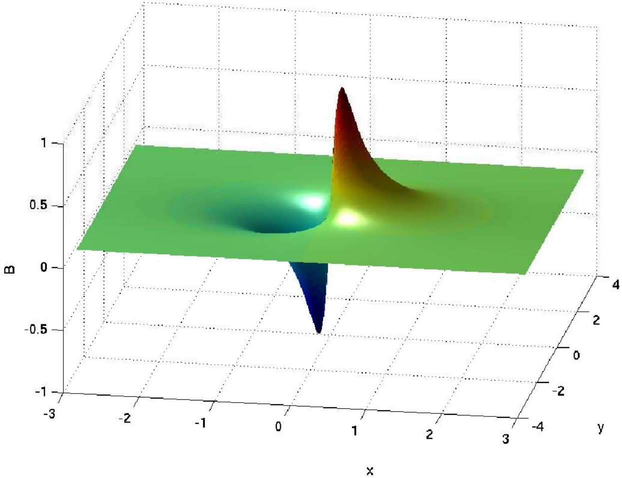
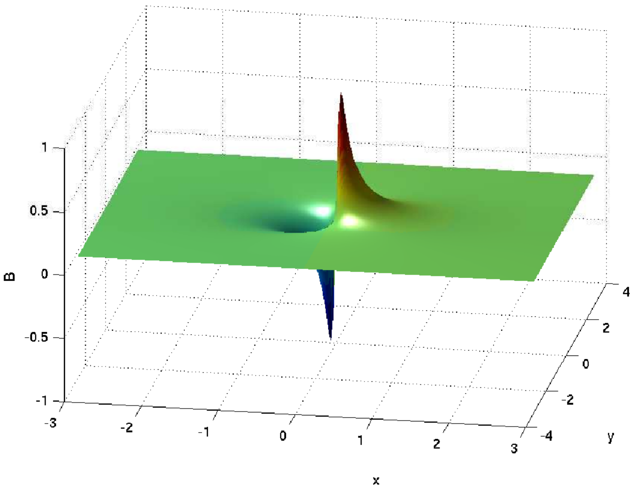


$\varepsilon = 0.15$

(1,1) vortices



$\varepsilon = 0.06$

Vortex dynamics

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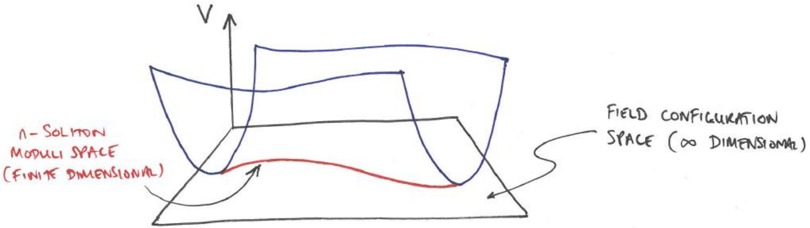

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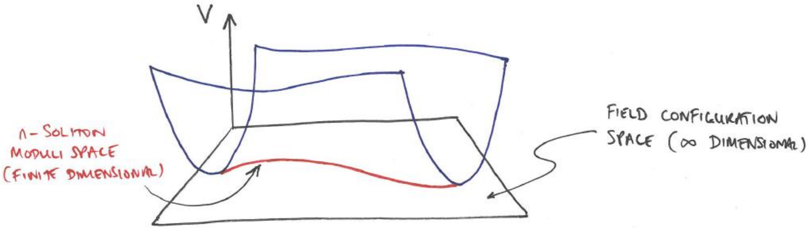

- Adiabatic approximation: assume $(\mathbf{n}(t), A(t)) \in M_{n_+,n_-}$ for all time

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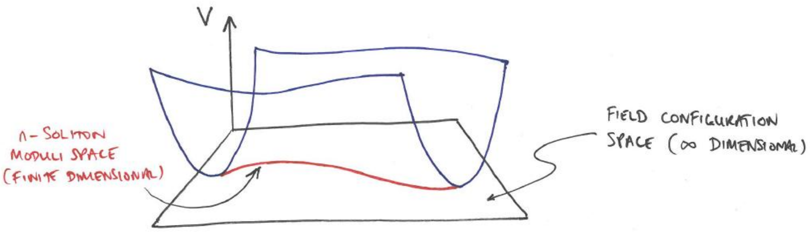

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$$S = \int \left(T - 2\pi(n_+ + n_-)\right) dt$$

- **Geodesic** motion in M_{n_+,n_-} w.r.t. the L^2 metric.

