L^2-geometry of moduli spaces of vortices and lumps

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Abstract

The low energy classical dynamics of topological solitions can often be modelled as geodesic motion in the space of static solitons with respect to a natural Riemannian metric called the L^2 metric. In this minicourse we will develop techniques to calculate this metric, and extract information about the resultant dynamics, for sigma model lumps and Abelian vortices. A recurrent theme will be interesting pheonomena arising due to noncompactness of the moduli space of static solitons.

1 Lumps

A word of caution: moduli spaces in this minicourse will generally be noncompact and exhibit noncompact properties (such as geodesic incompleteness). We will first consider lumps.

Consider a map $\varphi : \mathbb{R}^{2,1} \to \mathbb{S}^2$, where the Minkowski metric $\eta = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2$. Consider the dynamics of the field to be governed by the Lagrangian density

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$$

i.e., ϕ being a (formal) critical point of the action functional $S(\phi) = \int_{\mathbb{R}^{2,1}} L$, yielding the Euler–Lagrange equations

$$\Box\phi - (\phi \cdot \Box\phi)\phi = 0 \tag{1}$$

where $\Box = \partial_t^2 - \partial_x^2 - \partial_y^2$. There is a conserved quantity, the energy $E(\phi) = \int_{\mathbb{R}^2} \frac{1}{2}(\phi_t^2 + \phi_x^t + \phi_y^2)$, where the integral is over a time slice. Impose the energy to be finite and you get the boundary condition that the field should be constant when approximating infinity, choose without loss of generality that $\phi \to (0,0,1)$ for $|(x,y)| \to \infty$. This allows for interpreting ϕ as a map from $\mathbb{R}^2 \cup \{\infty\}$ to $\mathbb{S}^2$, i.e. a map of $\mathbb{S}^2 \to \mathbb{S}^2$. These kinds of map are classified up to homotopy by the degree, which can be computed as

$$\deg\phi = \frac{1}{4\pi}\int_{\mathbb{R}^2} \phi \cdot (\phi_x \times \phi_y).$$

Note that the integrand corresponds to the area form of $\mathbb{S}^2$ times the Jacobian of ϕ. One may wonder how does the energy $E(\phi)$ depend on the degree of the map. This result is known as the Polyakov bound (1974), although it was known to Lichnerowicz in 1970.

$$\begin{aligned} 0 \le \frac{1}{2}\int_{\mathbb{R}^2} |\phi_x + \phi \times \phi_y|^2 &= \int_{\mathbb{R}^2} \frac{1}{2}(|\phi_x|^2 + |\phi_y|^2) + \int_{\mathbb{R}^2} \phi_x \cdot \phi \times \phi_y \\ &= \int_{\mathbb{R}^2} \frac{1}{2}(|\phi_x|^2 + |\phi_y|^2) - \int_{\mathbb{R}^2} \phi \cdot \phi_x \times \phi_y = E(\phi) - 4\pi\deg(\phi) \end{aligned}$$

Thus we deduce that $E(\phi) \ge 4\pi\deg(\phi)$ and equality holds if and only if

$$\phi_x = -\phi \times \phi_y. \tag{2}$$

This is a first order differential equation characterizing the static solutions to the wave map equation (1). We can easily see that it is equivalent to (since $\phi \perp \phi_y$) to $\phi_y = \phi \times \phi_x$. This establishes an almost complex structure $J_{\mathbb{S}} : T\mathbb{S}^2 \to T\mathbb{S}^2$, $J_{\mathbb{S}^2} : v \mapsto \phi \times v$ with $J_{\mathbb{S}^2}^2 = -\mathrm{Id}$. Similarly, $\mathbb{R}^2 \equiv \mathbb{C}$ has an almost complex structure $J_{\mathbb{C}} : \partial_x \mapsto \partial_y$, $\partial_y \mapsto -\partial_x$. Then (2) coincides with

$$J_{\mathbb{S}^2} \circ d\phi = d\phi \circ J_{\mathbb{C}}. \tag{3}$$

So, $E(\phi) \geq 4\pi \deg(\phi)$ and equality holds iff ϕ is holomorphic.

We can now choose complex coordinates z in $\mathbb{R}^2$ and w the stereographic coordinate (from the South pole) in $\mathbb{S}^2$, characterizing the holomorphicity of ϕ by the Cauchy–Riemann equation $\frac{\partial(w\circ\phi)}{\partial\bar{z}} = 0$. Note that in coordinates we have merely a meromorphic function $w(\phi(z))$ as the poles are not accesible. The general solution to this equation is the rational function

$$w(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_n z^n}$$

with numerator and denominator with no common roots. The boundary condition $w(\infty) = 0$ forces $a_n = 0$ and $b_1 = 1$ without loss of generality. The "moduli" space is thus M_n the degree n rational maps with $w(\infty) = 0$, which is an open subset of $\mathbb{C}^{2n}$ once you remove the set of coefficients which yield common roots of numerator and denominator, which is a codimension 1 subset.

