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Near BPS Skyrme models and restricted harmonic maps

Skyrme model: $\varphi : (M^3, g) \longrightarrow (N^3, h)$, $M^3 = \mathbb{R}^3$, $N^3 = S^3$

$$E(\varphi) = \frac{1}{2}\int_M \Big(c_0\, U(\varphi)^2 + \underbrace{c_2 |d\varphi|^2 + c_4 |\varphi^*\omega|^2}_{\text{usual}} + c_6 |\varphi^*\Omega|^2 \Big)$$

extra

$U : N \to [0,\infty)$ potential

Ω = volume form on N.

Adam, Sánchez-Guillén, Wereszczyński (ASW):

<u>BPS</u> Skyrme model: $c_2 = c_4 = 0$, $c_0 = c_6 = 1$

$$0 \leq \frac{1}{2}\int_M |\varphi^*\Omega - *U\circ\varphi|^2 = E(\varphi) - \int_M \varphi^*(U\Omega)$$

$$\Rightarrow E \geq \int_M \varphi^*(U\Omega) = \langle U\rangle \int_M \varphi^*\Omega = \langle U\rangle \operatorname{Vol}(N) \underbrace{\deg\varphi}_{B}$$

Equality $\iff$ $\varphi^*\Omega = *U\circ\varphi$

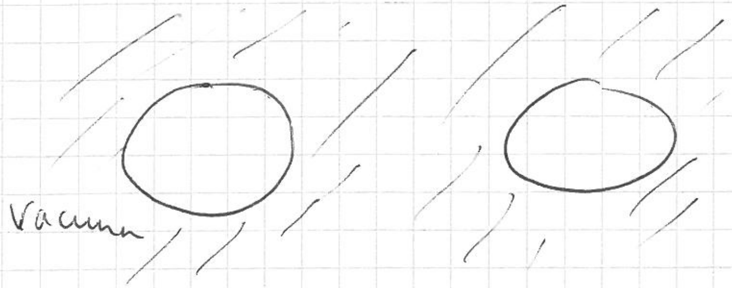
i.e. $\varphi^*\left(\frac{\Omega}{U}\right) = \operatorname{vol}_M$

φ a volume preserving map $M\setminus\{\text{critical pts}\} \to \underbrace{N\setminus\{\text{vacua}\}}_{N'}$

Key features (i) If $\left(N', \frac{\Omega}{U}\right)$ has finite volume,
BPS solutions are <u>COMPACTONS</u>

(ii) Given any volume preserving diffeomorphism
$\psi : M \to M$ $\quad (\psi^* \mathrm{vol}_M = \mathrm{vol}_M)$,

$$E(\varphi \circ \psi) = E(\varphi) \quad \forall \varphi \qquad \text{PLASTIC}$$



Can trivially superpose.

"Nuclei" can be plastically deformed (cf liquid drop model)
and have exactly zero binding energy per nucleon

<u>But</u> 3+1 time dependent model pathologically bad.

ASW proposal: Take c_2 (~~and c_4~~) (keep $c_4 = 0$) > 0 & small,
use BPS ($c_2 = 0$) solutions as a
first approx.

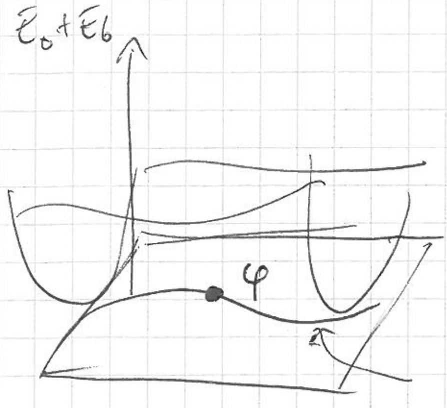
~~They took $\frac{1}{2} U(\varphi)^2 = \mu^2 (1 - \varphi_0)$ — usual mass term.~~

~~Problem: $E_2(\varphi) = \infty$ ($d\varphi$ has non-integrable singularities).~~

~~Bonenfant – Marleau: $\frac{1}{2} U(\varphi)^2 = \frac{1}{2^4} (1 + \varphi_0)(1 - \varphi_0)^3$~~

~~Find BPS solutions of "suspension" type:~~

But BPS solutions come in ∞ dim families!



Diff$_0$ orbit of φ

Q: Which φ is the right one to base approx on?

A: Should minimize $E_2(\varphi)$ among all maps in Diff$_0$ orbit.

