Knot solitons in two-component Ginzburg-Landau theory

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TCGL theory

- $\Psi : M \to \mathbb{C}^2$, $M =$ Riemannian mfd (e.g. $\mathbb{R}^3$)
  $A \in \Omega^1(M)$, em gauge potential
  $B = \mathrm{d}A \in \Omega^2(M)$, magnetic field
  $\mathrm{d}_A\Psi = \mathrm{d}\Psi - iA\Psi$
- GL energy

$$E = \frac{1}{2}\|\mathrm{d}_A\Psi\|^2 + \frac{1}{2}\|B\|^2 + \int_M U(\Psi)$$

where $\|C\|^2 = \langle C, C\rangle$, $\langle B, C\rangle = \int_M B \wedge *C$

- Field equations

$$\begin{aligned} \delta_A \mathrm{d}_A\Psi + \frac{\partial U}{\partial \Psi^\dagger} &= 0 \\ -\delta B &= J \\ J &= -\mathrm{Im}(\Psi^\dagger \mathrm{d}_A\Psi) = \text{supercurrent} \end{aligned}$$

Babaev-Faddeev-Niemi decomposition

- Gauge invariant fields

$$\begin{aligned} \rho &= |\Psi| : M \to \mathbb{R}_+ \\ \varphi &= [\Psi_1, \Psi_2] : M \to \mathbb{C}P^1 \equiv S^2 \\ C &= J/\rho^2 \in \Omega^1(M) \end{aligned}$$

- GL energy is

$$E = \frac{1}{8}\|\rho \mathrm{d}\varphi\|^2 + \frac{1}{2}\|\mathrm{d}C + \frac{1}{2}\varphi^*\omega\|^2 + \frac{1}{2}\|\mathrm{d}\rho\|^2 + \frac{1}{2}\|\rho C\|^2 + \int_M U(\rho,\varphi)$$

$\omega =$ kähler form on S^2.

- Truncation 1: assume U enforces $\rho \approx \text{constant} = 1$ WLOG

$$E \approx \frac{1}{8}\|\mathrm{d}\varphi\|^2 + \frac{1}{8}\|\varphi^*\omega\|^2 + \frac{1}{2}\langle \varphi^*\omega, \mathrm{d}C\rangle + \frac{1}{2}\|\mathrm{d}C\|^2 + \frac{1}{2}\|C\|^2$$

- Truncation 2: C is massive, so assume $C \approx 0$

$$\begin{aligned} E &\approx \frac{1}{8}\|d\varphi\|^2 + \frac{1}{8}\|\varphi^*\omega\|^2 \\ &= \frac{1}{8}\int_{\mathbb{R}^3} \sum_i \left| d\varphi \frac{\partial}{\partial x_i} \right|^2 + \sum_{i<j} \left((\varphi \times d\varphi \frac{\partial}{\partial x_i}) \cdot d\varphi \frac{\partial}{\partial x_j} \right)^2 \\ &= \frac{1}{8}\int_{\mathbb{R}^3} \sum_i \left| \frac{\partial \varphi}{\partial x_i} \right|^2 + \sum_{i<j} \left(\varphi \cdot (\frac{\partial \varphi}{\partial x_i} \times \frac{\partial \varphi}{\partial x_j}) \right)^2 \\ &= \frac{1}{4} E_{FS}(\varphi) \end{aligned}$$

- E_{FS} certainly has knot solitons. BFN conjecture TCGL does too (Phys.Rev.B65:100512,2002).

Why should we care?

- Very influential: $>$100 citations
- If true, gives Faddeev field $\varphi : \mathbb{R}^3 \to S^2$ concrete physical interpretation
- Qualitatively similar arguments have been made (for nonabelian gauge theories) by Niemi and others to address questions of fundamental importance: e.g. quark confinement, quantum gravity

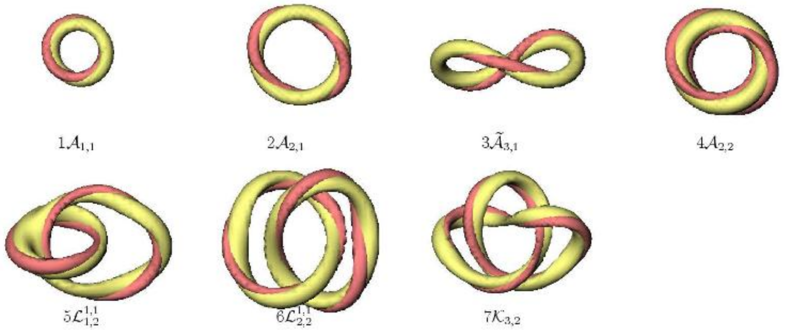
Knot solitons in the Faddeev-Skyrme model

$$E_{FS}(\varphi) = \frac{1}{2}\|\mathrm{d}\varphi\|^2 + \frac{1}{2}\|\varphi^*\omega\|^2$$

- $\varphi : \mathbb{R}^3 \to S^2$, b.c. $\varphi(\infty) = (0,0,1)$
- Hopf degree $Q = \frac{1}{16\pi^2}\int_{\mathbb{R}^3} A \wedge \mathrm{d}A$ where $\varphi^*\omega = \mathrm{d}A$
- $\varphi^{-1}(\text{reg. value}) =$ oriented link in $\mathbb{R}^3$.
  $Q =$ linking number of different regular preimages.
- Numerics: for some Q, $\varphi^{-1}(0,0,-1)$ is knotted.
- Vakulenko-Kapitanskii bound: $E_{FS}(\varphi) \geq c|Q|^{\frac{3}{4}}$. The power is sharp.

Knot solitons in the Faddeev-Skyrme model

Studied numerically by Battye and Sutcliffe, Hietarinta and Salo, and many others



[Sutcliffe, Proc. Roy. Soc. Lond. **A463** (2007) 3001]

Testing the BFN conjecture

- E_{FS} has knot solitons, but do they survive unfreezing of $C = 0$ and $\rho = 1$?
- Test in most favourable case, hard confining potential, e.g.

$$U = \lambda(1 - |\Psi|^2)^2, \qquad \lambda \to \infty$$

so truncation 1 holds exactly,

$$E = \frac{1}{4}E_{FS}(\varphi) + \frac{1}{2}\|\mathrm{d}C\|^2 + \frac{1}{2}\|C\|^2 + \frac{1}{2}\langle \varphi^*\omega, \mathrm{d}C\rangle$$

Supercurrent coupled Faddeev-Skyrme (SCFS) model

- Truncation 2 amounts to neglecting φ-C coupling

Testing the BFN conjecture

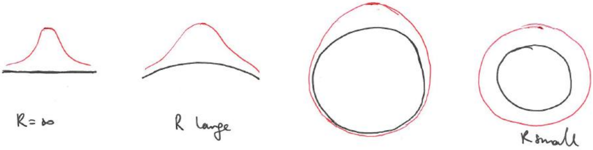
- Definitive test: introduce parameter $0 \leq \alpha \leq 1$

$$E_\alpha = \frac{1}{4}E_{FS}(\varphi) + E_{KG}(C) + \frac{\alpha}{2}\langle \varphi^* \omega, \mathrm{d}C \rangle$$

  Know this has knot solitons when $\alpha = 0$. Do any of them continue to $\alpha = 1$?
- On $M = \mathbb{R}^3$ need numerics (even E_{FS} not analytically tractable)
- On $M = S^3_R$ can answer question exactly ($Q = 1$, $0 < R < 2$).

