Vortex lattices in anisotropic multicomponent GL models

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joint with
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Nordita 28/7/26

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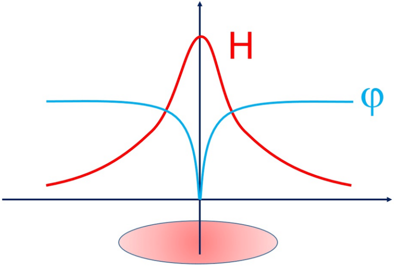
Vortices in superconductors



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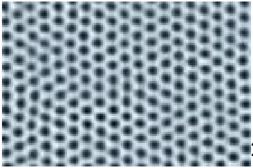


2

$$F = \frac{1}{2}|D_i\psi|^2 + \frac{1}{2}|B|^2 + \frac{\lambda}{8}(1 - |\psi|^2)^2$$

[1] Picture credit: H. Suganuma , Y. Nakagawa , K. Matsumoto (Kyoto)

[2] Picture credit: Somesh Chandra Ganguli (Aalto Un., Helsinki)

Multicomponent anisotropic GL theory

- Several condensates ψ_α, $\alpha = 1, 2, \ldots, N$.

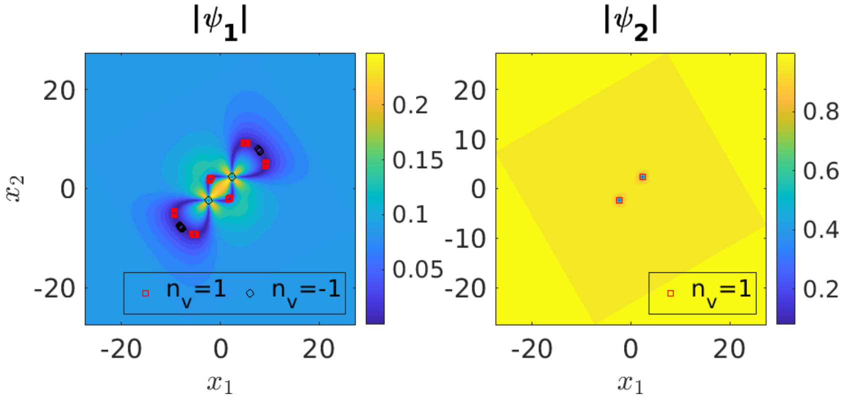
$$F = \frac{1}{2} Q_{ij}^{\alpha\beta} \overline{D_i \psi_\alpha} D_j \psi_\beta + V(\psi) + \frac{1}{2}|B|^2$$

- $Q_{ij}^{\alpha\beta} = \bar{Q}_{ji}^{\beta\alpha}$, $V(e^{i\theta}\psi) = V(\psi)$
- $V \geq 0$, $V(u) = 0$

$$\begin{aligned} -Q_{ij}^{\alpha\beta} D_i D_j \psi_\beta + 2\frac{\partial V}{\partial \bar{\psi}_\alpha} &= 0 \\ -\partial_j(\partial_j A_i - \partial_i A_j) &= \mathrm{Im}\left(Q_{ij}^{\alpha\beta} \bar{\psi}_\alpha D_j \psi_\beta\right) \end{aligned}$$

- Flux quantization: $\psi \sim u e^{i\chi(\theta)}$, $A \sim d\chi$

It's complicated (sometimes)



[3]

$s + id$ model (Zhang et al Phys. Rev. B 101, 064501 (2020))

[3] Picture credit: Alex Wormald

What's going on?

- $s + id$ model, potential tuned so that $u_1 \ll u_2$

$$
\begin{aligned}
F &= \cdots + Q_{ij}^{12}\overline{D_i\psi_1}D_j\psi_2 + \cdots \\
&= \cdots + (dx^2 - dy^2)(\nabla\psi_1, \nabla\psi_2) + \cdots
\end{aligned}
$$

- Contributes **negatively** if $\psi_2 \sim re^{i\theta}$, $\psi_1 \sim -re^{-i\theta}$

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- Contributes **negatively** if $\psi_2 \sim re^{i\theta}$, $\psi_1 \sim -re^{-i\theta}$
- Why would vortices like this form a triangular lattice?
- In general, for given Q, V and applied field $\mathbf{H}$, what is the energetically optimal vortex lattice?

