$$\mathcal{L} = \frac{1}{2}\partial_\mu\bar\varphi\,\partial^\mu\bar\varphi - \frac{1}{4}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)(\partial^\mu\omega^\nu - \partial^\nu\omega^\mu) + \frac{1}{2}\omega_\mu\omega^\mu - \beta\,\omega^\mu B_\mu$$

$$\bar\varphi : \underset{dt^2 - g}{\mathbb{R}\times M^n} \longrightarrow \underset{h}{N^n} \qquad \omega \in \Omega^1(M)$$

~~Static problem~~

Static problem: $\bar\varphi(t,x) = \varphi(x)$, $\omega = f(x)\,dt$ $\qquad \varphi : M \to N$, $f : M \to \mathbb{R}$

Stationary pt of $S = \int_{\mathbb{R}\times M}\mathcal{L} \iff$

$$\boxed{\begin{aligned} &\tau(\varphi) + \beta * (df \wedge \Xi_\varphi) = 0 \quad &-(1)\ -(*)\\ &(\Delta+1)f = \beta * \varphi^*\Omega \quad &-(2)\end{aligned}}$$

where Ω = volume form on (N,h)

$\tau(\varphi)$ = tension field $= \mathrm{Pr}_{**}\,\Delta\varphi \qquad (N \subset \mathbb{R}^p)$

$\in \Gamma(\varphi^* TN)$

$\Xi_\varphi \in \Omega^{n-1}(M) \otimes \varphi^* TN$

$$h(X, \Xi_\varphi(A_1,\dots,A_{n-1})) := \Omega(X, d\varphi(A_1), \dots, d\varphi(A_{n-1}))$$

FACT (1),(2) equivalent to constrained variational problem

Extremize $E_\omega(\varphi,f) = \frac{1}{2}\|d\varphi\|^2 - \frac{1}{2}\|df\|^2 + \frac{1}{2}\|f\|^2$

subject to constraint $(\Delta+1)f = *\varphi^*\Omega$

N.B.: φ uniquely determines f so $E_\omega = E_\omega(\varphi)$ really.

~~Compare~~

Compare with $E_6(\varphi) = \frac{1}{2}\|d\varphi\|^2 + \frac{1}{2}\beta^2\|\varphi^*\Omega\|^2$

Euler-Lagrange $\tau(\varphi) + \beta * (d(*\varphi^*\Omega)\wedge \Xi_\varphi) = 0$

Energy bounds (M compact)

E_ω

$\langle 1, (\Delta+1)f\rangle = \beta\langle 1, *\varphi^*\Omega\rangle$

$\|1\|\,\|f\| \geq \langle 1, f\rangle = \beta B\,\mathrm{Vol}(N)$

$E_\omega(\varphi) \geq \frac{1}{2}\|f\|^2 \geq \frac{1}{2}\beta^2\left(\frac{\mathrm{Vol}(N)^2}{\mathrm{Vol}(M)}\right)B^2$

$M = \mathbb{R}^3$??

N.B.

$\beta^2\|\varphi^*\omega\|^2 \quad \|(\Delta+1)f\|^2 \geq \|df\|^2 + \|f\|$

So $E_\omega(\varphi) \leq E_6(\varphi)$

E_6

$\|1\|\,\|\varphi^*\Omega\| \geq \langle 1, *\varphi^*\Omega\rangle = B\,\mathrm{Vol}(N)$

$E_6(\varphi) \geq \frac{\beta^2}{2}\|\varphi^*\Omega\|^2 \geq \frac{1}{2}\beta^2\left(\frac{\mathrm{Vol}(N)^2}{\mathrm{Vol}(M)}\right)B^2$

$M = \mathbb{R}^3$

$E_6(\varphi) = \frac{1}{2}\int_{\mathbb{R}^3}\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \beta^2\lambda_1^2\lambda_2^2\lambda_3^2$

$\geq 2\sqrt{\beta}\int_{\mathbb{R}^3}|\lambda_1\lambda_2\lambda_3| \geq 2\sqrt{\beta}\,\mathrm{Vol}(N)\,B$

Existence of solutions?

E_ω	E_6
???	direct method of calculus of variations works

Derrick scaling?

