Geometry and dynamics of vortex solitons

Martin Speight
University of Leeds

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http://www.maths.leeds.ac.uk/∼speight/talks
Chapter 7 “Topological Solitons” Manton and Sutcliffe

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  - antisolitons
- Examples:
  - $d = 1$ kinks
  - $d = 2$ vortices, lumps
  - $d = 3$ monopoles, skyrmions, hopfions
  - $d = 4$ instantons

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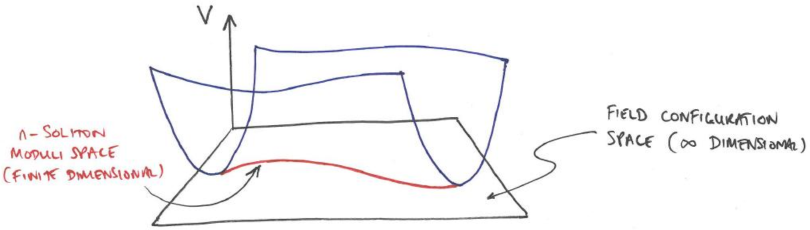
  Equality iff ϕ satisfies system of nonlinear *first order* PDEs (Bogomol'nyi equations)
- Moduli space of charge n solutions of Bogomol'nyi equation M_n a smooth manifold, $\dim M_n = n \dim M_1$

Geodesic approximation (method of Manton)



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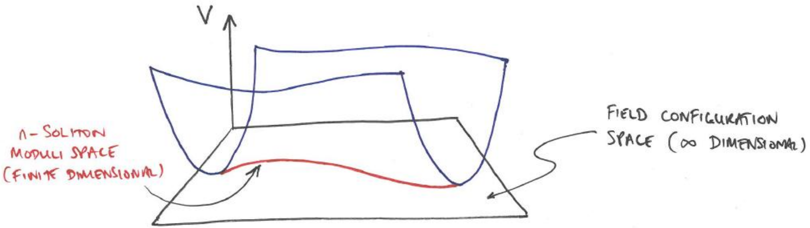

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  - Quantum: Schrödinger eqn on M_n
  - Statistical mechanics ($n \to \infty$)

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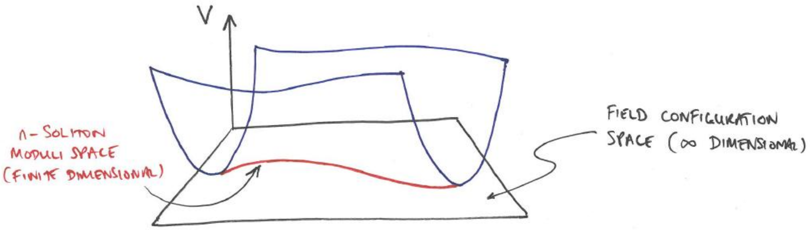

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  - Statistical mechanics ($n \to \infty$)
- We'll look in detail at case of *vortices*.

Abelian Higgs model

- Scalar field $\phi : \mathbb{R}^{(2,1)} \to \mathbb{C}$, gauge field $A \in \Omega^1(\mathbb{R}^{(2,1)})$

$$\mathcal{L} = \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{8}(1 - |\phi|^2)^2$$

$D\phi = \mathrm{d}\phi - iA\phi$, $F = \mathrm{d}A$

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- Gauge invariant: $\phi \mapsto e^{i\chi}\phi$, $A \mapsto A + \mathrm{d}\chi$
- Field equations:

$$\begin{aligned} D_\mu D^\mu \phi + \frac{\lambda}{2}(1 - |\phi|^2)\phi &= 0 \\ -\partial_\mu F^{\mu\nu} &= J^\nu \\ J_\nu &= \frac{i}{2}(\overline{\phi} D_\nu \phi - \phi \overline{D_\nu \phi}) \end{aligned}$$