Homogenization of solitons on shrinking domains

- Generic phenomenon: topological solitons on compact domains undergo a phase transition as the domain shrinks – they **gain** symmetry



- Happens for Skyrme model, vector meson Skyrme model, Faddeev-Skyrme model on S^3, and abelian Higgs model on any compact Riemann surface
- **Theorem** (Ward, JMS-Svensson, Isobe) The Hopf map

$$\pi : \mathbb{C}^2 \supset S^3 \to S^2 \equiv \mathbb{C}P^1, \qquad (z_1, z_2) \mapsto [z_1, z_2]$$

is a stable critical point of $E_{FS}(\varphi)$ if and only if $0 < R < 2$.

First variation of E_α

$$E_\alpha(\varphi, C) = \frac{1}{8}\|\mathrm{d}\varphi\|^2 + \frac{1}{8}\|\varphi^*\omega\|^2 + \frac{1}{2}\|\mathrm{d}C\|^2 + \frac{1}{2}\|C\|^2 + \frac{\alpha}{2}\langle \mathrm{d}C, \varphi^*\omega\rangle$$

- Makes sense for $\varphi : M \to N$, $C \in \Omega^1(M)$, M Riemannian, N Kähler.
- Demand
$$\left.\frac{dE_\alpha(\varphi_s, C_s)}{ds}\right|_{s=0} = 0$$
for all smooth variations (φ_s, C_s)

$$\begin{aligned} \delta(\mathrm{d}C + \frac{\alpha}{2}\varphi^*\omega) + C &= 0, \\ \tau(\varphi) - \frac{2}{\alpha} J\mathrm{d}\varphi\sharp[C + (1-\alpha^2)\delta\mathrm{d}C] &= 0. \end{aligned}$$

First variation of E_α

$$\begin{aligned} \delta(\mathrm{d}C + \frac{\alpha}{2}\varphi^*\omega) + C &= 0, \\ \tau(\varphi) - \frac{2}{\alpha} J \mathrm{d}\varphi \sharp [C + (1-\alpha^2)\delta \mathrm{d}C] &= 0. \end{aligned}$$

- **Fact:** For fixed $\varphi : M \to N$, there can be at most one C s.t. (φ, C) is critical.
  - Assume (φ, C') also a solution. Then $C'' = C - C'$ solves $\delta \mathrm{d}C'' + C'' = 0$
  - Hence $0 = \langle C'', \delta \mathrm{d}C'' + C'' \rangle = \|\mathrm{d}C''\|^2 + \|C''\|^2$
- Hopf map has a unique continuation for all $\alpha \in [0, 1]$

$$(\varphi, C) = \left(\pi, \frac{2\alpha}{4 + R^2}\sigma_3\right)$$

where $\pi : G \to G/K$, $G = SU(2)$, $K = \{\mathrm{diag}(\lambda, \bar{\lambda}) : \lambda \in U(1)\}$ and $\pi : x \to xK$.

- Great. But is it stable at $\alpha = 1$?
- Need second variation formula. . .

Second variation of E_α

- Given smooth variation (φ_s, C_s) of critical point (φ, C), define variation section $(X, Y) \in \Gamma\mathscr{E} = \Gamma(\varphi^{-1}TN \oplus T^*M)$

$$X = \frac{\partial \varphi_s}{\partial s}\bigg|_{s=0}, \qquad Y = \frac{\partial C_s}{\partial s}\bigg|_{s=0}$$

- Then

$$\frac{d^2 E_\alpha(\varphi_s, C_s)}{ds^2}\bigg|_{s=0} = \langle (X, Y), \mathscr{H}_\alpha(X, Y)\rangle$$

where $\mathscr{H}_\alpha$ is a self-adjoint, 2nd order linear diff-op $\Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$

- Spectrum of $\mathscr{H}_\alpha$ determines stability of (φ, C): unstable if $\mathscr{H}_\alpha$ has negative eigenvalue(s).

Second variation of E_α

- For our energy E_α

$$\mathscr{H}_\alpha \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\mathscr{J} + \frac{1}{4}\mathscr{L} + \alpha\mathscr{C} & \frac{1}{2}\alpha\mathscr{A} \\ \frac{1}{2}\alpha\mathscr{B} & \delta d + 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

  where

$$\begin{aligned} \mathscr{A} &: \Omega^1(M) \to \Gamma(\varphi^{-1}TN) & \mathscr{A} &: Y \mapsto -Jd\varphi\sharp\delta d Y \\ \mathscr{B} &: \Gamma(\varphi^{-1}TN) \to \Omega^1(M) & \mathscr{B} &: X \mapsto \delta d(\varphi^* \iota_X \omega) \\ \mathscr{C} &: \Gamma(\varphi^{-1}TN) \to \Gamma(\varphi^{-1}TN) & \mathscr{C} &: X \mapsto -\frac{1}{2} J \nabla^{\varphi}_{\sharp\delta d C} X. \end{aligned}$$

- For $(\varphi, C) =$ continued Hopf map, can compute spectrum of $\mathscr{H}_\alpha$ exactly. $\mathscr{H}_1$ has a negative eigenvalue of index 10, for all $0 < R < 2$. So supercurrent coupling **destabilizes** the unit hopfion, at least on S^3.
- Back to $M = \mathbb{R}^3$...

Energy bounds

- Recall VK bound $E_{FS}(\varphi) \geq c_0|Q|^{\frac{3}{4}}$
- Have similar bound for E_α

$$
\begin{aligned}
E_\alpha &= \frac{1}{8}\|\mathrm{d}\varphi\|^2 + \frac{1}{8}(1-\alpha)\|\varphi^*\omega\|^2 + \frac{1}{2}(1-\alpha)\|\mathrm{d}C\|^2 + \\
&\quad \frac{\alpha}{2}\|\mathrm{d}C + \frac{1}{2}\varphi^*\omega\|^2 + \frac{1}{2}\|C\|^2 \\
&\geq \frac{1}{8}\|\mathrm{d}\varphi\|^2 + \frac{1}{8}(1-\alpha)\|\varphi^*\omega\|^2 \\
&= \frac{1}{8}\sqrt{1-\alpha}\left(\|\mathrm{d}\hat{\varphi}\|^2 + \|\hat{\varphi}^*\omega\|^2\right) \\
&\geq \frac{1}{4}\sqrt{1-\alpha}c_0|Q^{\frac{3}{4}}| \qquad (1)
\end{aligned}
$$

where $\hat{\varphi} = \varphi \circ \mathscr{D}_{\sqrt{1-\alpha}}$ and $\mathscr{D}_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathscr{D}_\lambda(x) = \lambda x$

- Expect E_α to have smooth minimizer in every homotopy class for $0 < \alpha < 1$.
- Bound trivial at $\alpha = 1$. Is this sharp?