All of this is statics of lumps. But what about dynamics? In general it is very difficult to write the t-dependent solution, so we will attack the problem with another approach, not trying to directly solve the parabolic PDE.

Consider the Cauchy problem $\phi(0) \in M_n$, $\phi_t(0) \in T_{\phi(0)}M_n$ and assume that the latter initial derivative is small, in some sense yet to be made precise. Recall that the energy is conserved, and we have that

$$E(t) = E(0) = \frac{1}{2}\int_{\mathbb{R}^2} |\phi_t(0)|^2 + 4\pi n$$

because $\phi(0)$ is a static solution (the kinetic kerms in ∂_x, ∂_y vanish).

The conjecture is that $\phi(t)$ never strays far from M_n.

Approximation: assume that $\phi(t) \in M_n$ for all t, the *adiabatic approximation*. In that case the action restricts to

$$\begin{aligned} S(\phi) &= \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}^2} \frac{1}{2}\left(|\phi_t|^2 - |\phi_x|^2 - |\phi_y|^2\right) \\ &= \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{C}} \frac{1}{2}|\phi_t|^2 - 4\pi n \end{aligned}$$

This will yield an ODE in our moduli space. We consider the new variational problem and propose a curve

$$\phi(t) = \frac{a_0(t) + a_1(t)z + \cdots + a_{n-1}(t)z^{n-1}}{b_0(t) + b_1(t)z + \cdots + b_n(t)z^n}$$

and plug it into the action. Call the free parameters $(a(t), b(t)) \equiv q^i(t)$.

$$\begin{aligned} S &= \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{C}} \mathrm{d}x\mathrm{d}x \frac{1}{2}\frac{4}{(1+|w|^2)^2}\left|\frac{\partial w}{\partial t}\right|^2 \\ &= \int_{\mathbb{R}} \mathrm{d}t \underbrace{\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y \frac{2}{\left(1+|w|^2\right)^2}\frac{\partial w}{\partial q^i}\frac{\partial w}{\partial \overline{q}^j}}_{g_{ij}(q)} \dot{q}^i\dot{\overline{q}}^j \end{aligned}$$

and we have a functional which is determined simply by geodesic motion with respect to some Hermitian metric. There is an associated Kähler form

$$\omega = \frac{i}{2} g_{ij} \mathrm{d}q^i \wedge \mathrm{d}\overline{q}^j$$

which can be shown to be closed as follows. Rewrite the metric as

$$g_{ij}(q) = \int_{\mathbb{C}} \frac{\partial}{\partial q^i}\frac{\partial}{\partial \overline{q}^j} \log\left(1+|w|^2\right) \mathrm{d}x\mathrm{d}y$$

and from this it is easy to see that $\frac{\partial g_{ij}}{\partial q^k} = \frac{\partial g_{kj}}{\partial q^i}$ and that $\frac{\partial g_{ij}}{\partial \overline{q}^k} = \frac{\partial g_{ij}}{\partial \overline{q}^j}$. This is equivalent to $\mathrm{d}\omega = 0$. We thus have a Kähler metric in our moduli space M_n.

Problem: however, the metric is not well defined. For example, for $n = 1$, lumps of degree 1, we have that the moduli space is given by

$$M_1 = \left\{ w(z) = \frac{\lambda}{z - z_0} : (\lambda, z_0) \in \mathbb{C}^* \times \mathbb{C} \right\}$$

and when considering the $\frac{\partial}{\partial \lambda}$ direction we obtain a divergent matrix element

$$\begin{aligned} g\left(\partial_\lambda, \partial_\lambda\right) &= 4 \int_{\mathbb{C}} \frac{\left(\frac{\partial w}{\partial \lambda}\right)^2}{\left(1 + |w|^2\right)^2} \\ &= 4 \int_{\mathbb{C}} \frac{1/z}{\left(1 + \frac{|\lambda|^2}{|z|^2}\right)^2} = 2\pi \int_0^\infty \mathrm{d}r r \frac{4/r^2}{\left(1 + \lambda^2/r^2\right)^2} \end{aligned}$$

and the last integrand goes like $1/r$, which diverges upon integration over $(0, \infty)$.

The physical message is that λ has *infinite inertia*, it is infinitely costly to modify it. One can then restrict to λ =constant and foliate the moduli space, with z_0-metric being the Euclidean metric, and thus lumps will "travel in straight lines".