Vortex lattices (warm up)

- Assume **H** is applied along a crystal symmetry direction
- Problem reduces to 2 dimensions $\psi : \mathbb{R}^2 \to \mathbb{C}^N$, $A = A_1 dx_1 + A_2 dx_2$

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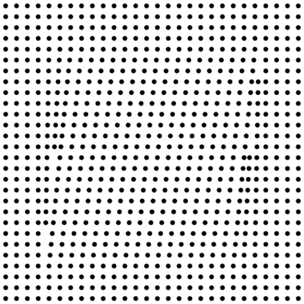

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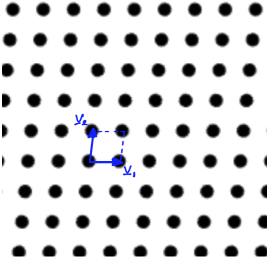

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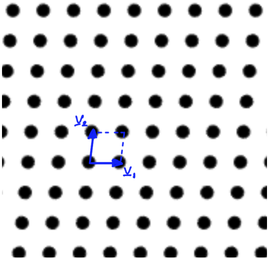

- If we knew period vectors $\mathbf{v}_1, \mathbf{v}_2$ we could be much more efficient (and precise).

Vortex lattices (warm up)

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$$F = \int_\Omega \frac{1}{2} Q_{ij}^{\alpha\beta} \overline{D_i \psi_\alpha} D_j \psi_\beta + \frac{1}{2} |B|^2 + V(\psi)$$

on torus $\Omega = \mathbb{R}^2 / \{ k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 : (k_1, k_2) \in \mathbb{Z}^2 \}$?

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- (ψ, A) periodic **up to gauge**

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$$G=\int_\Omega\left\{\frac{1}{2}Q_{ij}^{\alpha\beta}\overline{D_i\psi_\alpha}D_j\psi_\beta+\frac{1}{2}|B|^2+V(\psi)\right\}-2H\int_\Omega B$$

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$$\langle G \rangle = \frac{1}{|\Omega|} \int_\Omega \left\{ \frac{1}{2} Q_{ij}^{\alpha\beta} \overline{D_i \psi_\alpha} D_j \psi_\beta + \frac{1}{2}|B|^2 + V(\psi) \right\} - 2\pi n \frac{H}{|\Omega|}$$

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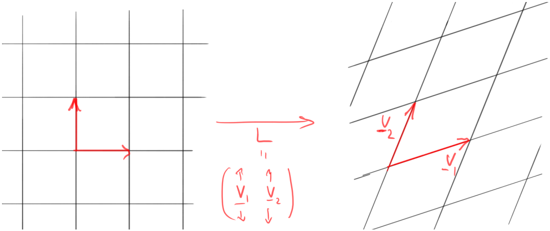

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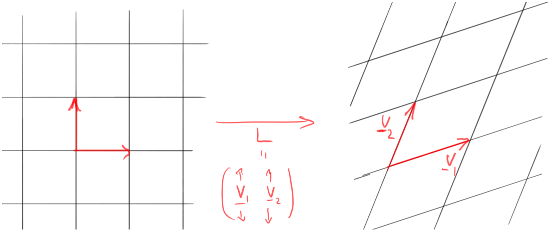

- New coordinates (X_1, X_2): $\mathbf{x} = X_1\mathbf{v}_1 + X_2\mathbf{v}_2 = L\mathbf{X}$

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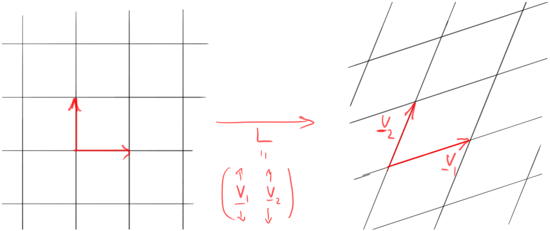

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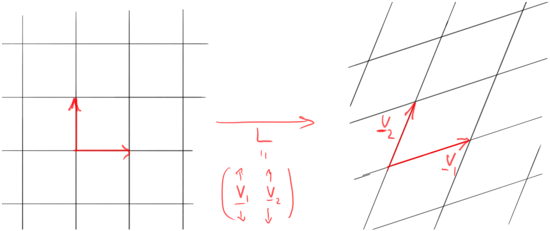

