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Geometry of Near-BPS Skyrme Models

Skyrme model : $\varphi: (\underset{\mathbb{R}^3}{\underset{\shortparallel}{M^3}}, g) \to (\underset{S^3}{\underset{\shortparallel}{N^3}}, h)$

$$E(\varphi) = \frac{1}{2}\int_M \Big(c_0 U(\varphi)^2 + \underbrace{c_2|d\varphi|^2 + c_4|\varphi^*\omega|^2}_{\text{usual}} + c_6|\varphi^*\Omega|^2 \Big)$$

extra

$U: N \to [0,\infty)$ potential.

$\Omega = \mathrm{vol}_N$ – volume form on N^3.

Adam, Sánchez-Guillén, Wereszczyński (ASW):
<u>BPS</u> Skyrme model : $c_2 = c_4 = 0, \quad c_0 = c_6 = 1.$

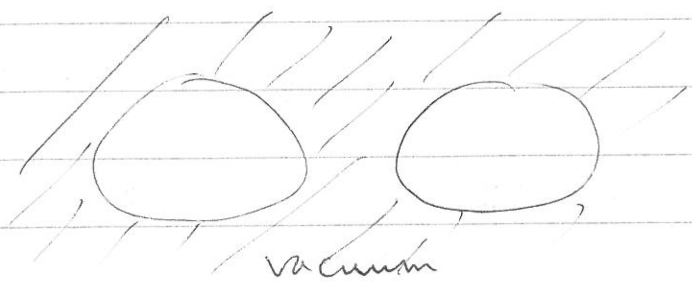
$$0 \le \frac{1}{2}\int_M |\varphi^*\Omega - *U\circ\varphi|^2 = E(\varphi) - \int_M \varphi^*(U\Omega)$$

$$\Rightarrow E \ge \int_M \varphi^*(U\Omega) = \langle U\rangle \int_M \varphi^*\Omega = \langle U\rangle \mathrm{Vol}(N)\underbrace{\deg\varphi}_{B\in\mathbb{Z}}$$

Equality $\iff \varphi^*\Omega = *U\circ\varphi$
$\iff \varphi^*\left(\frac{\Omega}{U}\right) = \mathrm{vol}_M$

$\varphi: M\setminus\{\text{critical pts}\} \to \underbrace{N\setminus\{\text{vacua}\}}_{N',\ \frac{\Omega}{U}}$ volume preserving.

__Key features__ (i) If $(N', \frac{\Omega}{\mu})$ has finite volume, COMPACTONS



can trivially superpose.

(ii) Given any volume preserving diffeo $\psi : M \to M$
$(\psi^* \mathrm{vol}_M = \mathrm{vol}_M)$

$$E(\varphi \circ \psi) = E(\varphi) \quad \forall \varphi.$$

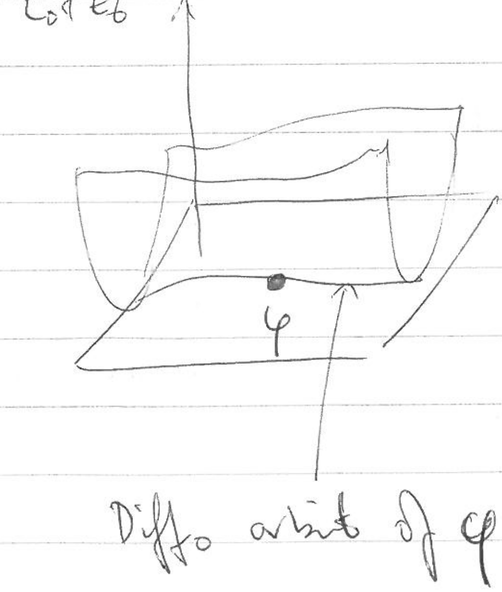
__PLASTIC__

"Nuclei" can be plastically deformed (cf liquid drop model) and have exactly zero binding energy per nucleon.

Problem: 3+1 time dependent model pathologically bad.

ASW proposal: Take $c_2 > 0$ small, use BPS ($c_2 = 0$) solutions as a first approx.

But BPS solutions come in ∞ dim families!



Q: Which φ is the right one to base approx on?

A: Should __minimize__ $E_2(\varphi)$ among all maps in Diffo orbit.

Defn $\varphi : (M,g) \to (N,h)$ is a restricted harmonic map

if $E_2(\varphi) = \frac{1}{2}\int_M |d\varphi|^2$ is critical wrt all variations

of φ arising from vol. preserving diffeos of (M,g). □

Defn Given a symmetric $(0,2)$ tensor ρ on (M,g)

define $\operatorname{div}\rho \in \Omega^1(M)$ s.t. $(\operatorname{div}\rho)(X) = \sum_i (\nabla_{e_i}\rho)(e_i, X)$

eg $\operatorname{div} g = 0$, $\operatorname{div} fg = df$.