Energy bounds

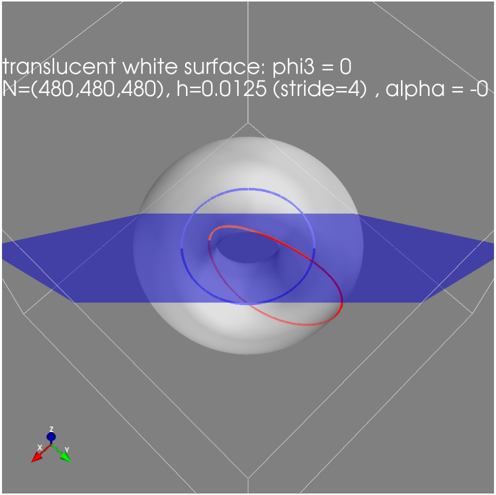
- Yes:
$$\inf\{E_1(\varphi, C)\ :\ Q(\varphi) = n\} = 0 \quad \text{for all } n \in \mathbb{Z}$$

- **Proof:** $\exists\ \varphi$ with $\varphi = (0,0,1)$ outside $\overline{B}$.
$\exists\ C$ s.t. $\varphi^*\omega = -2\mathrm{d}C$ ($H^2(M) = 0$).
Can assume $C = 0$ outside $\overline{B}$ ($H^1(M \backslash \overline{B}) = 0$).
$$E_1(\varphi \circ \mathscr{D}_\lambda, \mathscr{D}_\lambda^* C) = \frac{1}{2\lambda}\|\mathrm{d}\varphi\|^2 + 0 + \frac{1}{2\lambda}\|C\|^2$$

Numerics (joint work with Juha Jäykkä)

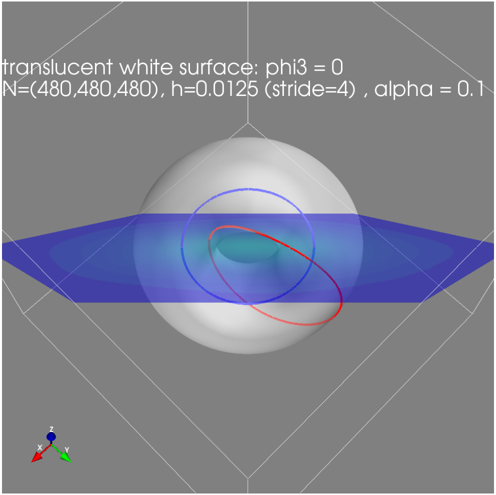
- Write down lattice approximant for $E_\alpha(\varphi, C)$
- Fix Q. We've done $Q = 1, 2$. $Q = 3$ still working.
- Starting at $\alpha = 0$, minimize E_α using gradient-based minimization scheme (e.g. conjugate gradient method).
- At $\alpha = 0$, get usual charge Q knot soliton, with $C = 0$.
- Increment α slightly. Use old minimizer as new initial guess. Minimize E_α
- Get curve of minimizers, parametrized by $\alpha \in [0, 1]$
- Knot solitons shrink and disappear as $\alpha \to 1$

Numerics (joint work with Juha Jäykkä)



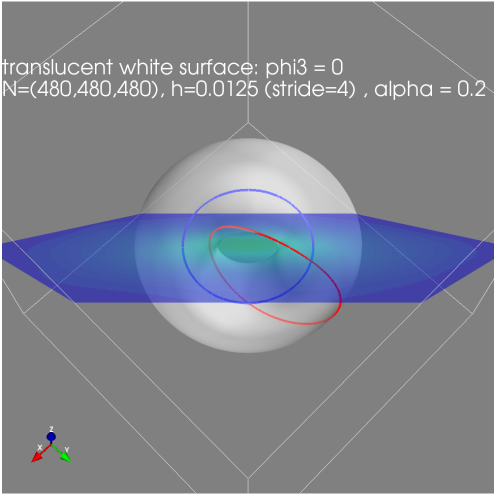
$Q = 1$, $\alpha = 0$

Numerics (joint work with Juha Jäykkä)



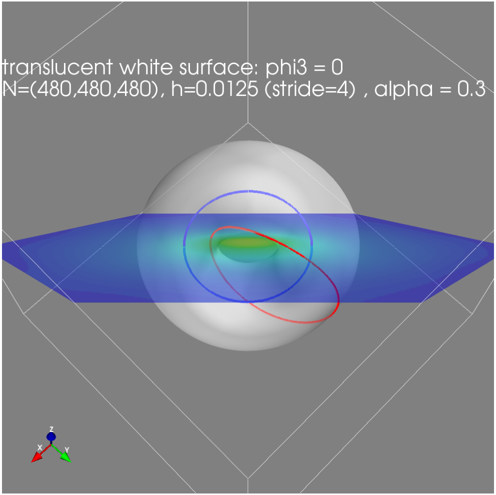
$Q = 1$, $\alpha = 0.1$

Numerics (joint work with Juha Jäykkä)



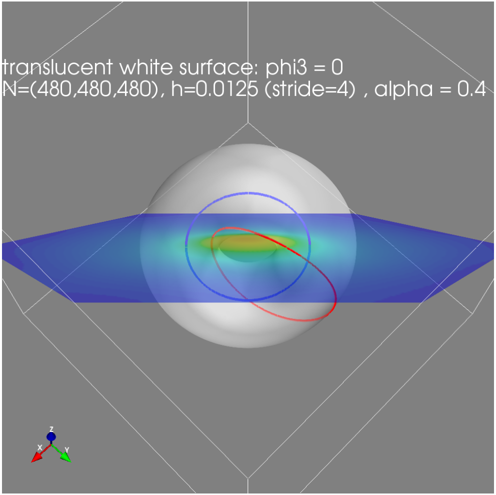
$Q = 1$, $\alpha = 0.2$

Numerics (joint work with Juha Jäykkä)