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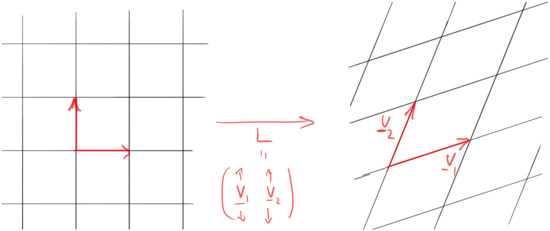

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- $|\Omega| = |\det L| = |L|$

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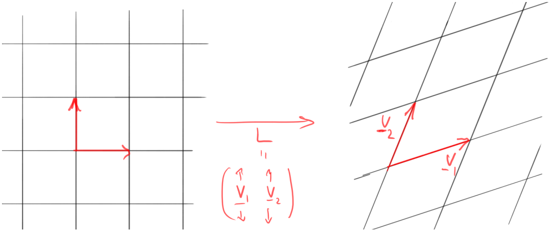

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$$\langle G \rangle = \int_{\square} \left\{ \frac{1}{2} L^{-1}_{ki} Q^{\alpha\beta}_{ij} L^{-1}_{lj} \overline{D_k \psi_\alpha} D_l \psi_\beta + \frac{F_{12}^2}{2|L|^2} + V(\psi) \right\} dX_1 dX_2 - 2\pi n \frac{H}{|L|}$$

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- For fixed $L \in GL(2, \mathbb{R})$ can minimize this over degree n fields (ψ, A) on $\square = [0, 1]^2$

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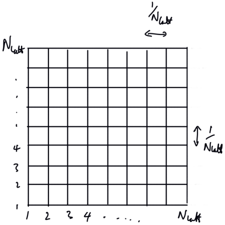
- For fixed **fields** (ψ, A) can minimize this over $L \in GL(2, \mathbb{R})$ – **explicitly!**

Optimizing the fields: Arrested Newton flow

- Discretize unit square



$$G_{dis} : \mathbb{R}^{(2N+2)N_{latt}^2} \to \mathbb{R}$$

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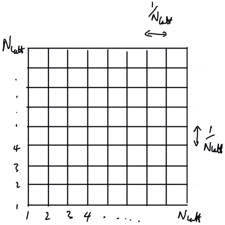

$$G_{dis} : \mathbb{R}^{(2N+2)N_{latt}^2} \to \mathbb{R}$$

- Choose initial config $q(0) \in \mathbb{R}^K$, solve

$$\ddot{q} = -(\mathrm{grad}\, G_{dis})(q), \qquad \dot{q}(0) = 0$$

- At each time step, if

$$\dot{q}(t) \cdot \mathrm{grad}\, G_{dis}(q(t)) > 0$$

ARREST! Set $\dot{q} = 0$ and restart the flow.

Optimizing the fields: Arrested Newton flow

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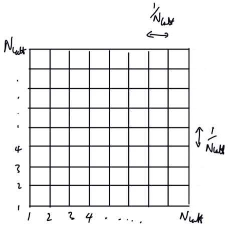

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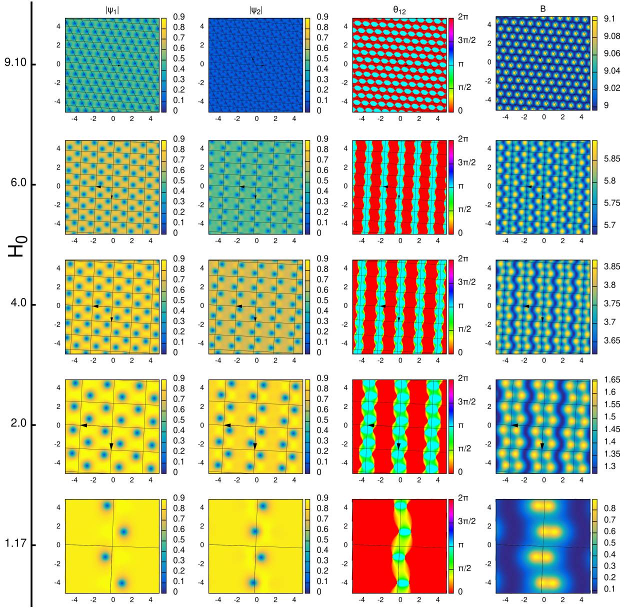
- Converges much faster than gradient flow.