Defn $\varphi:(M,g)\to(N,h)$ is a restricted harmonic map if $E_2(\varphi)=\frac{1}{2}\int_M |d\varphi|^2$ is critical w.r.t. all variations of φ arising from volume preserving diffeos of M. $\square$

Defn Given a symmetric $(0,2)$ tensor ρ on (M,g), define $\operatorname{div}\rho\in\Omega^1(M)$ s.t.

$$(\operatorname{div}\rho)(X) := \sum_i (\nabla_{e_i}\rho)(e_i,X)$$

e.g. $\operatorname{div} g = 0$!

e.g. $\operatorname{div} fg = df$.

$f: M\to\mathbb{R}$

Prop $\varphi:(M,g)\to(N,h)$ is R.H. only if $d(\operatorname{div}\varphi^*h)=0$
(iff when $H^{m-1}(M)=0$)

Proof: Require $\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi\circ\psi_t)=0$

where ψ_t is the flow of any divergenceless vector field $X\in\Gamma(TM)$.

For *any* diffeo $\psi: M \to M$, ~~$E_2(\varphi, g)$~~

$$E_2(\varphi \circ \psi, g) \equiv E_2(\varphi, \tilde{\psi}^* g) \qquad (\tilde{\psi} = \psi^{-1}).$$

Hence
$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi \circ \psi_t, g) = \left.\frac{d}{dt}\right|_{t=0} E_2(\varphi, \tilde{\psi}_t^* g)$$

$$= \frac{1}{2} \Big\langle \underset{\substack{\uparrow \\ \text{stress tensor} \\ \text{of } \varphi}}{S(\varphi)}, \partial_t\big|_{t=0} \tilde{\psi}_t^* g \Big\rangle_{L^2}$$

$$= -\frac{1}{2} \langle S, \mathcal{L}_X g \rangle_{L^2}$$

$$= -\frac{1}{2} \int_M 2\{\delta(S(X, \cdot)) - (\operatorname{div} S)(X)\} \operatorname{vol}_M$$

$$= \langle \flat X, \operatorname{div} S \rangle_{L^2} = 0$$

for all X with $\operatorname{div} X = 0$ i.e. $\forall \flat X \in \Omega^1(M)$ s.t. $\delta(\flat X) = 0$

$$\delta \flat X = 0 \quad \Longleftarrow \quad \flat X = \delta \nu \qquad \nu \in \Omega^2(M)$$
$$(\Longrightarrow \text{ if } H^{m-1}(M) = 0)$$

So φ RH $\underset{(\Leftarrow)}{\Longrightarrow}$ $\langle \delta \nu, \operatorname{div} S \rangle_{L^2} = 0 \quad \forall \nu \in \Omega^2(M)$

$$\Longleftrightarrow \quad \langle \nu, d(\operatorname{div} S) \rangle_{L^2} = 0$$

Fact: for Dirichlet energy, $S(\varphi) = \underbrace{\tfrac{1}{2}|d\varphi|^2}_{f} g - \varphi^* h$

$$\Rightarrow \operatorname{div} S = df - \operatorname{div}(\varphi^* h)$$

$$\Rightarrow d(\operatorname{div} S) = -d(\operatorname{div} \varphi^* h) \qquad \square$$

Note: argument works for *any* natural geometric energy E

φ restricted "harmonic" $\underset{(\Leftarrow)}{\Longrightarrow}$ $d \operatorname{div} S_E(\varphi) = 0$

Corollary Let $\varphi : (M,g) \to (N,h)$ be weakly conformal $\boxed{5}$
(i.e. $\varphi^* h = \lambda^2 g$ $\lambda : M \to \mathbb{R}$).
Then φ is R.H.

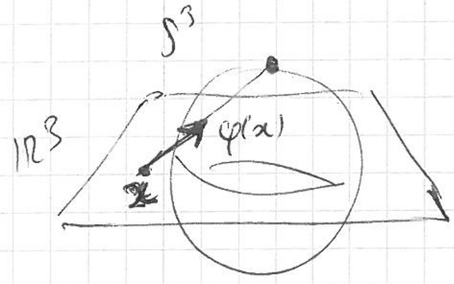
Proof: $S = \left(\frac{m}{2} - 1\right)\lambda^2 g = f g \Rightarrow \operatorname{div} S = df$