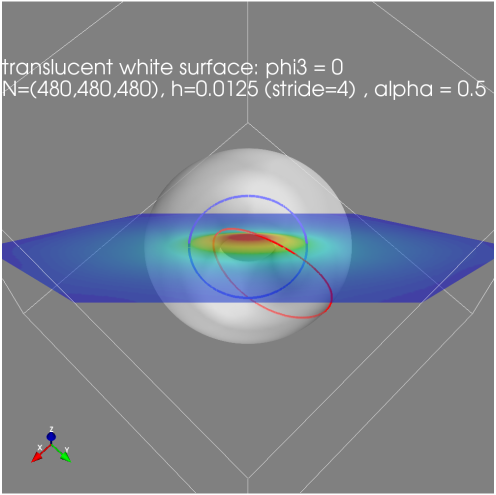
$Q = 1$, $\alpha = 0.3$

Numerics (joint work with Juha Jäykkä)



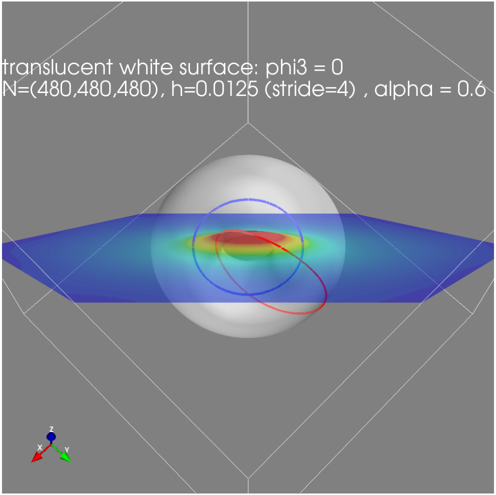
$Q = 1$, $\alpha = 0.4$

Numerics (joint work with Juha Jäykkä)



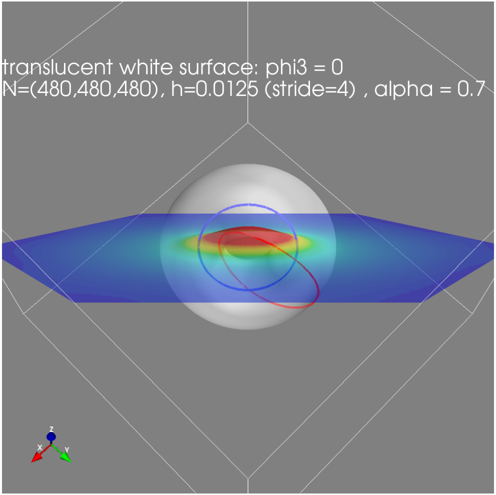
$Q = 1$, $\alpha = 0.5$

Numerics (joint work with Juha Jäykkä)



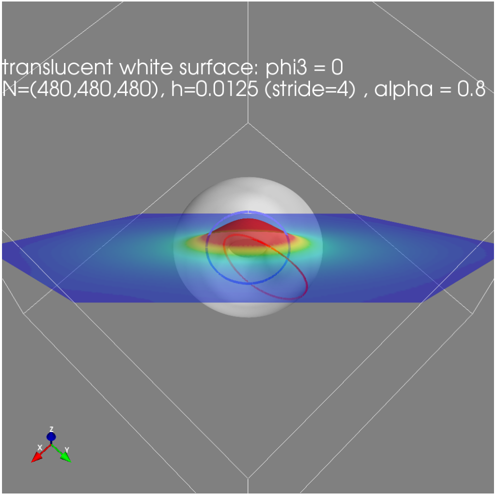
$Q = 1$, $\alpha = 0.6$

Numerics (joint work with Juha Jäykkä)



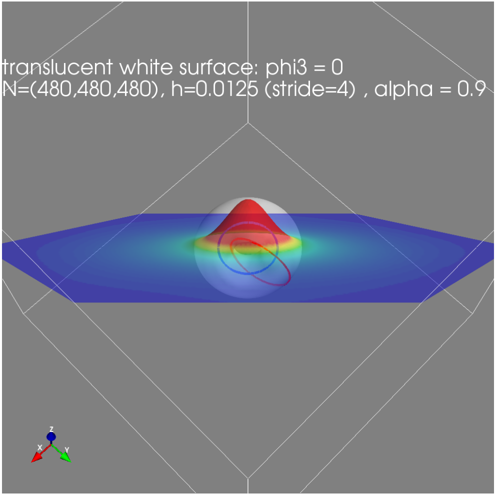
$Q = 1$, $\alpha = 0.7$

Numerics (joint work with Juha Jäykkä)



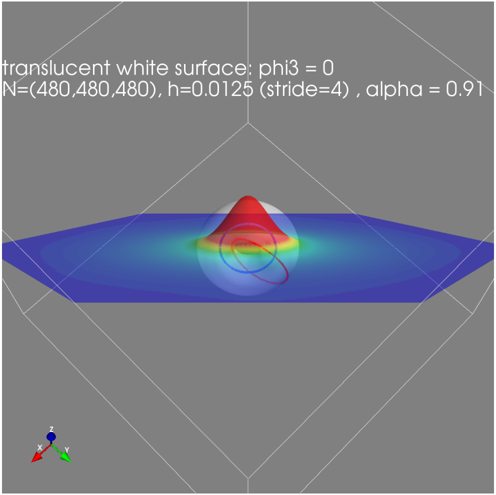
$Q = 1$, $\alpha = 0.8$

Numerics (joint work with Juha Jäykkä)



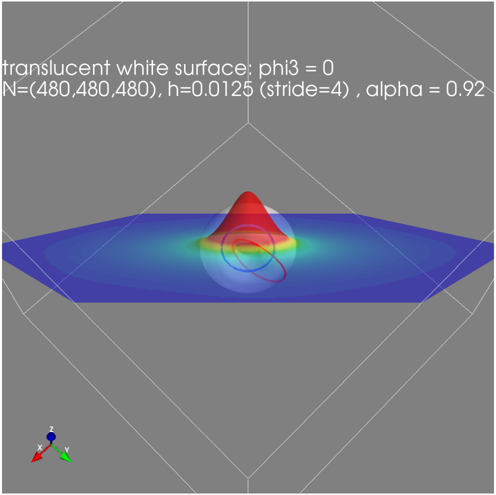
$Q = 1$, $\alpha = 0.9$

Numerics (joint work with Juha Jäykkä)



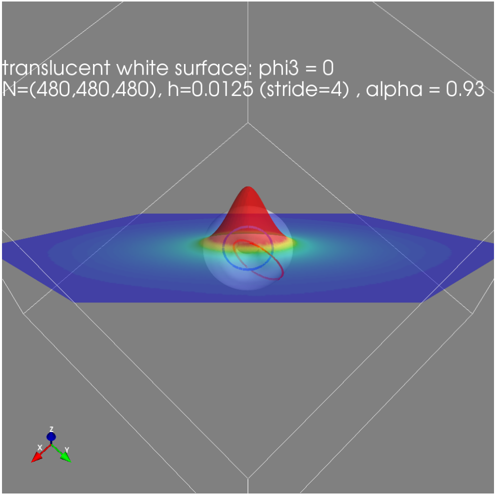
$Q = 1$, $\alpha = 0.91$

Numerics (joint work with Juha Jäykkä)