Abelian Higgs model

- Temporal gauge: $A_0 = 0$
- Energy: $E = T + V$

$$
\begin{aligned}
T &= \frac{1}{2}\int_{\mathbb{R}^2} |\partial_0\phi|^2 + (\partial_0 A_1)^2 + (\partial_0 A_2)^2 \\
V &= \frac{1}{2}\int_{\mathbb{R}^2} |D_i\phi|^2 + F_{12}^2 + \frac{\lambda}{4}(1-|\phi|^2)^2
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- Evolution equations for $\phi(t), \mathbf{A}(t)$

$$\begin{aligned} \partial_0^2\phi &= D^2\phi - \frac{\lambda}{2}(1-|\phi|^2)\phi \\ \partial_0^2\mathbf{A} &= \nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A}) - \mathbf{J} \\ \nabla\cdot(\partial_0\mathbf{A}) &= \frac{i}{2}(\overline{\phi}\partial_0\phi - \phi\partial_0\overline{\phi}) \end{aligned}$$

Flux quantization

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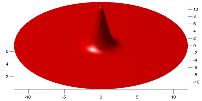
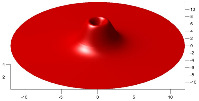
$$\int_{\mathbb{R}^2} F_{12} = \int_{\mathbb{R}^2} \mathrm{d}A = \int_{S^1_\infty} A = \chi(2\pi) - \chi(0) = 2\pi n$$

Vortices

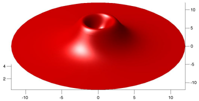
- $\phi = f(r)e^{in\theta}$, $A = a(r)\mathrm{d}\theta$



$n = 1$



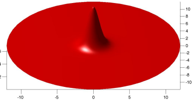
$n = 2$



$n = 3$

Vortices

- $\phi = f(r)e^{in\theta}$, $A = a(r)\mathrm{d}\theta$



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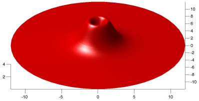

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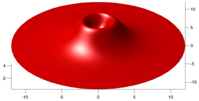

$n = 3$

- $n > 1$ unstable if $\lambda > 1$
- Cross section through cosmic string, or magnetic flux tube in superconductor

Bogomol'nyi bound ($\lambda = 1$)

- For a static field with winding n,

$$\begin{aligned} 0 &\leq \frac{1}{2}\int_{\mathbb{R}^2} |D_1\phi + iD_2\phi|^2 + (B - \frac{1}{2}(1-|\phi|^2))^2 \\ &= E - \frac{1}{2}\int_{\mathbb{R}^2} B + i(\partial_1(\overline{\phi}D_2\phi) - \partial_2(\overline{\phi}D_1\phi)) \\ &= E - \pi n \end{aligned}$$

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\end{aligned}
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- So $E \geq \pi n$, with equality iff

$$
\begin{aligned}
(BOG1) &\qquad D_1\phi + iD_2\phi = 0 \\
(BOG2) &\qquad B = \frac{1}{2}(1 - |\phi|^2)
\end{aligned}
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Taubes's existence theorem

- Define $h = \log|\phi|^2$ (gauge invariant), so $\phi = e^{\frac{1}{2}h + i\chi}$

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  Valid away from zeroes of ϕ

- If ϕ has winding n, it has n zeroes (counted with multiplicity) $X_1, X_2, \ldots, X_n$ say.

$$\Delta h + 1 - e^h = 4\pi \sum_{r=1}^{n} \delta(x - X_r) \qquad (*)$$

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- Interpretation: given any collection of points $X_1, \ldots, X_n$ there is a unique (up to gauge) n-vortex solution of the Bogomol'nyi equations with $\phi = 0$ precisely at $X_1, \ldots, X_n$. Roughly, $X_r =$ vortex positions.