E_ω

$$\tilde{E}_\omega(\varphi) = \frac{1}{2} R^{n-2} \|d\varphi\|_g^2 + \frac{1}{2} R^{n-2} \|d\tilde{f}\|_g^2 + \frac{1}{2} R^n \|\tilde{f}\|_g^2$$

where $\left(\frac{1}{R^2}\Delta_g + 1\right)\tilde{f} = \frac{\beta}{R^n} *_g \varphi^*\Omega$

[cf $(\Delta_g + 1) f = \beta *_g \varphi^*\Omega$]

$$R\downarrow \not\equiv \beta\uparrow$$

E_6

$$\tilde{g} = R^2 g$$

$$\tilde{E}_6(\varphi) = \frac{1}{2} R^{n-2} \|d\varphi\|_g^2 + \frac{1}{2} R^{-n} \beta^2 \|\varphi^*\Omega\|_g^2$$

$$R \downarrow \equiv \beta \uparrow$$

~~Compact M "homogeneous" solutions:~~

$\cancel{\varphi^*\Omega = \lambda \, vol_g}$

$\cancel{\tau(\varphi) = 0}$

~~E_ω~~

$\cancel{(\Delta+1) f = \beta\lambda \Rightarrow f = \beta\lambda}$

$\cancel{\theta = 0}$ ✓

~~E_6~~

$\cancel{0 = 0}$ ✓

$$\tau(\varphi) + \beta * (df \wedge \Xi_\varphi) = 0$$

$$(\Delta + 1) f = \beta * \varphi^*\Omega$$

$$\tau(\varphi) + \beta^2 * (d(*\varphi^*\Omega) \wedge \Xi_\varphi) = 0$$

Continuation from $\beta = 0$ $\quad \tau(\varphi) = 0$ (φ harmonic) $[f = 0]$?

Problem: every harmonic map $\varphi: M^n \to S^{n \geq 3}$ is <u>unstable</u>

Maybe there's a stability transition as β gets large?

Manton: happens for standard Skyrme on $\mathbb{T}_L S^3 \to S^3$

2nd Variation of $E(\varphi)$

φ_t, $\varphi_0 = \varphi$

$\partial_t|_{t=0} \varphi_t = X \in \Gamma(\varphi^{-1}TN)$

$\mathrm{Hess}_\varphi(X,X) = \frac{d^2}{dt^2}\Big|_{t=0} E(\varphi_t)$ Assume $\frac{d}{dt}\Big|_{t=0} E(\varphi_t) = 0$ (φ a c.p.)

Stable $\Leftrightarrow$ $\mathrm{Hess}_\varphi(X,X) \geq 0$ $\forall X \in \Gamma(\varphi^{-1}TN)$

~~$\varphi = Id : N \to N$~~, $E(\varphi) = E_6(\varphi) = \frac{1}{2}\|d\varphi\|^2 + \frac{1}{2}\beta^2\|\varphi^*\Omega\|^2$

~~$X \in \Gamma(TN) \leftarrow$ a vector field on $N = M$~~

$\varphi : Id : N \to N$ $X \in \Gamma(TN) \leftarrow$ vector field on N

$$\mathrm{Hess}^{(6)}_{Id}(X,X) = \langle X, J_{Id}X\rangle + \beta^2 \underbrace{\|d(\iota_X\Omega)\|^2}_{\|\mathrm{div}\, X\|^2}$$

$$J_{Id}X = \#\Delta \flat X - 2\,\mathrm{Ric}(X)$$

Bochner-Yano formula $\langle X, J_{Id}X\rangle = \frac{1}{2}\|\mathcal{L}_X h\|^2 - \|\mathrm{div}\,X\|^2$

$\Rightarrow \mathrm{Hess}^{(6)}_{Id}(X,X) = \frac{1}{2}\|\mathcal{L}_X h\|^2 + (\beta^2 - 1)\|\mathrm{div}\,X\|^2$

$\Rightarrow Id : N \to N$ is certainly E_6 stable if $\beta^2 \geq 1$

More refined calculation for $N = S^n \Rightarrow$ Id is E_6 stable if $\beta^2 \geq \frac{n-2}{n}$

$$E(\varphi) = E_\omega(\varphi)$$

$\varphi = Id : N \to N$, $(\Delta + 1)f = \beta * \varphi^*\Omega = \beta \Rightarrow f = \beta$

Variation φ_t, $\partial_t|_{t=0}\varphi_t = X \in \Gamma(TN)$ $\partial_t|_{t=0} f_t = \alpha_X \in C^\infty(N)$

$$(\Delta + 1)\alpha_X = \beta * d(Id^*\iota_X\Omega) = \beta\,\mathrm{div}\,X$$

$$\frac{d^2}{dt^2} E(\varphi_t)\Big|_{t=0} = \langle X, J_{Id} X\rangle + \langle \alpha_X, (\Delta+1)\alpha_X\rangle$$

$$\mathrm{Hess}^{\omega}_{Id}(X,X) = \underbrace{\langle X, J_{Id} X\rangle}\; + \underbrace{\|d\alpha_X\|^2 + \|\alpha_X\|^2}$$