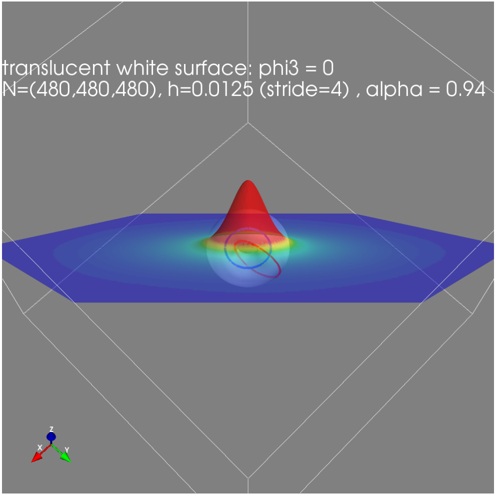
$Q = 1$, $\alpha = 0.92$

Numerics (joint work with Juha Jäykkä)



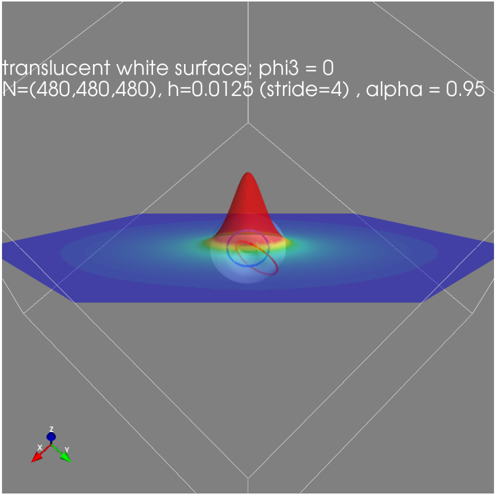
$Q = 1$, $\alpha = 0.93$

Numerics (joint work with Juha Jäykkä)



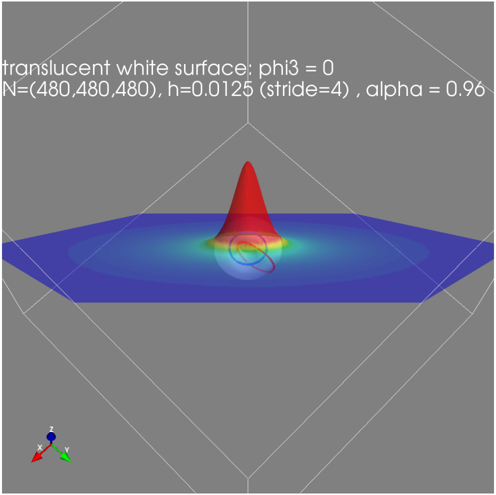
$Q = 1$, $\alpha = 0.94$

Numerics (joint work with Juha Jäykkä)



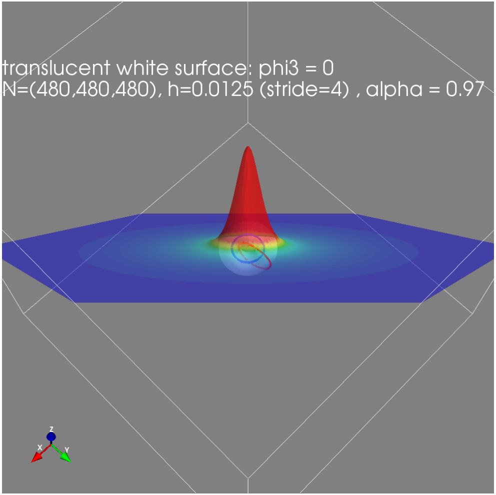
$Q = 1, \alpha = 0.95$

Numerics (joint work with Juha Jäykkä)



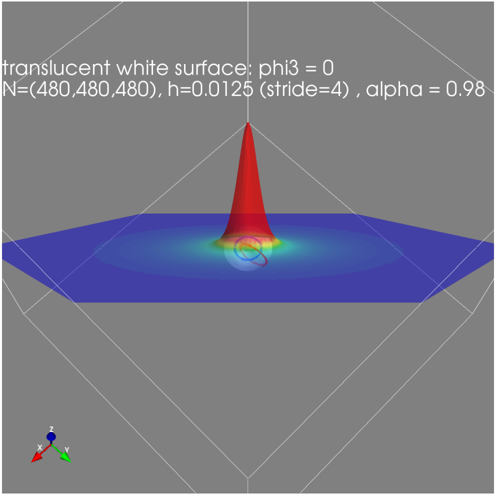
$Q = 1$, $\alpha = 0.96$

Numerics (joint work with Juha Jäykkä)



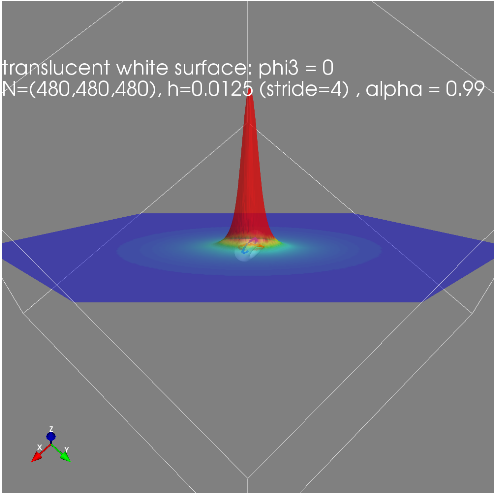
$Q = 1, \alpha = 0.97$

Numerics (joint work with Juha Jäykkä)



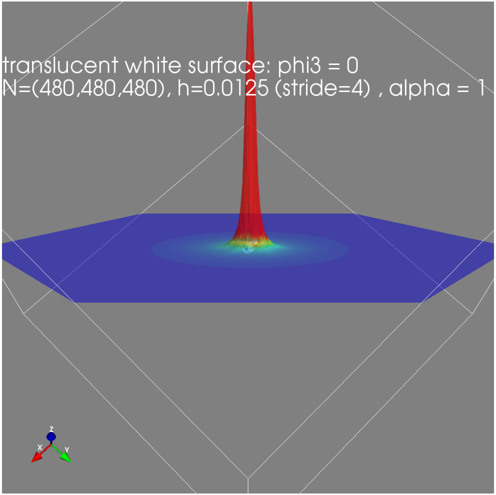
$Q = 1$, $\alpha = 0.98$

Numerics (joint work with Juha Jäykkä)