So for all X s.t. $\operatorname{div} X = 0$,

$$\frac{d}{dt}\Big|_{t=0} E(\varphi \circ \psi_t) = \langle \flat X, df \rangle_{L^2} = \langle \delta \flat X, f \rangle_{L^2} = 0$$

$\square$

Example Inverse stereographic projection $\mathbb{R}^3 \to S^3$



R.H., E_2 finite ($E_2 = 6\pi^2$)

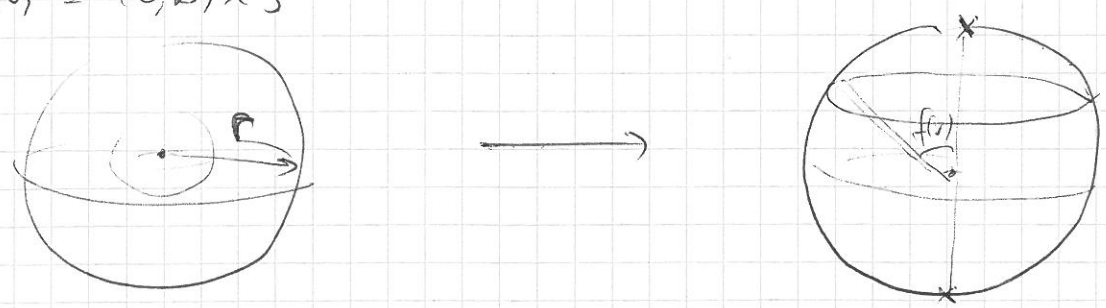
$B=1$ BPS minimizer for $E_0 + E_6$ with $U(\varphi) = (\varphi_0 - 1)^3$

$$\varphi = (\varphi_0, \varphi) = \left(\frac{|x|^2 - 1}{|x|^2 + 1}, \frac{2x}{|x|^2+1} \right)$$

$\square$

This example is of __suspension type__

$\mathbb{R}^3 \setminus \{0\} = (0,\infty) \times S^2$



$$\varphi(r, n) = (\cos f(r), \sin f(r) R(n))$$

$f(0) = \pi$, $f(\infty) = 0$ $R : S^2 \to S^2$ a fixed map of degree B.

Eg: $R = \mathrm{Id}$, $f(r) = \cos^{-1}\left(\frac{r^2-1}{r^2+1}\right)$ Stereo Proj.

$R = \mathrm{Id}$, $f = ?$ Hedgehog Ansatz

$R =$ holomorphic, $f = ?$ Rational map ansatz

$$g_{\mathbb{R}^3} = dr^2 + r^2 g_{S^2}$$

$$d\varphi \, \partial/\partial r = f'(-\sin F, \cos F \, R(n))$$

$\forall X \in T_n S^2 \quad d\varphi X = (0, \sin F \, dR_n X) \quad \perp d\varphi \, \partial/\partial r$

$$\Rightarrow \quad \varphi^* h = (F'(r)^2) \, dr^2 + \sin^2 F \, R^* g_{S^2}$$

General fact $\quad \underset{(\mathcal{T}M)}{\mathrm{div}(f\rho)} = \nabla f \lrcorner \rho + f \, \mathrm{div} \rho$

$$\Rightarrow \mathrm{div}(\varphi^* h) = \alpha(r) \, dr + \sin^2 F \, \mathrm{div}(R^* g_{S^2})$$

Eg: <u>Rational map ansatz</u>: $\quad R$ holo $\Rightarrow R$ conformal

$$\Rightarrow \quad R^* g_{S^2} = \lambda^2 g_{S^2} \quad (\lambda : S^2 \to \mathbb{R})$$
$$= \frac{\lambda^2}{r^2} (g_{\mathbb{R}^3} - dr^2)$$

~~div~~

Now $\quad \mathrm{div} \frac{dr^2}{r^2} = 0$ ~~$\mathrm{div}\, g_{\mathbb{R}^3} = 0$~~

$$\Rightarrow \mathrm{div} R^* g_{S^2} = d\left(\frac{\lambda^2}{r^2}\right) - \nabla(\lambda^2) \lrcorner \frac{dr^2}{r^2} = d\left(\frac{\lambda^2}{r^2}\right)$$

$$\Rightarrow \mathrm{div} \, \varphi^* h = \alpha(r) \, dr + \underbrace{\sin^2 F(r) \, d\left(\frac{\lambda^2}{r^2}\right)}_{\text{not closed unless } \lambda = \text{const}}$$

i.e. $R = Id$

Hence rational map ansatz is not RH except when it reduces to hedgehog. $\square$.