Strachan-Samols localization formula

- Consider a curve in M_{n_+,n_-} along which all vortex positions $z_r^{\pm}(t)$ remain distinct

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$$T = \pi \left\{ \sum_r |\dot{z}_r|^2 + \sum_{r,s} \frac{\partial b_s}{\partial z_r} \dot{z}_r \dot{\bar{z}}_s \right\}$$

where sums over all (anti)vortex positions and, in a nbhd of $z_s^{\pm}$,

$$h = \pm \left\{ \log |z - z_s^{\pm}|^2 + a_s + \frac{1}{2} \bar{b}_s (z - z_s^{\pm}) + \frac{1}{2} b_s (\bar{z} - \bar{z}_s^{\pm}) + \cdots \right\}$$

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where sums over all (anti)vortex positions and, in a nbhd of $z_s^\pm$,

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- T real $\Longrightarrow$ g hermitian, in fact kähler

Strachan-Samols localization formula

- Consider a curve in M_{n_+,n_-} along which all vortex positions $z_r^\pm(t)$ remain distinct
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- Can compute g if we know $b_r(z_1^+, \ldots, z_{n_-}^-)$

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- $M_{1,1} = (\mathbb{C} \times \mathbb{C}) \backslash \Delta = \mathbb{C}_{com} \times \mathbb{C}^{\times}$

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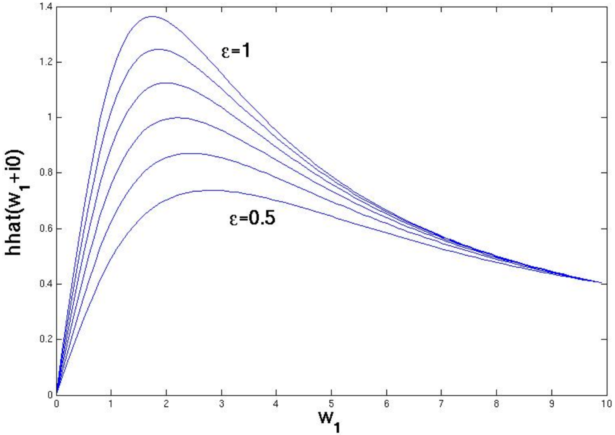

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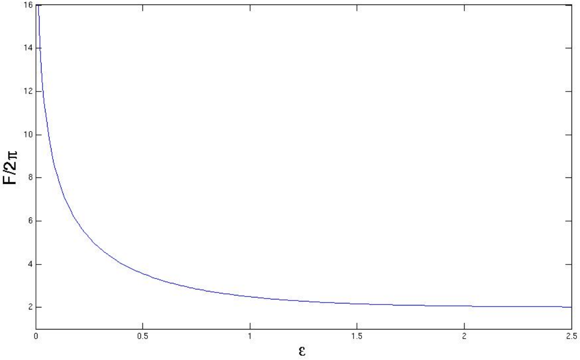
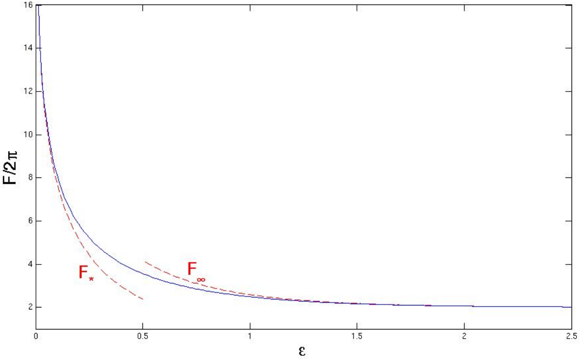


$$F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d(\varepsilon b(\varepsilon))}{d\varepsilon} \right)$$

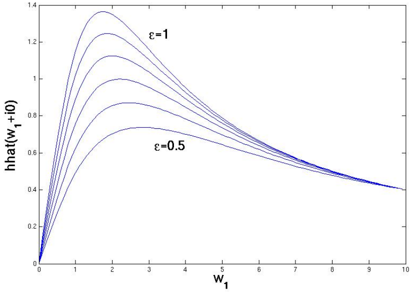
The metric on $M_{1,1}$: conjectured asymptotics



$$F_*(\varepsilon) = 2\pi\left(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)\right) \sim -8\pi \log \varepsilon$$

$$F_\infty(\varepsilon) = 2\pi\left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon)\right)$$

Self similarity as $\varepsilon \longrightarrow 0$



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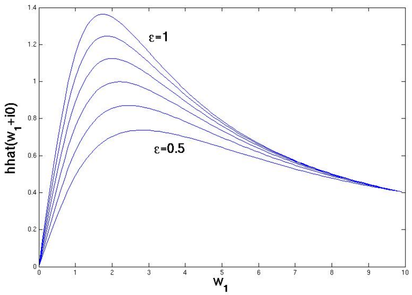