However, this is not a very good model of lump dynamics: although (1) does have solutions in which a single lump just moves at constant velocity without changing shape, it also much more interesting solutions, in which the lump shrinks and forms a singularity in finite time (i.e. $\lambda(t) \to 0$). So the parameter we're freezing here has interesting dynamics in the real wave map flow. So let us now work generally over a Riemann surface Σ; the metric on M_n is then guaranteed to be well-defined.

The Polyakov bound is obtained in exactly the same manner, and now the L^2-metric is well defined because kinetic energies are finite, and is now properly a Kähler metric. Choose $\Sigma = \mathbb{P}^1$ for the simplest case of study. Again the moduli space is

$$M_n = \left\{ \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_n z^n} \right\},$$

rational maps $\mathbb{S}^2 \to \mathbb{S}^2$. Now the equivalence class of coefficients is identified upon $\mathbb{C}^*$ action, we should focus not in particular tuples (a, b) but on their projective rays $[a : b]$. Thus, $M_n \simeq \mathbb{CP}^{2n+1}$. Now with our previously studied metric, (M_n, g) is geodesically incomplete, which is a feature rather than a bug.

We now consider the flow induced by the wavemap equation for lumps (1).

Theorem 1.1 (Speight, 2012)**.** *Consider a one-parameter family of Cauchy problems given by $\phi(0) \in M_n$, and $\phi_t(0) = \varepsilon v$ for some $v \in T_{\phi(0)} M_n$. Then $\exists \varepsilon_*, \tau_* > 0$ such that $\exists!$ solution to the wave map equation*

$$\phi : \left[0, \frac{\tau_*}{\varepsilon_*}\right] \times \Sigma \to \mathbb{S}^2$$

with these initial date, for all $\epsilon \in (0, \epsilon_*)$. Furthermore, if we define the time re-scaled solution

$$\begin{aligned} \phi_\varepsilon : [0, \tau] \times \Sigma &\longrightarrow \mathbb{S}^2 \\ (\tau, p) &\longrightarrow \phi\left(\tfrac{\tau}{\varepsilon}, p\right) \end{aligned}$$

this converges in C^1 to the geodesic $\psi(t) \in (M_n, g)$ with $\psi(0) = \phi(0), \psi_t(0) = v$.

This gives rigorous support for our intuition that low energy lump dynamics should be well approximated by geodesic motion in (M_n, g).

2 Vortices

We will now talk about vortices. Again we consider the Minkowski spacetime $\mathbb{R}^{1,2}$ and consider Higgs maps $\phi : \mathbb{R}^{1,2} \to \mathbb{C}$ and a connection $A = A_\mu \mathrm{d}x^\mu$, with the induced covariant derivative $D_\mu \phi = \partial_\mu \phi - i A_\mu \phi$. Take the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \overline{D^\mu \phi} D_\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{8} \left(1 - |\phi|^2\right)^2 .$$

It's invariant under gauge transformation, and the Euler–Lagrange equations are given by these nonlinear coupled 2nd order PDEs

$$D_\mu D^\mu \phi - \frac{1}{2}\left(1-|\phi|^2\right)\phi = 0$$
$$\partial_\mu F^{\mu\nu} + \frac{i}{2}\left(\overline{\phi} D^\nu \phi - \phi \overline{D^\nu \phi}\right) = 0$$

There is a conserved energy given by

$$E = \int_{\mathbb{R}^2} \underbrace{\frac{1}{2}|D_0\phi|^2 + \frac{1}{2}F_{0i}F_{0i}}_{\text{kinetic energy } T} + \underbrace{\frac{1}{2}F_{12} + \frac{1}{2}|D_i\phi|^2 + \frac{1}{8}(1-|\phi|^2)^2}_{\text{potential energy } V}$$

We want solutions with finite energy. Note that $F_{0i} = E_i$, the electric field, while $F_{12} = B$, the magnetic field. At infinity we would like $|\phi| \to 1$ and $|D_i\phi| \to 0$. On a large circle, thus, $\phi \sim e^{i\chi}$ and A must look like $\mathrm{d}\chi$. The Higgs field could wrap around a lot , and thus we count the degree of the map as $\chi(2\pi) - \chi(0) = 2\pi n$ for some $n \in \mathbb{Z}$. Then,

$$\Phi = \int_{\mathbb{R}^2} B = \int_{\mathbb{R}^2} \mathrm{d}A = \oint_{\mathbb{S}^1_\infty} A = 2\pi n$$