An example: a nematic model

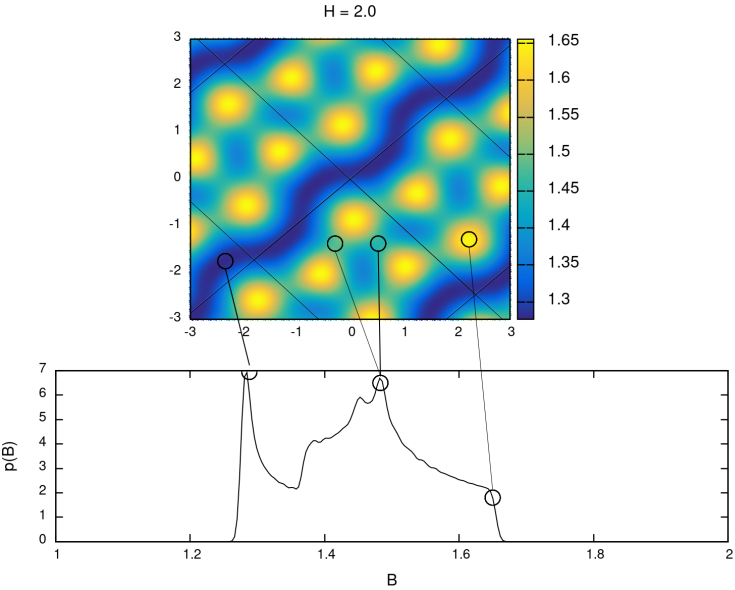
$$V = \frac{\eta^2}{2}\left(-|\psi_1|^2 - |\psi_2|^2 + \frac{1}{2}(|\psi_1|^4 + |\psi_2|^4) + \gamma|\psi_1|^2|\psi_2|^2\right).$$

$$Q^{11} = Q^{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q^{12} = \beta_\perp \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

An example: a nematic model



Experimental signature?



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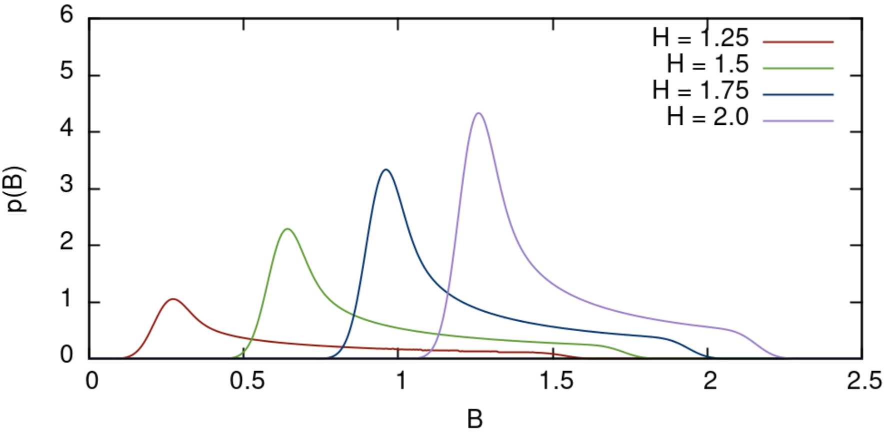


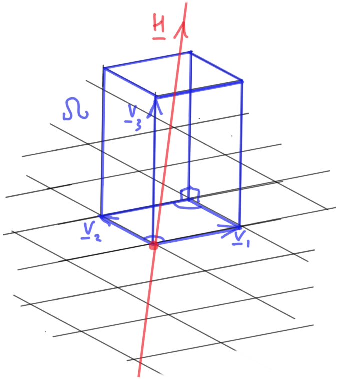
The general problem

What if **H** isn't along a symmetry direction?

$$
\begin{aligned}
-Q_{ij}^{\alpha\beta} D_i D_j \psi_\beta + 2\frac{\partial V}{\partial \bar{\psi}_\alpha} &= 0 \\
-\partial_j(\partial_j A_i - \partial_i A_j) &= \mathrm{Im}\,(Q_{ij}^{\alpha\beta} \bar{\psi}_\alpha D_j \psi_\beta)
\end{aligned}
$$

- Can certainly seek solutions translation invariant in any given direction $\mathbf{v}_3$
- Can seek solutions doubly periodic in some plane transverse to $\mathbf{v}_3$. WLOG $\mathbf{v}_3^\perp$
- **Cannot** assume $A(\mathbf{v}_3) = 0$
- **Cannot** assume that $\mathbf{v}_3 = \hat{\mathbf{H}}$ is energetically favoured