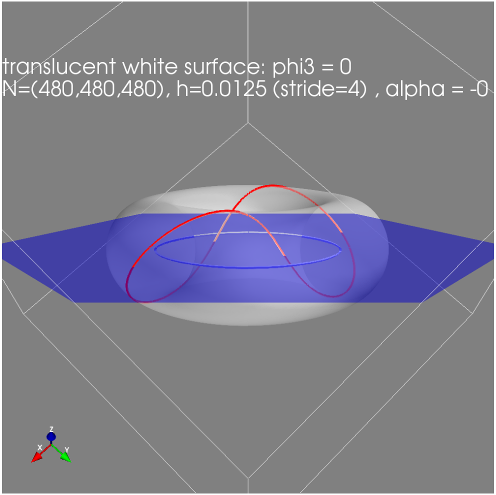
$Q = 1$, $\alpha = 0.99$

Numerics (joint work with Juha Jäykkä)



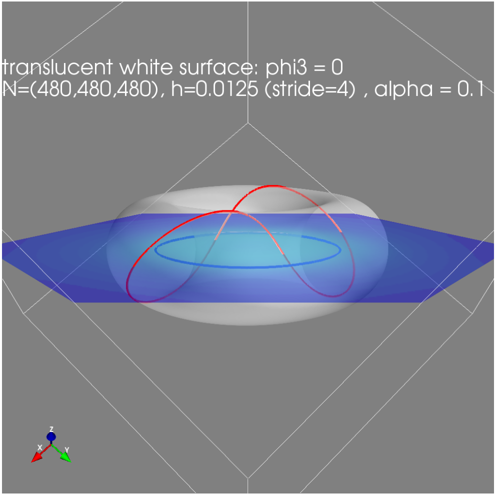
$Q = 1$, $\alpha = 1$

Numerics (joint work with Juha Jäykkä)



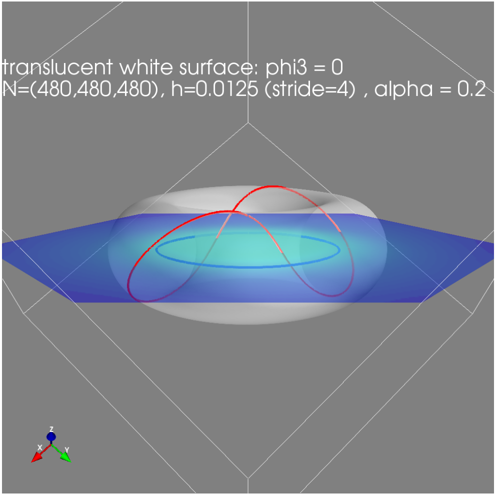
$Q = 2$, $\alpha = 0$

Numerics (joint work with Juha Jäykkä)



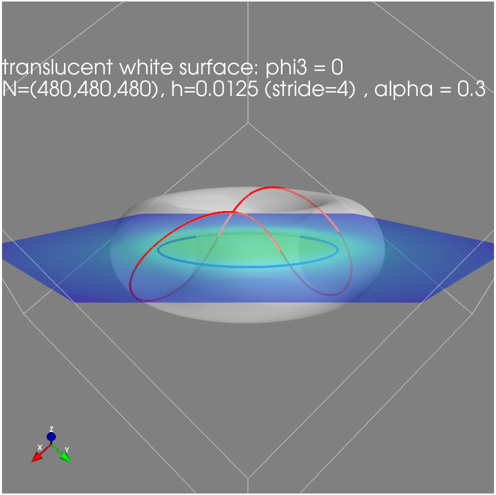
$Q = 2$, $\alpha = 0.1$

Numerics (joint work with Juha Jäykkä)



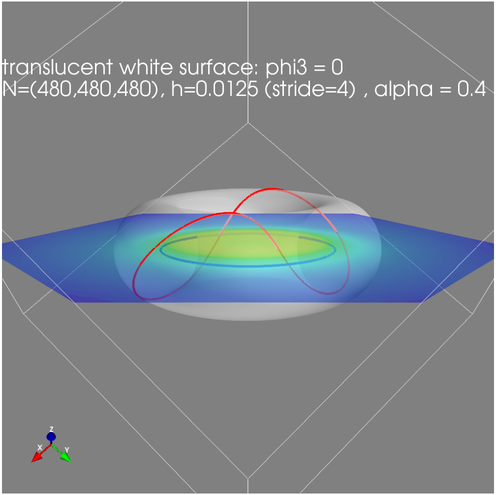
$Q = 2$, $\alpha = 0.2$

Numerics (joint work with Juha Jäykkä)



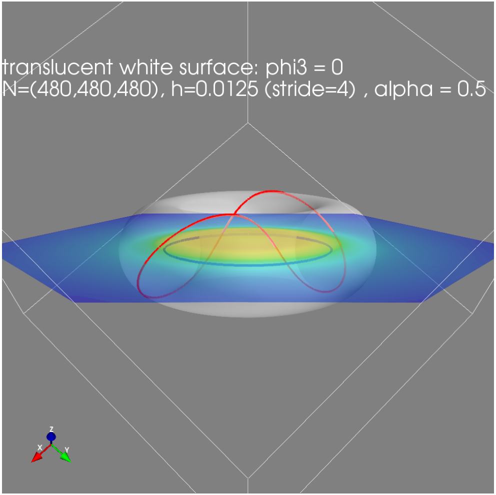
$Q = 2$, $\alpha = 0.3$

Numerics (joint work with Juha Jäykkä)



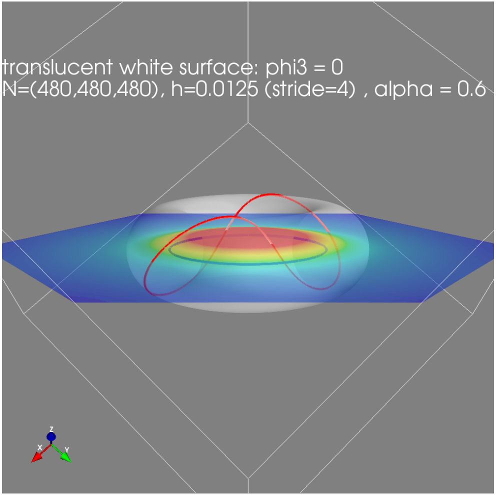
$Q = 2$, $\alpha = 0.4$

Numerics (joint work with Juha Jäykkä)



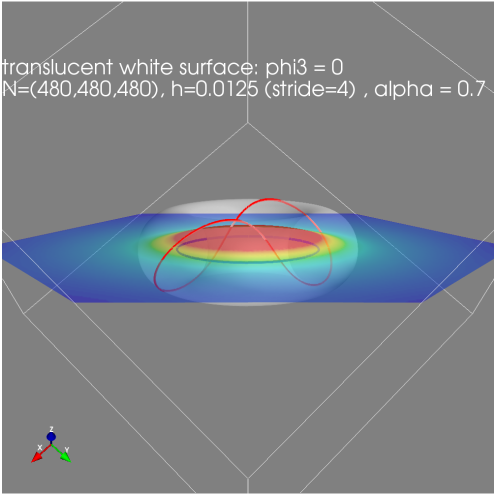
$Q = 2$, $\alpha = 0.5$

Numerics (joint work with Juha Jäykkä)