~~Bochner-Yano~~ $\cancel{= \frac{1}{2}\|\mathcal{L}_X h\|^2 - \|\operatorname{div} X\|^2 + \|d\alpha_X\|^2 + \|\alpha_X\|^2}$

negative only on a finite dim subspace of $\Gamma(TN)$

certainly ≥ 0

> 0 unless $\operatorname{div} X = 0$

Bochner-Yano. $\underbrace{\frac{1}{2}\|\mathcal{L}_X h\|^2 - \|\operatorname{div} X\|^2}_{\geq 0 \text{ if } \operatorname{div} X = 0}$

$$(\Delta+1)\alpha_X^{(\beta)} = \beta \operatorname{div} X \quad \Rightarrow \quad \alpha_X^{(\beta)} = \beta\,\alpha_X^{(1)}$$

If $\langle X, J_{Id} X\rangle < 0$ then $\operatorname{div} X \neq 0 \Rightarrow \alpha_X^{(1)} \neq 0$

$\Rightarrow \mathrm{Hess}^{\omega}_{Id}(X,X) > 0$ for β suff large.

If $\operatorname{div} X = 0$ $\quad \mathrm{Hess}^{\omega}_{Id}(X,X) = \frac{1}{2}\|\mathcal{L}_X h\|^2 \geq 0$

— Looks like $\mathrm{Hess}^{\omega}_{Id}(X,X) \geq 0$ for all X. But we can't quite conclude this

Knowing $\mathrm{Hess}(X,X) \geq 0 \quad \forall X \in V$

$\mathrm{Hess}(Y,Y) \geq 0 \quad \forall Y \in W$ where $V \oplus W = \Gamma(TN)$

$\not\Rightarrow \mathrm{Hess}(X,X) \geq 0 \quad \forall X \in \Gamma(TN)$

Assume N is Einstein: $\rho = ch$

$\Rightarrow J_{Id} X = \#\Delta\flat X - 2cX$

Hodge decomposition. $X = \underset{\substack{\uparrow \\ \operatorname{div}=0}}{X_0} + \nabla l$ $\leftarrow$ gradient $\quad L^2 \perp$

$\#\Delta\flat$ preserves splitting. $\Rightarrow J_{Id}$ preserves splitting.

$$\begin{aligned}
\mathrm{Hess}^{\omega}_{Id}(X,X) &= \langle X_0, JX_0\rangle + \langle \nabla l, J\nabla l\rangle + \langle \alpha_{\nabla l}, (\Delta+1)\alpha_{\nabla l}\rangle \\
&= \tfrac{1}{2}\|\mathcal{L}_X h\|^2 + \langle dl, (\Delta - 2c)\, dl\rangle + \langle \alpha_{\nabla l}, \beta \operatorname{div} \nabla l\rangle \\
&= \tfrac{1}{2}\|\mathcal{L}_X h\|^2 + \langle \Delta l, (\Delta - 2c)\, l\rangle + \underbrace{\beta\langle \alpha_{\nabla l}, \operatorname{div}\nabla l\rangle}_{-\beta\langle \alpha_{\nabla l}, \Delta l\rangle}
\end{aligned}$$

Spectral decomposition $\quad \Delta f_n = \lambda_n f_n \quad \langle f_n, f_p \rangle = \delta_{np}$ (5)

$$l = \sum a_m f_m$$

$$(\Delta + 1)\alpha_{\nabla l} = \beta \operatorname{div} \nabla l = -\beta \Delta l$$

$$\alpha_{\nabla l} = -\beta \sum_n \frac{\lambda_m a_m}{1+\lambda_m} f_m$$

$$\mathrm{Hess}^{\omega}_{Id}(X,X) = \frac{1}{2}\|\mathcal{L}_X h\|^2 + \sum_m{}' \left\{ \lambda_m(\lambda_m - 2c) + \frac{\beta^2 \lambda_m^2}{1+\lambda_m} \right\} a_m^2$$

$$\geq \sum_{m=1}^{q} \left\{ \lambda_m(\lambda_m - 2c) + \frac{\beta^2 \lambda_m^2}{1+\lambda_m^2} \right\} a_m^2$$

where $q = \max\{m : \lambda_m < 2c\}$

$$\cancel{\frac{\lambda_m^2}{1+\lambda_m^2}\left\{\beta^2 + \frac{(\lambda_m - 2c)(1+\lambda_m^2)}{\lambda_m}\right\} \geq 0}$$

So if $\quad \beta^2 \geq \max\left\{ \dfrac{(2c-\lambda_m)(1+\lambda_m^2)}{\lambda_m} : m = 1, \dots, q \right\}$

then $\mathrm{Hess}^{\omega}_{Id}(X,X) \geq 0 \quad \forall X \qquad$ Stable

More refined calculation for $N = S^n \Rightarrow$

$$\beta^2 \geq \frac{(n-2)(n+1)}{n} = \frac{4}{3} \quad \text{for } n = 3$$