The general problem

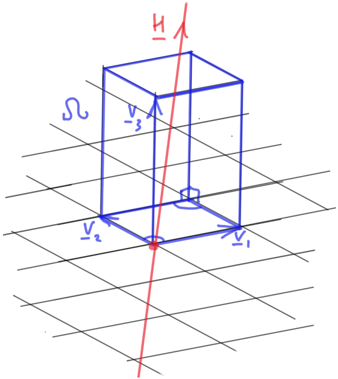


$$L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$\mathbf{x} = L\mathbf{X}$$

- Choose $\mathbf{v}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|^2}$

The general problem



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- Choose $\mathbf{v}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|^2}$
- Work with fields $\psi_\alpha(X_1, X_2)$, $A_i(X_1, X_2)$, $i = 1, 2, 3$ on unit square $[0, 1]^2$

Optimal lattice geometry?

Should minimize

$$G = \int_\Omega \left\{ \frac{1}{2} Q(D\psi, D\psi) + V(\psi) + \frac{1}{2}|B - H|^2 \right\}$$

w.r.t. ψ, A **and** L (**and** n):

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w.r.t. ψ, A **and** L (**and** n):

$$G = \frac{1}{2} L_{ki}^{-1} \mathbb{Q}_{ki,lj} L_{lj}^{-1} + \frac{1}{2}\mathrm{tr}\left(L \mathbb{F} L^T\right) - 2n\pi H_i L_{i3} + \int_\Box V(\psi),$$

where

$$\begin{aligned} \mathbb{Q}_{ki,lj} &= \mathrm{Re} \int_\Box Q_{ij}^{\alpha\beta} \overline{D_k \psi_\alpha} D_l \psi_\beta \\ \mathbb{F}_{ij} &= \int_\Box B_i B_j. \end{aligned}$$

Optimal lattice geometry?

- Minimize

$$G = \frac{1}{2} L^{-1}_{ki} \mathbb{Q}_{ki,lj} L^{-1}_{lj} + \frac{1}{2} \mathrm{tr}\,(L \mathbb{F} L^T) - 2n\pi H_i L_{i3},$$

over codimension 3 submanifold of $GL(3,\mathbb{R}) \subset \mathbb{R}^9$

$$\mathscr{C} = \{L \in Mat_{3\times 3} : \det L = 1, L_{i1}L_{i3} = 0, L_{i2}L_{i3} = 0\}$$

- Use ANF for L as well as (ψ, A)

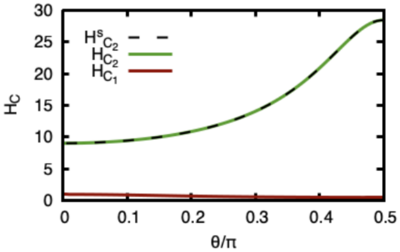
Vortex line tilting

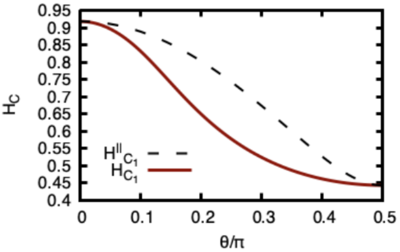
- **Single** component model:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}, \qquad V = \frac{9}{4}(1 - |\psi|^2)^2$$

- Optimal lattice does **not** have $\mathbf{v}_3 \parallel H$ if H not an eigenvector of Q

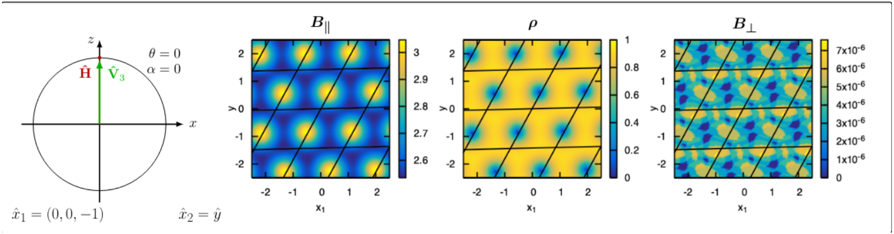
H_{c1} and H_{c2} are anisotropic





Vortex line tilting

$$|H| = 3$$



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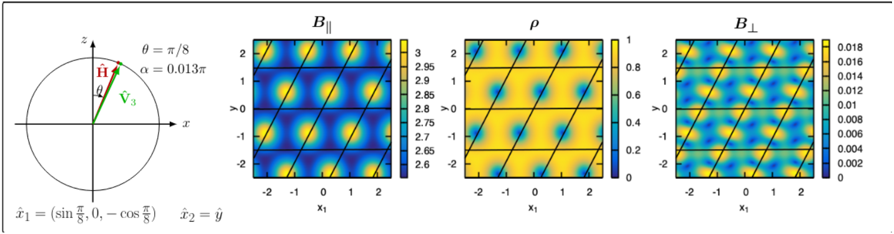