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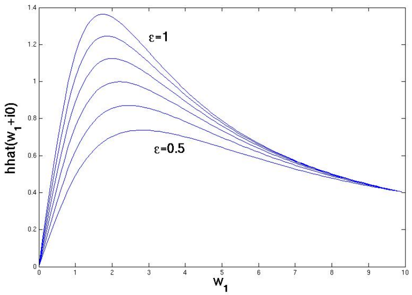

- Suggests $\widehat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w)$ for small ε, where f_* is fixed?
- Define $f_\varepsilon(z) := \varepsilon^{-1}\widehat{h}_\varepsilon(\varepsilon^{-1}z)$

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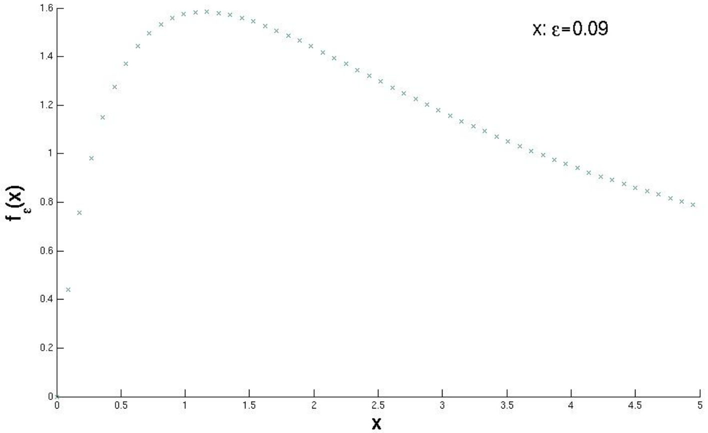

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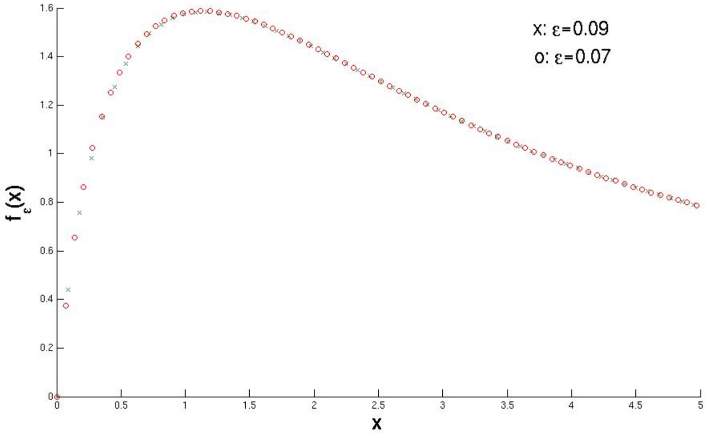

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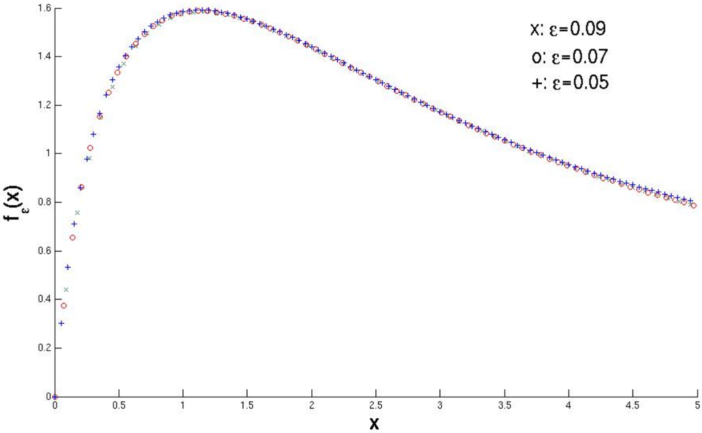

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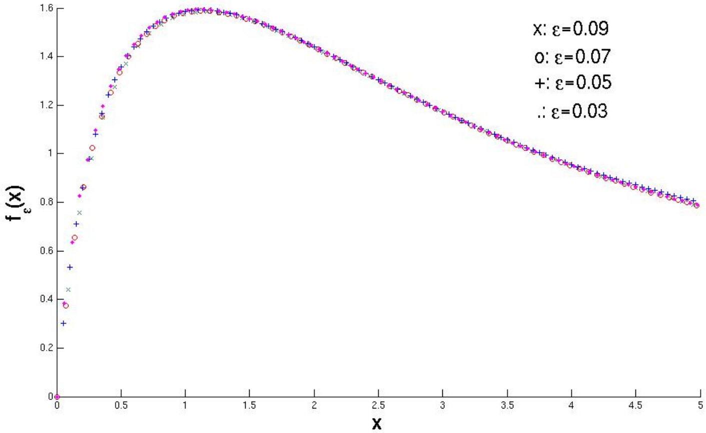