Magnetic flux is quantized. We'd like something similar to the Polyakov bound in energy for lumps. The analogue is the 1976 Bogomol'nyi bound. Interestingly, this was discovered a couple of years earlier by a German condensed matter physicist, and the result was forgotten.

$$\begin{aligned} 0 &\le \frac{1}{2}\int_{\mathbb{R}^2} |D_1\phi + iD_2\phi|^2 + \left(B - \frac{1}{2}(1-|\phi|^2)\right)^2 \\ &= \frac{1}{2}\int_{\mathbb{R}^2} |D_1\phi|^2 + |D_2\phi|^2 + i\left(\overline{D_1\phi} D_2\phi - D_1\phi\overline{D_2\phi}\right) + B^2 - B + |\phi|^2 B + \frac{1}{4}(1-|\phi|^2) \\ &= V - \frac{1}{2}\int_{\mathbb{R}^2} B = V - \pi n \end{aligned}$$

Thus $V \ge \pi n$ and equality holds if and only if

$$\begin{cases} D_1\phi + iD_2\phi = 0 & (V1) \\ B = \frac{1}{2}(1-|\phi|^2) & (V2) \end{cases} \tag{4}$$

which are known as the vortex equations. Note that the first condition is a holomorphicity condition. The first (V1) implies that ϕ vanishs at exactly n points counted with multiplicity. Denote them by $z_1, \dots, z_n$. This defines an effective divisor of degree n. The converse is given by Taubes's theorem, which characterizes the moduli space of vortices in $\mathbb{R}^2$.

Theorem 2.1 (Taubes)**.** *Given a degree n effective divisor D, there exists a solution to the vortex equations with $(\phi) = D$. Furthermore, the solution is unique up to gauge transformation, smooth, and we have the following localization result:* $\forall \delta \in (0,1)$ $\exists C > 0$ *such that*

$$|D\phi|, ||\phi| - 1|, |B| \le Ce^{-(1-\delta)|(x,y)|}$$

Proof. More of a sketch. Taubes reformulated the equations into a 2nd order PDE with singularities. Define h, χ implicitly by

$$\phi = e^{\frac{1}{2}h + i\chi}$$

noting that h is well defined on $\mathbb{C} \setminus D$ although the phase need is defined up to $2\pi\mathbb{Z}$, in a branched way. These will have singularities. One rewrites the fields in terms of h via

$$A = -\frac{1}{2}\star \mathrm{d}h + \mathrm{d}\chi$$
$$B = \star \mathrm{d}A = -\frac{1}{2}\star \mathrm{d} \star \mathrm{d}h = -\frac{1}{2}\Delta h$$

Then note that the vortex equations summarize as $\Delta h = e^h - 1$, valid on $\mathbb{C} \setminus D$. One can extend it to the entire plane by considering the equation in a distributional sense, adding the adequate delta terms at the zeroes

$$\nabla^2 h = e^h - 1 + 4\pi \sum_r \delta(z - z_r).$$

One can regularize the equation by absorbing the delta terms

$$h_0 := -\sum_r \log\left(1 + \frac{\mu}{|z - z_r|}\right) A$$

from which

$$\nabla^2 h_0 = 4\pi \sum_r \delta(z - z_r) - \underbrace{4\pi \sum_r \frac{\mu}{(|z - z_r|^2 \mu^2)}}_{g_0}$$

and then define v by $h = h_0 + v$, and rewrite the equation in terms of v alone. Taubes defines an action for which the equation for v

$$-\nabla^2 v + e^{h_0} e^v + (g_0 - 1) = 0$$

is the Euler–Lagrange equation, given by

$$a(v) = \int_{\mathbb{R}^2} \frac{1}{2} |\mathrm{d}v|^2 + v(g_0 - 1) + e^{h_0}(e^v - 1).$$

Now some analytical considerations are in order. First of all, a is C^1 map $H^1(\mathbb{R}^2) \to \mathbb{R}$, and indeed the (formal) critical points are determined by the equation for v. Here H^1 refers to the Sobolev space.

Secondly, a is strictly convex, meaning $a(tv + (1-t)w) < ta(u) + (1-t)a(w)$ for all $v, w \in H^1(\mathbb{R}^2), t \in (0, 1)$.

And third, the functional is coercive, roughly meaning that $a(v) \to \infty$ as $\|v\|_{H^1} \to \infty$.