- Use complex coord $z = x_1 + ix_2$ on space. Vortex positions $Z_1, \dots, Z_n$

Moduli space

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- Taubes: for each monic degree n polynomial

$$p(z) = \prod_{r=1}^{n}(z - Z_r) = z^n + p_1 z^{n-1} + \cdots + p_n$$

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  there exists a unique n-vortex with zeros at roots of p.
- Moduli space of n-vortices: $M_n \equiv \mathbb{C}^n$
- Global coords $p_1, \ldots, p_n$
- Local coords $Z_1, \ldots, Z_n$ on $\mathrm{M}_n \backslash \Delta$

Geodesic approximation

- Think of field equations as ODEs for motion of a point $(\phi(t), \mathbf{A}(t))$ in field configuration space

$$\begin{aligned}
\ddot{\phi} &= D^2\phi - \frac{\lambda}{2}(1-|\phi|^2)\phi \\
\ddot{\mathbf{A}} &= \nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A}) - \mathbf{J} \\
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- Configuration space:

$$\begin{aligned}
\mathcal{A} &= \{\text{finite energy maps } (\phi, \mathbf{A}) : \mathbb{R}^2 \to \mathbb{C}\times\mathbb{R}^2 \equiv \mathbb{R}^4\} \\
\mathcal{C} &= \mathcal{A}/\mathcal{G}
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identify gauge equivalent fields

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  identify gauge equivalent fields

- $\mathcal{A}$ has a natural Riemannian metric, assigns to tangent vectors $a, b \in T_{(\phi,\mathbf{A})}\mathcal{A} \equiv \mathcal{A}$ inner product

$$\Gamma(a, b) = \int_{\mathbb{R}^2} a \cdot b$$

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- Descends to true configuration space $\mathcal{C}$: project a orthogonal to gauge orbit through $(\phi, \mathbf{A})$

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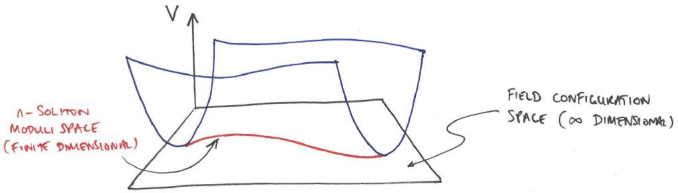
- Descends to true configuration space $\mathcal{C}$: project a orthogonal to gauge orbit through $(\phi, \mathbf{A})$
- Infinitesimal gauge transform: $\lambda = (i\chi\phi, -\nabla\chi)$

$$\begin{aligned}\Gamma((\dot{\phi}, \dot{\mathbf{A}}), \lambda) &= \int_{\mathbb{R}^2} Re(\overline{\dot{\phi}} i\chi\phi) - \dot{\mathbf{A}} \cdot \nabla\chi \\ &= \int_{\mathbb{R}^2} \chi \left\{ \frac{i}{2}(\phi\overline{\dot{\phi}} - \overline{\phi}\dot{\phi}) + \nabla \cdot \dot{\mathbf{A}} \right\}\end{aligned}$$

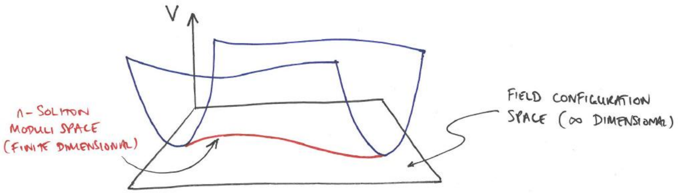
Gauss's law $\Leftrightarrow$ trajectory orthogonal to gauge orbits

Geodesic approximation



- **Restrict** dynamics to $M_n \subset \mathcal{C}$

$$S = \int (T - V)dt = \int (T - \pi n)dt$$



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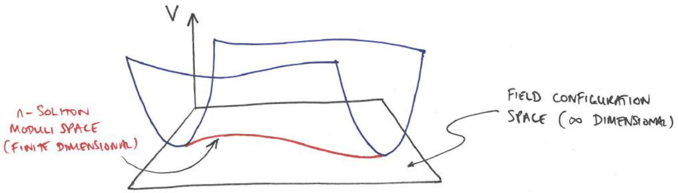
Geodesic approximation



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- Geodesic motion w.r.t. metric induced on M_n by Γ. Denote this metric γ, the L^2 metric

The metric

- Consider time varying field $(\phi(t), \mathbf{A}(t))$ which at each fixed time satisfies the Bogomol'nyi equations, with distinct vortex positions $Z_r(t)$ varying with time, and whose tangent vector $(\dot{\phi}, \dot{\mathbf{A}})$ satisfies Gauss's law