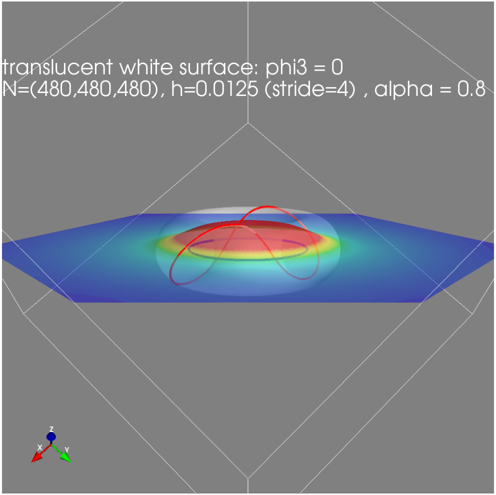
$Q = 2$, $\alpha = 0.6$

Numerics (joint work with Juha Jäykkä)



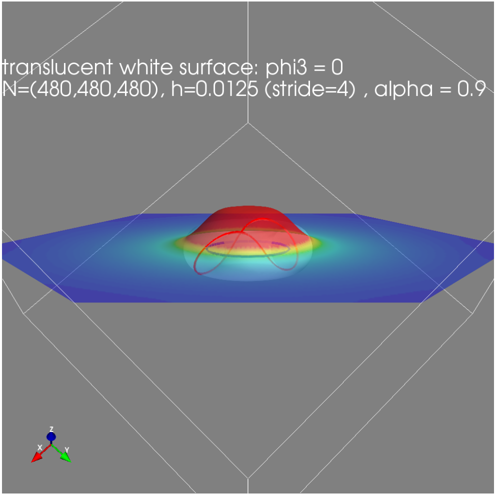
$Q = 2$, $\alpha = 0.7$

Numerics (joint work with Juha Jäykkä)



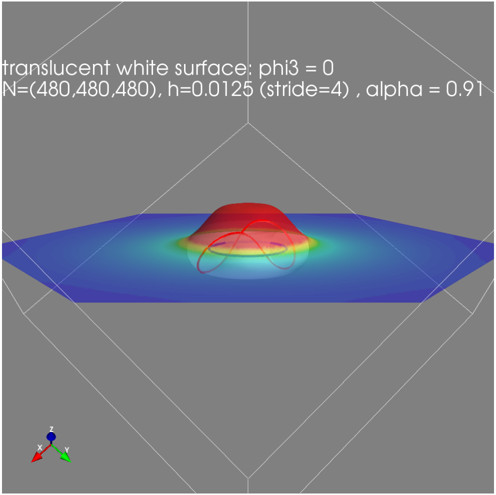
$Q = 2$, $\alpha = 0.8$

Numerics (joint work with Juha Jäykkä)



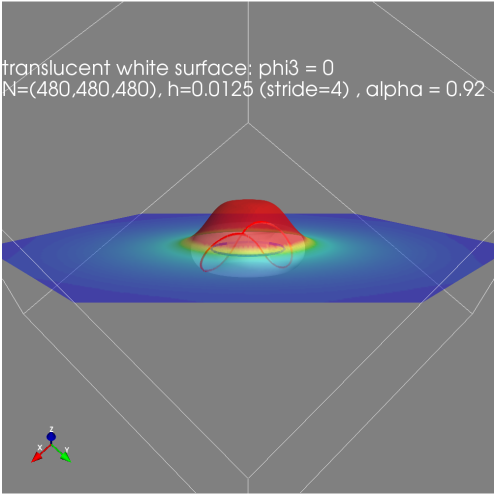
$Q = 2$, $\alpha = 0.9$

Numerics (joint work with Juha Jäykkä)



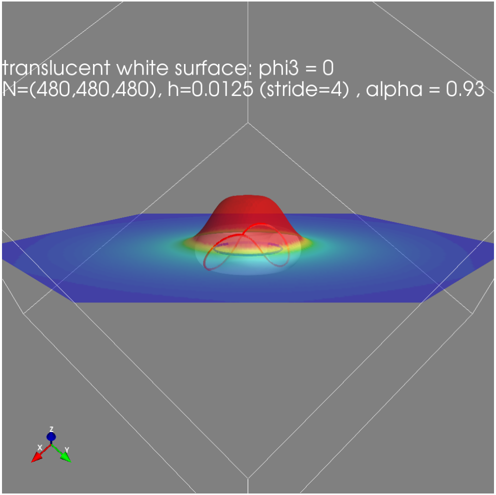
$Q = 2$, $\alpha = 0.91$

Numerics (joint work with Juha Jäykkä)



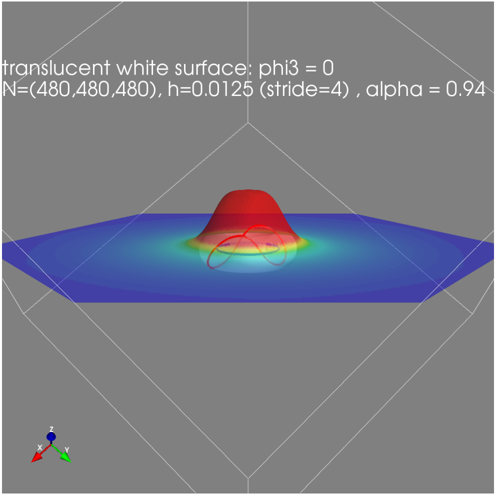
$Q = 2$, $\alpha = 0.92$

Numerics (joint work with Juha Jäykkä)



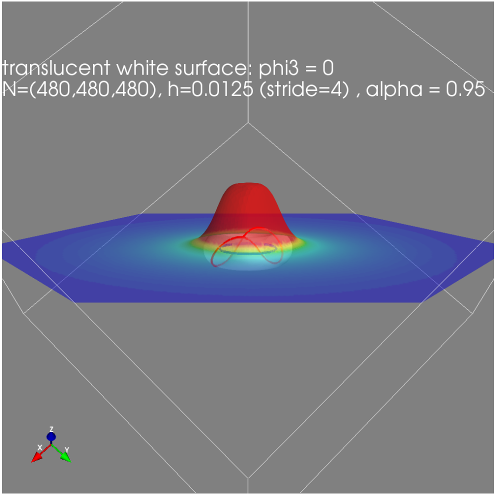
$Q = 2$, $\alpha = 0.93$

Numerics (joint work with Juha Jäykkä)