$f : M \to \mathbb{R}$

Prop $\varphi : (M,g) \to (N,h)$ is R.H. only if $d(\operatorname{div}\varphi^* h) = 0$

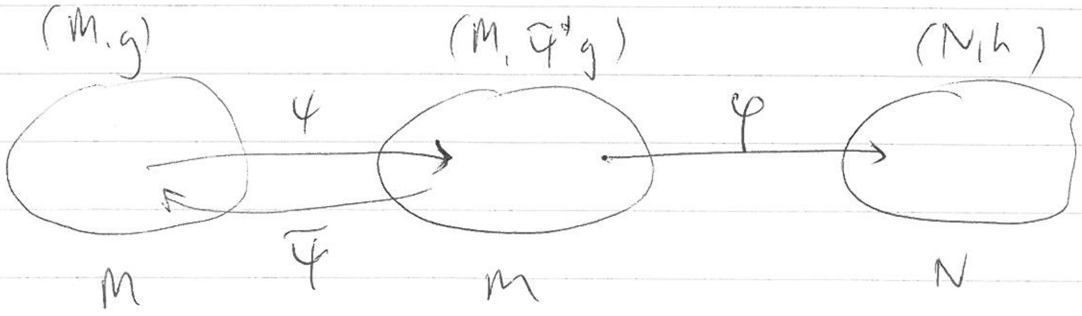
(iff when $H^{m-1}(M) = 0$)

Proof: Require $\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi \circ \psi_t) = 0$

where ψ_t is the flow of any divergenceless vector field X on M.

Nice fact: for any diffeo $\psi : M \to M$

$$E_2(\varphi \circ \psi, g) \equiv E_2(\varphi, \tilde{\psi}^* g) \qquad \tilde{\psi} = \psi^{-1}$$



$$\Rightarrow \frac{d}{dt}\Big|_{t=0} E_2(\varphi \circ \psi_t, g) = \frac{d}{dt}\Big|_{t=0} E_2(\varphi, \tilde{\psi}_t^* g)$$

$$= \frac{1}{2} \Big\langle \underset{\substack{\uparrow \\ \text{stress tensor of } \varphi}}{S(\varphi)}, \partial_t\big|_{t=0} \tilde{\psi}_t^* g \Big\rangle_{L^2}$$

$$= -\frac{1}{2} \langle S(\varphi), \mathcal{L}_X g \rangle_{L^2}$$

$$= -\frac{1}{2} \int_M 2 \{ \delta(S(X, \cdot)) - (\operatorname{div} S)(X) \} \operatorname{vol}_M$$

$$= \langle \flat X, \operatorname{div} S \rangle_{L^2} = 0$$

for all X with $\operatorname{div} X = 0$ i.e. $\forall \flat X \in \Omega^1(M)$ s.t. $\delta(\flat X) = 0$

$$\delta(\flat X) = 0 \quad \Leftarrow \quad \flat X = \delta \nu \qquad \nu \in \Omega^2(M)$$

$$(\Rightarrow \text{ if } H^{m-1}(M) = 0)$$

So φ R.H. $\underset{(\Leftarrow)}{\Rightarrow}$ $\langle \delta \nu, \operatorname{div} S \rangle_{L^2} = 0 \quad \forall \nu \in \Omega^2(M)$

$$\Longleftrightarrow \quad \langle \nu, d(\operatorname{div} S) \rangle_{L^2} = 0$$

<u>Fact</u> for E_2, $\quad S(\varphi) = \underbrace{\tfrac{1}{2}|d\varphi|^2}_{f} g - \varphi^* h$

$$\Rightarrow \quad \operatorname{div} S = df - \operatorname{div}(\varphi^* h)$$

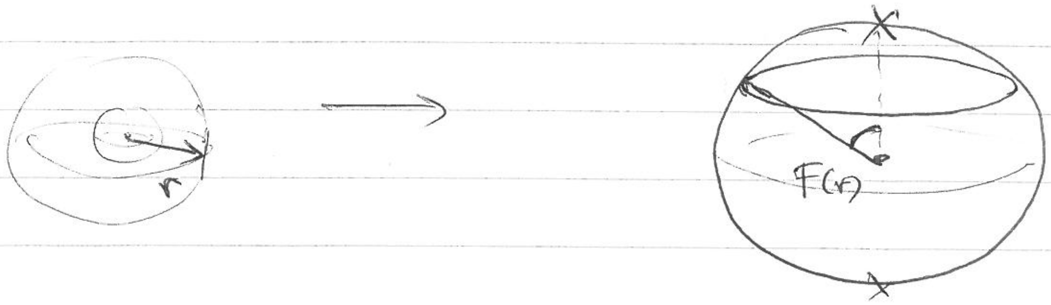
$$\Rightarrow \quad d(\operatorname{div} S) = - d(\operatorname{div} \varphi^* h) \qquad \square$$

N.B. Argument works for any geometrically natural energy E:

$$\varphi \text{ restricted "harmonic" } \underset{(\Leftarrow)}{\Rightarrow} \quad d \operatorname{div} S_E(\varphi) = 0$$

Example Suspension maps $\mathbb{R}^3 \to S^3$

$\mathbb{R}^3 \setminus \{0\} = (0,\infty)\times S^2$



$$\varphi(r,n) = (\cos F(r), \sin F(r) R(n))$$

$F(0)=\pi$, $F(\infty)=0$ $\quad R : S^2 \to S^2$ fixed map of degree B.