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- Remarkable fact (Samols, after Strachan): kinetic energy integral "localizes" around zeros $Z_r(t)$ of ϕ

$$T = \lim_{\varepsilon \to 0} -i \sum_{r=1}^{n} \int_{C_\varepsilon(Z_r)} \overline{\eta} \partial_{\overline{z}} \eta$$

where η is defined by $\dot{\phi} = \eta\phi$

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- Taubes's equation for $h = \log |\phi|^2$ implies

$$\eta = \sum_{r=1}^{n} \dot{Z}_r \frac{\partial h}{\partial Z_r}$$

The metric

- Expand h in a neighbourhood of Z_r

$$h = 2\log|z - Z_r| + a_r + \frac{1}{2}\overline{b}_r(z - Z_r) + \frac{1}{2}b_r(\overline{z} - \overline{Z}_r) + \cdots$$

Defines coefficients $b_r(Z_1, \ldots, Z_n)$, $r = 1, 2, \ldots, n$

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- Subst in "localized" formula for T:

$$T = \frac{\pi}{2}\sum_{r,s=1}^{n}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right)\dot{Z}_r\dot{\bar{Z}}_s$$

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- T is manifestly real, so $\frac{\partial b_s}{\partial Z_r} = \frac{\partial \bar{b}_r}{\partial \bar{Z}_r}$ $\quad (KC)$

- Extract metric: $$\gamma = \pi\sum_{r,s=1}^{n}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right)\mathrm{d}Z_r\mathrm{d}\bar{Z}_s$$

The metric

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- Extract metric: $\quad \gamma = \pi\sum_{r,s=1}^{n}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right)\mathrm{d}Z_r\mathrm{d}\bar{Z}_s$
- Hermitian (since T real). Kähler form

$$\omega = \frac{i\pi}{2}\sum_{r,s=1}^{n}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right)\mathrm{d}Z_r \wedge \mathrm{d}\bar{Z}_s$$

  Closed by (KC). M_n is a Kähler manifold.

Reduced moduli space

- Translation invariance

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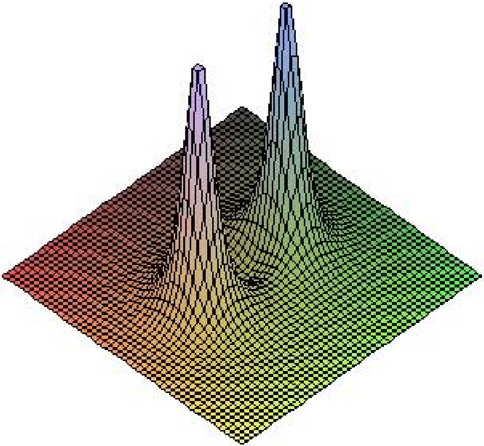
- Vortex positions = roots of $z^2 + p_2$
- As p_2 traverses real axis left to right, roots approach one another along x_1 axis, coalesce and scatter at 90°

Vortex scattering



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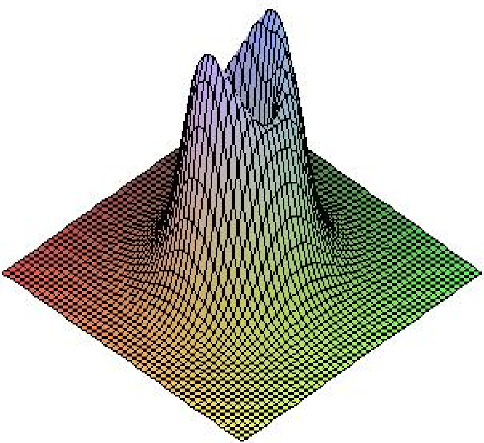

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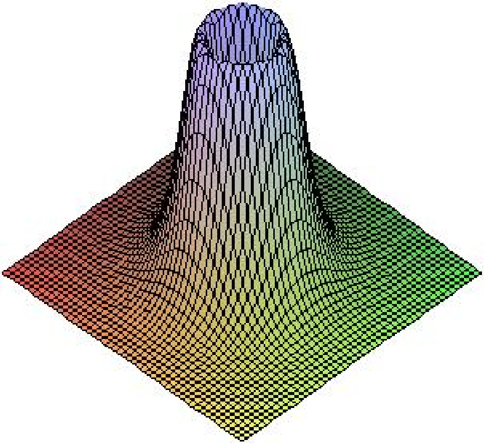