<u>Bonenfant and Marleau</u> studied case $\frac{1}{2} U^2 = \frac{1}{2^4} (1+\varphi_0)(1-\varphi_0)^3$

Find BPS solutions of suspension type with

$$R : S^2 \to S^2 \qquad R(\theta, \phi) = (\theta, n\phi) \quad !!$$

Computed $\quad E(\varphi^{(B)}_{BPS}) \quad 1 \leq B \leq 238 \quad$ (kind of)

Fitted C_0, C_2, C_4, C_6 to experimental binding energy data.

Problem 1 : φ have strings of conical singularities

Problem 2 : fits give $c_4 < 0 \Rightarrow E$ unbounded below !!

(suggests $c_4 = 0$ may be best)

Problem 3 : for $B > 1$, these φ's are not R.H.

$$\mathrm{div}(\varphi^* h) = \alpha(r)\,dr + (1-B^2)\underbrace{\beta(r)\cot\theta\,d\theta}_{\text{not closed.}}$$

<u>Open questions</u>

(i) Are there any smooth RH maps $\mathbb{R}^3 \to S^3$ of degree $|B| > 1$? E.g. toroidal $B = 2$?

(ii) Baby Skyrme model : $\varphi : \mathbb{R}^2 \to S^2$

- All axially symmetric maps are RH.
- Any others ?

(iii) Stability theory? Recall want φ to <u>minimize</u> E_2 in its Diff_0 orbit, not just extremize.

2nd variation? Very challenging.

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Nuclei as near BPS-Skyrmions

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Table II: Value of parameters for different fits

	Nucleus	Set Ia (N+^{4}He)	Set II (Masses)	Set III ($B.E./A$)
C_0	μ (MeV^2)	30174.2	29841.2	29475.7
(roughly) C_2	α (MeV^2)	0	0.00830341	0.0316869
C_4	β (dimensionless)	0	-5.48285×10^{-7}	-4.01085×10^{-7}
C_6	λ (MeV^{-1})	0.00491505	0.00496265	0.00503994

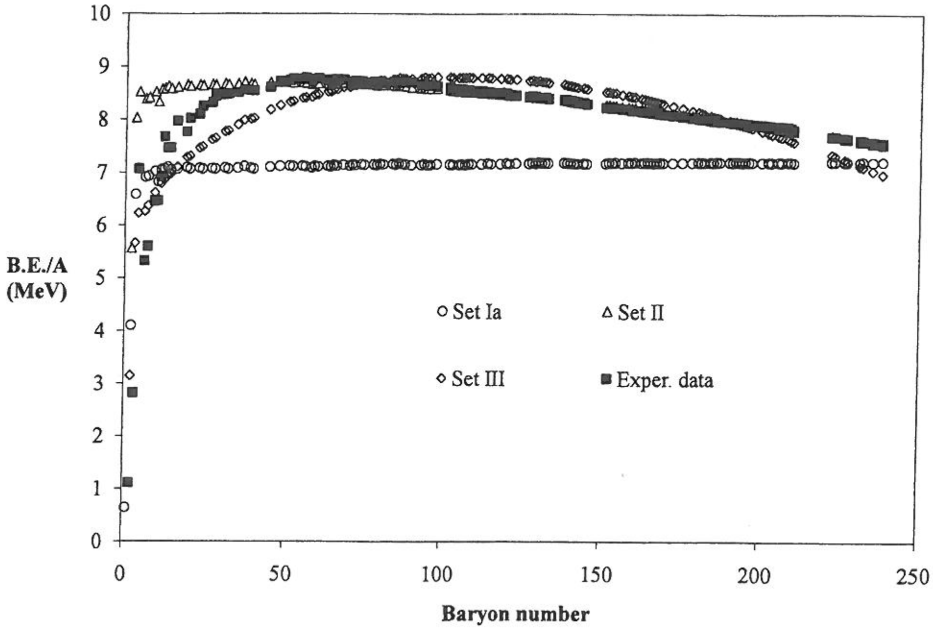

FIG. 1: Ratio of the binding energy ($B.E.$) over the atomic number A (or baryon number) as a function of A. The experimental data (black squares) are shown along with predicted value for parametization of Set Ia (empty circles), II (empty triangles) and III (empty diamonds) respectively.