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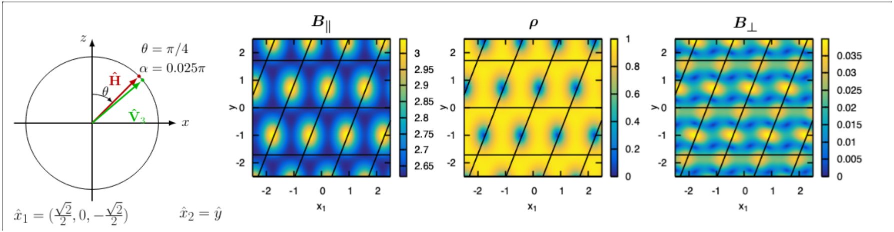

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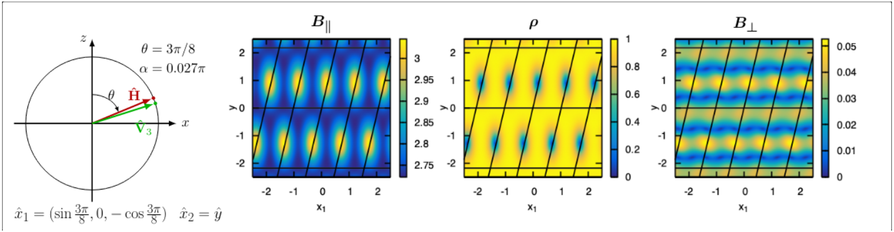
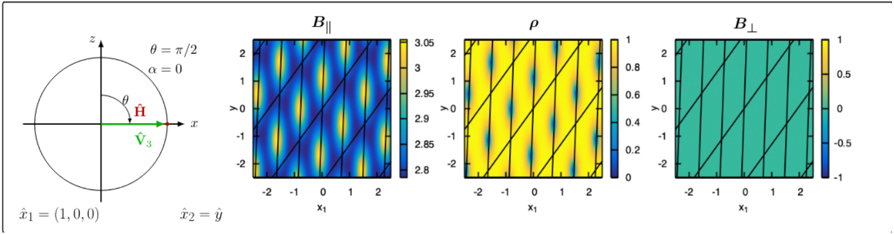
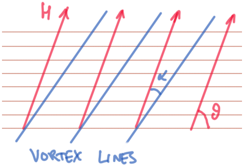


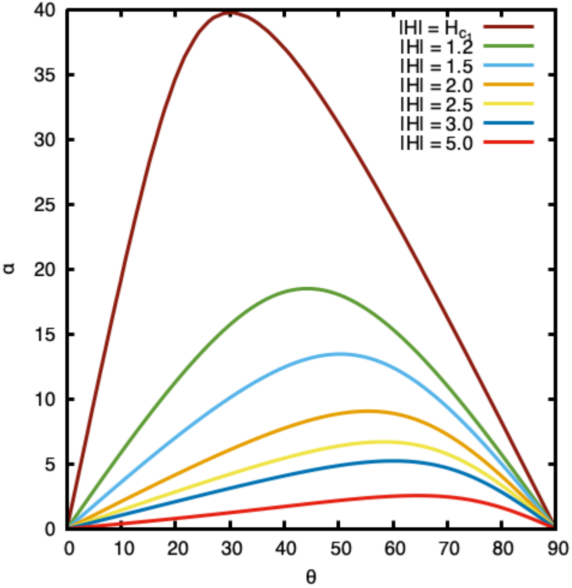
Vortex line tilting

$$|H| = 3$$



Vortex line tilting





Concluding remarks

- Vortex lattices in anisotropic superconductors are complicated
- Can't just scale the Abrikosov lattice
  - Vortex line tilting
  - Magnetic field deviation
  - Core splitting
- Need to minimize $\langle G \rangle$ over fields and period lattice $[v_1, v_2, v_3]$
- Assumption that $v_3 \parallel H$ is **not** well justified