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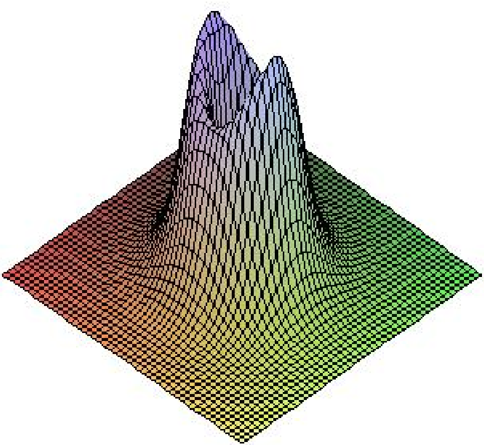

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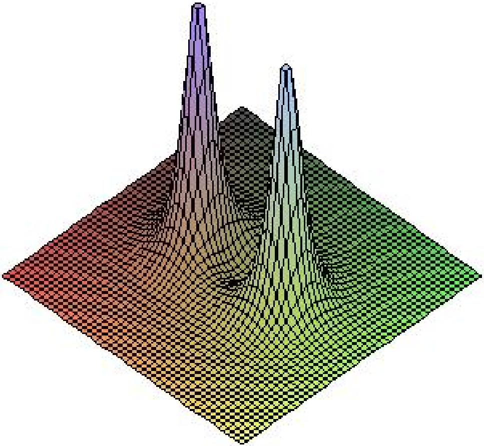

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- As vortices get close, lose identity, metric $\widetilde{\gamma}$ smooths out tip of cone
- Can construct $f(|p_2|)$ numerically: rounded cone

Vortex scattering

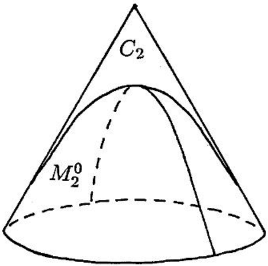
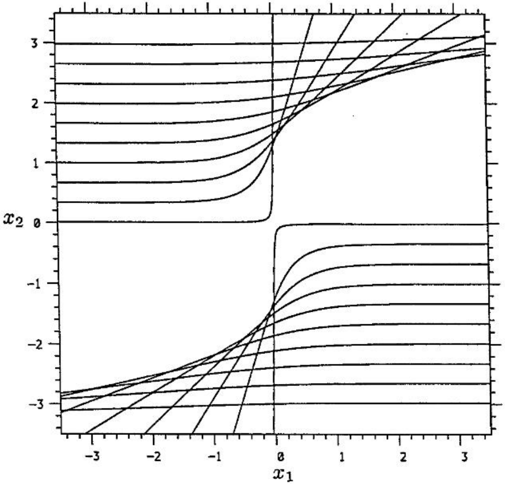


Fig. 2. A sketch of the smoothed cone representing M_2^0 as an embedding in $\mathbb{R}^3$, and the singular cone C_2 to which it is asymptotic. The difference in the areas of the cones is π. Also shown is a geodesic describing vortices in head-on collision

Samols, CMP **145** (1992) 149

General scattering



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Comparison with “experiment”

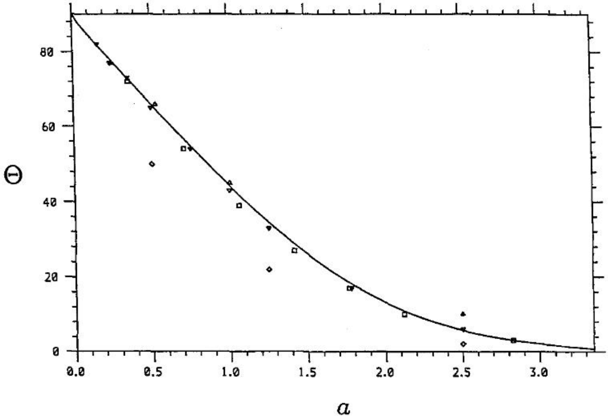

Fig. 8. The deflection angle as a function of impact parameter. The solid line is the geodesic prediction. The data points are from the numerical simulation of the full scattering problem at various impact speeds v: $v = 0.16$ (△), $v = 0.4$ (▽), $v = 0.85$ (◇) (from [17]); $v = 0.5$ (□) (from [18]). For estimates of the errors in some of these data points see [17]