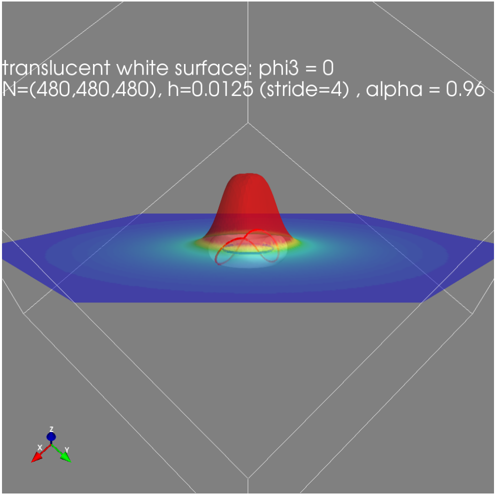
$Q = 2$, $\alpha = 0.94$

Numerics (joint work with Juha Jäykkä)



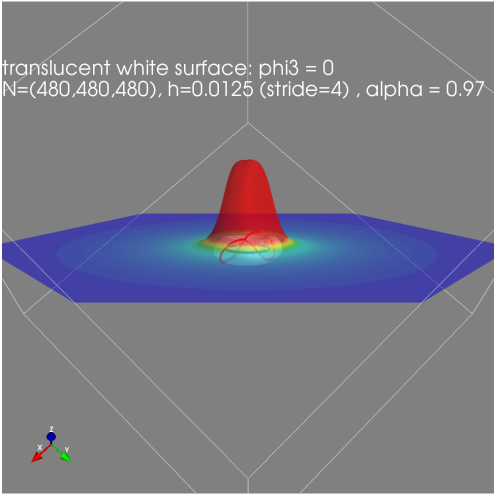
$Q = 2$, $\alpha = 0.95$

Numerics (joint work with Juha Jäykkä)



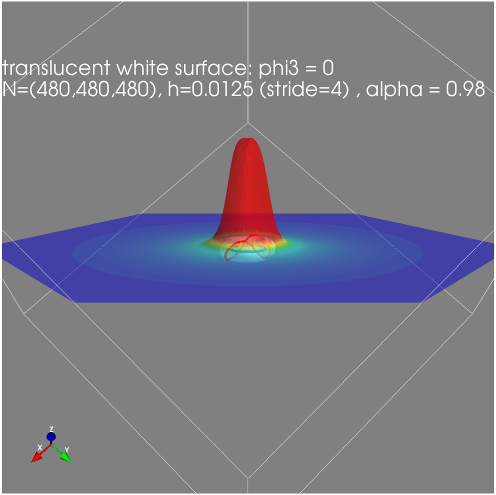
$Q = 2$, $\alpha = 0.96$

Numerics (joint work with Juha Jäykkä)



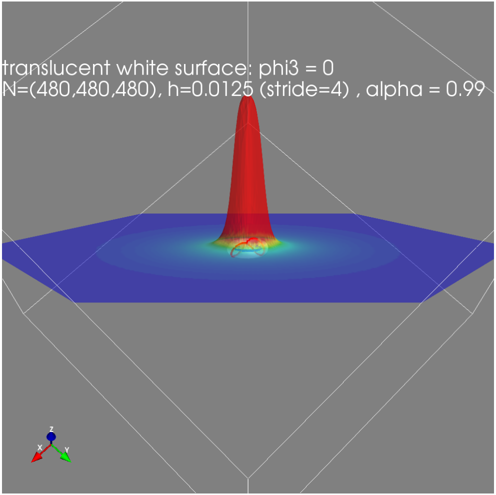
$Q = 2$, $\alpha = 0.97$

Numerics (joint work with Juha Jäykkä)



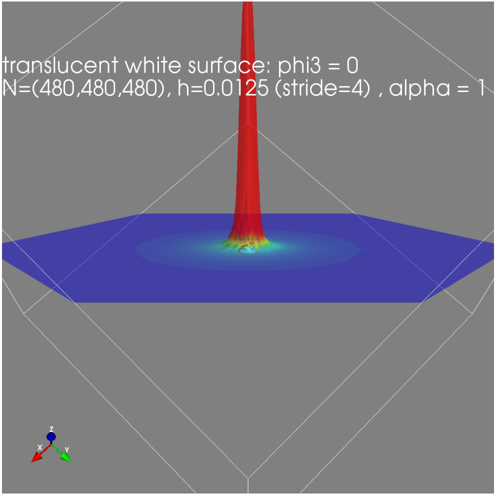
$Q = 2$, $\alpha = 0.98$

Numerics (joint work with Juha Jäykkä)



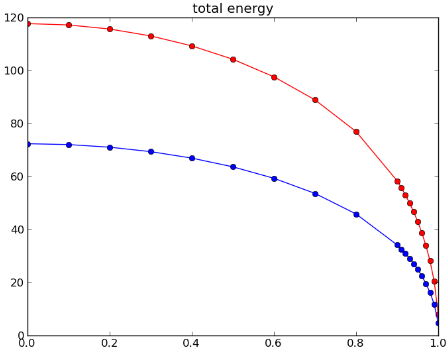
$Q = 2$, $\alpha = 0.99$

Numerics (joint work with Juha Jäykkä)



$Q = 2$, $\alpha = 1$

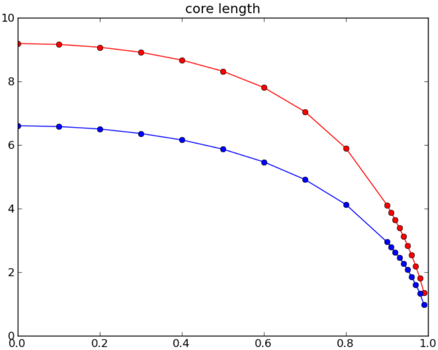
Energy versus α



$Q = 1$, $Q = 2$

Numerics (joint work with Juha Jäykkä)

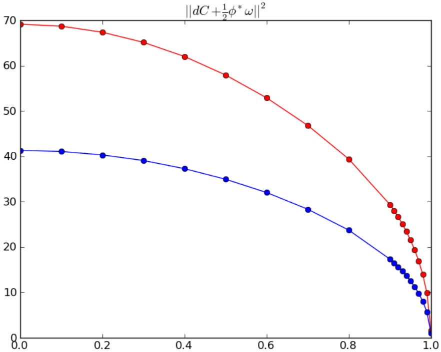
Core length versus α



$Q = 1$, $Q = 2$

Numerics (joint work with Juha Jäykkä)

L^2 norm of $\mathrm{d}C + \frac{1}{2}\varphi^*\omega$



$Q = 1$, $Q = 2$

Concluding remarks

- We have tested not only the BFN conjecture, but the reasoning underlying it: directly tested the assumption that “turning on the couplings” doesn’t destroy the knot solitons
- Even if we keep ρ frozen (so topology is preserved) we find that coupling to supercurrent alone destabilizes knot solitons
- Numerics on $\mathbb{R}^3$ supplemented by exact analysis on S^3_R
- Conclude BFN conjecture extremely unlikely to be true