Eg. $R = Id$: Hedgehog ansatz

$R =$ holomorphic : Rational Map ansatz.

$$d\varphi \,\partial/\partial r = F'(r)\,(-\sin F(r), \cos F(r) R(n))$$

$$\forall X \in T_n S^2 \quad d\varphi X = (0, \sin F(r)\, dR_n X) \perp d\varphi\,\partial/\partial r$$

$$\Rightarrow \varphi^* h = F'(r)^2 dr^2 + \sin^2 F(r) R^* g_{S^2}$$

General fact: $\operatorname{div}(f\rho) = \nabla f \lrcorner \rho + f \operatorname{div}\rho$

$$\Rightarrow \operatorname{div}\varphi^* h = \alpha(r)\, dr + \sin^2 F(r) \operatorname{div}_{\mathbb{R}^3}(R^* g_{S^2})$$

Eg Rational Map Ansatz $\quad R$ holo $\Rightarrow R$ conformal

$$\Rightarrow R^* g_{S^2} = \lambda^2 g_{S^2} \qquad (\lambda : S^2 \to \mathbb{R})$$

$$= \frac{\lambda^2}{r^2}\,(g_{\mathbb{R}^3} - dr^2)$$

$$\Rightarrow \operatorname{div} R^* g_{S^2} = d\left(\frac{\lambda^2}{r^2}\right) - \underbrace{\nabla(\lambda^2) \lrcorner \frac{dr^2}{r^2}}_{0} - \lambda^2 \underbrace{\operatorname{div}\left(\frac{dr^2}{r^2}\right)}_{0}$$

$$\Rightarrow \operatorname{div} \varphi^* h = \alpha(r)\, dr + \underbrace{\sin^2 F(r)\, d\left(\frac{\lambda^2}{r^2}\right)}$$

closed

closed $\Leftrightarrow$ λ = constant

$\Leftrightarrow$ $R = Id$ (up to symmetries)

i.e. ~~Hedgeh~~ Hedgehog maps are R.H.

Other maps in rational map ansatz are <u>not</u>!

~~Bonenfant and Marleau (and Adam)~~.

Eg $U(\varphi) = (1-\varphi_0)^3$ has hedgehog BPS maps

$\varphi : \mathbb{R}^3 \to S^3$ stereographic projection!

[General fact: any weakly conformal map $(M,g) \to (N,h)$ is R.H.]

<u>Bonenfant and Marleau</u> $\frac{1}{2} U(\varphi)^2 = \frac{1}{2^4}(1+\varphi_0)(1-\varphi_0)^3$

Found BPS solutions of suspension type with

$R : S^2 \to S^2$ $R(\theta, \phi) = (\theta, n\phi)$??

Computed $E\left(\varphi^{(B)}_{BPS}\right)$ $1 \leq B \leq 238$

Fitted c_0, c_2, c_4, c_6 to experimental data.

Problem 1: φ have strings of conical singularities $(B>1)$

Problem 2: Fits give $C_4 < 0 \Rightarrow E$ unbounded below!!

(suggests $c_4 = 0$ may be best)

Problem 3: for $B > 1$ these φ's are <u>not</u> R.H.

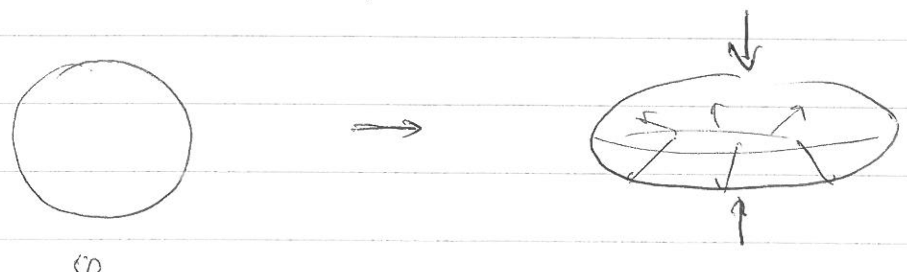
$$\operatorname{div} \varphi^* h = \underbrace{\alpha(r)\, dr}_{\text{closed}} + (1-B^2)\underbrace{\beta(r) \cot\theta \, d\theta}_{\text{not closed}}$$

$\Rightarrow$ do <u>not</u> minimize $E_2(\varphi)$ in their Diff_0 orbit.

Partial remedy for Prob 3: minimize

$E_2(\varphi_A)$ over all $A \in SL(3, \mathbb{R})$ ← linear vol. pres. diffeos $\mathbb{R}^3 \to \mathbb{R}^3$

where $\varphi_A(x) = \varphi(Ax)$.



φ_{1_3} suspension map

$\varphi_{A_{best}}$