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Long range intervortex forces

$$\mathcal{L} = \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\mu^2}{8}(1 - |\phi|^2)^2$$

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$$\phi = 1 + \frac{q}{2\pi} K_0(\mu r), \qquad \mathbf{A} = -\frac{m}{2\pi} \mathbf{k} \times \nabla K_0(r)$$

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- Note $K_0(r) \sim \sqrt{\frac{\pi}{2r}} e^{-r}$

Linearized model

- Coincides with solution of linearized AHM in presence of sources at vortex centre ($\phi = 1 + \psi$)

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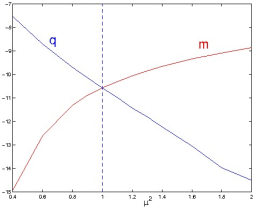
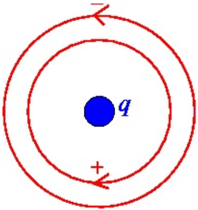
- Klein-Gordon-Proca theory: ψ scalar boson (Higgs) of mass μ, A^μ vector boson (photon) of mass 1
- Asymptotic vortex fields induced by

$$\begin{aligned} \kappa &= q\delta(\mathbf{x}) \qquad \text{scalar monopole, charge } q \\ \mathbf{j} &= -m\mathbf{k} \times \nabla\delta(\mathbf{x}) \qquad \text{magnetic dipole of moment } m\mathbf{k} \end{aligned}$$

Composite point source, "point vortex"

Point vortices





- At $\mu = 1$, $q \equiv m$ (from BOG eqns)

Point vortex interactions

- Interaction Lagrangian

$$L_{\text{int}} = \int_{\mathbb{R}^2} \{\kappa_{(1)}\psi_{(2)} - j^{\mu}_{(1)} A^{(2)}_{\mu}\} = L_{\psi} + L_A$$

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$$V_{\rm int} = -L_{\rm int} = \frac{1}{2\pi}[m^2 K_0(|\mathbf{y}-\mathbf{z}|) - q^2 K_0(\mu|\mathbf{y}-\mathbf{z}|)]$$

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  Reproduces typeI/II dichotomy of superconductivity

- Critical coupling ($\mu = 1$): $q = m \Rightarrow V_{\text{int}} = 0$. No **static** intervortex forces

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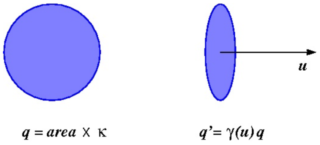
  Different tranformation properties under Lorentz boosts
  Do not balance for (point) vortices in relative motion
- Can compute L_{int} for point vortices moving along arbitrary trajectories $\mathbf{y}(t)$ and $\mathbf{z}(t)$, as an expansion in time derivatives

Moving point vortex

Point vortex moving along $\mathbf{y}(t)$ at constant velocity has

- $\kappa(\mathbf{x}, t) = q \left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right) \delta(\mathbf{x} - \mathbf{y}(t))$



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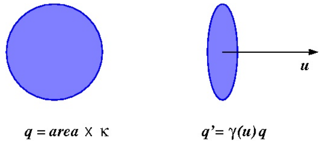
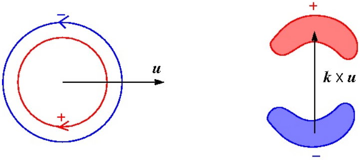


- $j^\mu = q(\mathbf{k} \times \dot{\mathbf{y}} \cdot \nabla, -\mathbf{k} \times \nabla + (\mathbf{k} \times \dot{\mathbf{y}})\dot{\mathbf{y}} \cdot \nabla)\delta(\mathbf{x} - \mathbf{y})$