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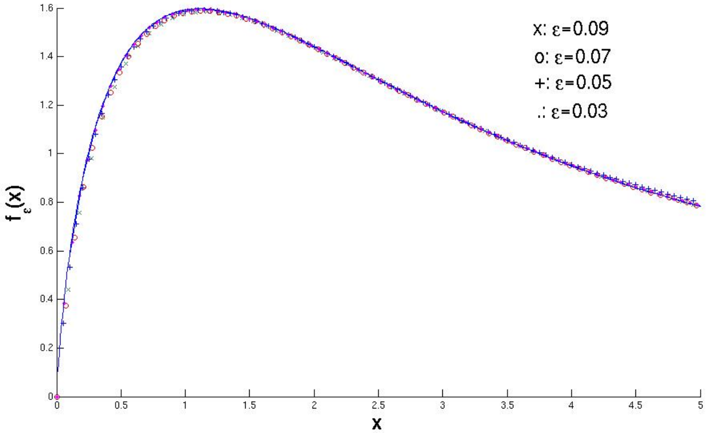

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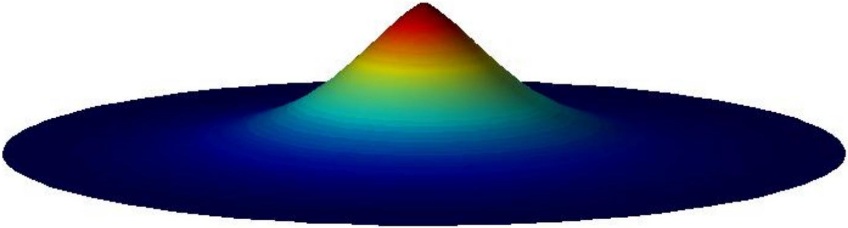

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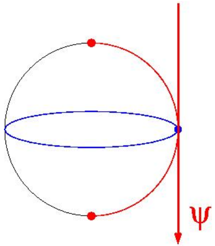
- $n_+ = 1$ vortex asymptotically indistinguishable from solution of *linearization* of model about vacuum $\mathbf{n} = (1, 0, 0)$



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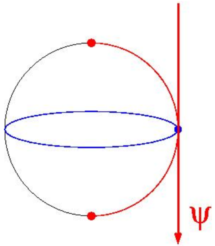

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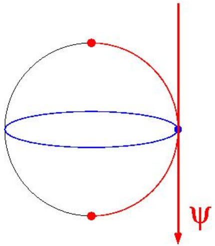

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in presence of **sources**:

$$\begin{aligned} \kappa &= q\delta(x) && \text{scalar monopole } q \\ (j^0, \mathbf{j}) &= (0, -q\mathbf{k}\times\nabla\delta(x)) && \text{magnetic dipole } q\mathbf{k} \end{aligned}$$

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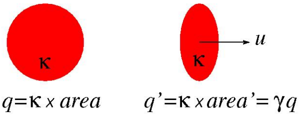

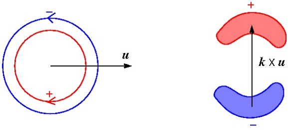

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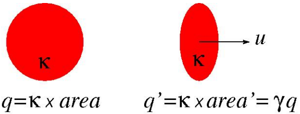

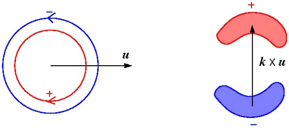

$$L = \pi(|\dot{\mathbf{x}}_1|^2 + |\dot{\mathbf{x}}_2|^2) \mp \frac{q^2}{4\pi} K_0(|\mathbf{x}_1 - \mathbf{x}_2|)|\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2|^2 \qquad \begin{cases} VV \\ V\overline{V} \end{cases}$$

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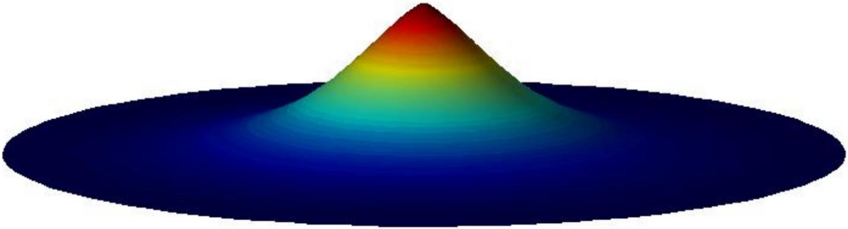
Asymptotically negatively curved



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    - Conjecture

$$Vol(M_{n,m}(S^2)) = \frac{(2\pi)^{n+m}}{n!m!}(Vol(S^2) - \pi(n-m))^n(Vol(S^2) + \pi(n-m))^m$$