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- Geodesic motion on $\mathbb{R}^2\times\mathbb{R}^2\backslash \mathrm{thick}(\Delta)$ wrt to metric

$$g = \pi\left(1-\frac{q^2}{2\pi^2}K_0(|\mathbf{y}-\mathbf{z}|)\right)(d\mathbf{y}\cdot d\mathbf{y}+d\mathbf{z}\cdot d\mathbf{z})+\frac{q^2}{\pi}K_0(|\mathbf{y}-\mathbf{z}|)d\mathbf{y}\cdot d\mathbf{z}$$

Asymptotic to the Samols metric

Vortices on compact spaces

- Motivation
  - You don't really understand a field theory until you understand it on a general background
  - It's mathematically interesting
  - Technical device to get nonzero soliton density without $n = \infty$ (soliton gas dynamics)
  - Torus $\implies$ spatially periodic solutions, cf Abrikosov lattice
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- $\phi : \Sigma \to \mathbb{C}$, $A \in \Omega^1(\Sigma)$ not good enough ($\int_\Sigma B = 0$!)
- Need more mathematical sophistication

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- $A = i\eta$ real local 1-form on Σ

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  e.g. $p = 0$: $\qquad (\mathrm{d}^{\nabla}\phi)(X) = \nabla_X \phi$

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- Energy of a pair $\phi \in \Gamma(L)$, ∇:

$$E = \frac{1}{2}\|\mathrm{d}^\nabla \phi\|^2 + \frac{1}{2}\|F^\nabla\|^2 + \frac{1}{8}\|1 - |\phi|^2\|^2$$

where $\|\cdot\| = L^2$ norm and $|\phi|^2 = h(\phi, \phi)$

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- Hence

$$\begin{aligned}
\|iF^\nabla + \frac{1}{2}(|\phi|^2 - 1)\omega\|^2 &= \|F^\nabla\|^2 + \|\partial^\nabla \phi\|^2 \\
&\quad -\|\bar{\partial}^\nabla \phi\|^2 + \frac{1}{4}\||\phi|^2 - 1\|^2 - \int_\Sigma F^\nabla
\end{aligned}$$

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- Hence $E \geq \pi n$ with equality iff

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- Integrate $(BOG2)$ over Σ

$$2\pi n = \frac{1}{2}\int_\Sigma (1 - |\phi|^2)\omega \leq \frac{1}{2}\mathrm{Vol}(\Sigma)$$

Roughly, think of each vortex as occupying volume 4π

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- Conversely, given operator $\overline{\partial} : \Omega^{p,q}(L) \to \Omega^{p,q+1}(L)$, this defines holomorphic structure on L

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$$(BOG1) \qquad \overline{\partial}^{\nabla}\phi = 0$$

$$(BOG2) \qquad iF^{\nabla} = \frac{1}{2}(1 - |\phi|^2)$$

- Choose and fix a holomorphic line bundle L of degree n over Σ, and a holomorphic section ϕ of L.
- Choose and fix a reference hermitian metric h_0 on L.
- Given any other hermitian metric $h = e^{2u}h_0$, where $u : \Sigma \to \mathbb{R}$, denote the corresponding compatible connexion ∇^u.

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- By construction, (ϕ, ∇^u) automatically solves $(BOG1)$

$$\overline{\partial}^u\phi = \overline{\partial}\phi = 0$$

- $(BOG2)$ reduces to a PDE for u

$$\Delta u + \frac{1}{2} h_0(\phi, \phi) e^{2u} + (i * F^0 - \frac{1}{2}) = 0$$

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- Simplest case: $\Sigma = S^2$

$$\begin{aligned} n\text{-vortex} \quad &\leftrightarrow \quad \text{unordered set of } n \text{ points on } S^2 \\ &\leftrightarrow \quad \text{polynomial of degree at most } n \\ & \quad p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \\ &\leftrightarrow \quad [a_0, a_1, \ldots, a_n] \in \mathbb{C}P^n \end{aligned}$$

Hence $M_n = \mathbb{C}P^n$ in this case

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- Let X be a 2-cycle generating $H_2(M_n) = \mathbb{Z}$, for example the projective line, $X = \{[a_0, a_1, 0, \ldots, 0] \ : \ [a_0, a_1] \in \mathbb{C}P^1\}$. Let ω_* be any closed two form on M_n with $\int_X \omega_* = 1$, for example the kähler form of a suitably chosen Fubini-Study metric

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- Since $H^2(M_n) = \mathbb{R}$, there is some $\alpha \in \mathbb{R}$ such that

$$\begin{aligned} \omega_{L^2} &= \alpha\omega_* + \mathrm{d}\beta \\ \Rightarrow \quad \mathrm{vol}_{L^2} &= \frac{\alpha^n}{n!}\omega_* n + \mathrm{d}\beta' \\ \Rightarrow \quad \mathrm{Vol}(M_n) &= \frac{\alpha^n}{n!}\int_{\mathbb{C}P^n} \omega_*^n \end{aligned}$$

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- Cohomology ring of $\mathbb{C}P^n$ is $H^* = (\mathbb{Z}, 0, \mathbb{Z}, 0, \ldots, 0, \mathbb{Z})$ freely generated by $[\omega_*] \in H^2$. Hence, $H^{2n}(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}$ is generated by ω_*^n, that is,

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- It remains to compute α. But this is just

$$\alpha = \int_X \omega_{L^2}$$

where X is any generator of $H_2(\mathbb{C}P^n)$

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$$p(z) = (z + w)^n, \qquad w \in \mathbb{C} \cup \{\infty\}$$

the sphere of **coincident** n-vortices. As a 2-cycle in $\mathbb{C}P^n$ this is

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- Flow on phase space T^*M_n with Hamiltonian $H(p) = \frac{1}{2}|p|^2$
- Gibbs distribution has partition function

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\begin{aligned}
Z &= \frac{1}{(2\pi\hbar)^{2n}} \int_{T^*M_n} e^{-H/T} \Omega^{2n} \\
&= \frac{1}{(2\pi\hbar)^{2n}} \int_{T^*M_n} e^{-\frac{1}{2}\gamma^{ij}(q) p_i p_j / T} \mathrm{d}^{2n} p \, \mathrm{d}^{2n} q
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- n vortices on a sphere of radius $A > 4\pi n$:

$$Z = \frac{1}{n!}(A - 4\pi n)^n \left(\frac{T}{2\pi\hbar^2}\right)^n$$

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- Coincides (to this order) with equation of state of gas of hard disks of area 2π.

Summary

- AHM supports topological solitons called vortices
- Critical coupling: $E \geq \pi n$, equality $\Leftrightarrow$ Bog. eqns
- Moduli space of static n-solitons, complex manifold of dimension $\dim_{\mathbb{C}} M_n = const \times n$ (actually, $const = 1$)
- Kinetic energy restricted to M_n equips it with natural Riemannian metric γ. Actually γ is kähler
- Geodesic motion in (M_n, γ) good approx to low energy n-soliton dynamics
- Point soliton model gives asymptotic formula for γ
- Case where space is a compact Riemann surface is mathemtically rich

Other developments

- Hyperbolic vortices:
  - $\Sigma = \mathbb{R} \times (0, \infty)$, $g = (dx_1^2 + dx_2)^2 / x_2^2$
  - Bogomol'nyi eqns become **integrable**
  - Dimensional reduction of SDYM

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- Chern-Simons variants $S_{CS} = \int_{[t_0, t_1] \times \Sigma} A \wedge \mathrm{d}A$
  - Vortices acquire electric charge
  - Manton's first order system: Hamiltonian flow on (M_n, ω_{L^2})
  - Collie-Tong: extra neutral scalar boson, vortex dynamics modelled by **Ricci geodesic flow** on M_n:

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- Baptista (after Salamon et al) studies big generalization
  - Gauged sigma models, kähler target X with hamiltonian G action
  - Formal limit $e^2 \to \infty$, (sometimes) M_n tends to $Hol_n(\Sigma, X//G)$
  - Conjectural formulae for volume of $Hol_n(\Sigma, \mathbb{C}P^k)$