Then, by standard results in analysis, there exists a unique global minimum. Since $h = \log|\phi|^2$, $h \leq 0 \iff |\phi| \leq 1$. Let us prove that ϕ lies always within the unit disk. Assume that $h \nleq 0$. Then $\exists z_*$ such that $h(z_*) > 0$, by the Maximum Principle, $h(z_*)$ has to be a local minimum, just by looking at the PDE: $\nabla^2 h = \operatorname{tr} \partial^2 h$, the trace of the Hessian. But $\nabla^2 h = e^h - 1$, which means that the Hessian is at the same time negative (by being a maximum) and positive (by the PDE).

This conclues the sketch of the proof. Nothing is said about how to prove the localization properties. □

Let M_n denote the space of vortex solutions modulo gauge group. We've just shown that these are divisors of degree n. Represent them by a monic polynomial

$$p(z) = (z - z_1) \dots (z - z_n) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

and notice how then $M_n \simeq \mathbb{C}^n$, the coefficients of the polynomial giving the coordinates. It has a nontrivial differentiable structure, since we are **not** parametrizing it by the roots, i.e. the divisor, but rather by the coefficients of the corresponding monic polynomial.

Now we want to understand the low energy dynamics on this moduli space, considering again the approximation in which we assume to "stay in" M_n. Restrict ourselves to fields such that $(A(t), \phi(t)) \in M_n \;\; \forall t$.

We use the temporal gauge $A_0 = 0$, but we **cannot** forget the corresponding equation, which will now be promoted to a constrain,

$$-\partial_i \dot{A} + \frac{i}{2}(\phi \overline{\dot{\phi}} - \overline{\phi} \dot{\phi})$$

$$\iff \delta \dot{A} + (i\phi, \dot{\phi})_{\mathbb{R}^2} = 0$$

where we are using the codifferential δ. Now we consider an infinitesimal gauge transformation

$$\phi \mapsto e^{i\mu}\phi \simeq \phi + i\mu\phi, A \mapsto A + \mathrm{d}\mu$$

and then the L^2-product

$$\begin{aligned}
\left\langle (\dot{\phi}, \dot{A}), (i\mu\phi, \mathrm{d}\mu) \right\rangle &= \int_{\mathbb{R}^2} (\dot{\phi}, i\mu\phi) + (\dot{A}, \mathrm{d}\mu) \\
&= \int_{\mathbb{R}^2} (\dot{\phi}, i\mu\phi) + \mu\delta\dot{A} = \left\langle \mu, \delta\dot{A} + (i\phi, \dot{\phi}) \right\rangle
\end{aligned}$$

so $(\dot{\phi}, \dot{A})$ is perpendicular to the gauge orbit at (ϕ, A). We then have the action with vanishing potential

$$S = \int_{\mathbb{R}} \mathrm{d}t(T - \cancel{V}) = \int_{\mathbb{R}} \mathrm{d} \int_{\mathbb{C}} \frac{1}{2}(|\dot{\phi}|^2 + |A|^2)$$

This quadratic form defines a Riemmanian metric on the modili space precisely because we have fixed the gauge properly.

We will now see the Strachan–Samols localization formula. Consider the zeroes $z_1(t), \ldots, z_n(t)$ moving in $\mathbb{C}$. Consider the corresponding transformation of fields

$$\phi(t) = e^{\frac{1}{2}h(t)+i\chi(t)}$$

where again we have that $\nabla^2 h(t) = e^{h(t)} - 1$ away from the zeroes. Differentiating with respect to t we obtain that

$$\nabla^2 \dot{h} = \dot{h}e^{h(t)}.$$

We have to choose a gauge such that Gauss' law still holds.

$$\dot{A} = -\frac{1}{2}\mathrm{d}\dot{h} + \mathrm{d}\dot{\chi}, \delta A = -\nabla^2\chi = e^h\dot{\chi}$$

and we define now η by $\eta\phi = \dot{\phi}$. Then

$$\eta = \frac{1}{2}\dot{h} - i\dot{\chi} \implies \nabla^2\eta = \eta e^h \tag{5}$$

away from the zeroes Trick: compute the kinetic energy

$$T = \frac{1}{2}\int_{\mathbb{C}} |\dot{A}|^2 + |\dot{\phi}|^2,$$

and encircle the moving zeroes by circles of radius ε and trivially obtain the first following identity

$$\begin{aligned} T &= \lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus S_\varepsilon} \frac{1}{2}(|\dot{A}|^2 + |\dot{\phi}|^2) = \lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus S_\varepsilon} \frac{1}{2}\mathrm{d}\overline{\eta}\wedge\mathrm{d}\eta + \frac{1}{2}e^h|\eta|^2 \star 1 \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus S_\varepsilon} \mathrm{d}(\overline{\eta}\wedge i\mathrm{d}\eta) - \nabla^2\eta + e^h\eta = \lim_{\varepsilon\to 0} -\frac{1}{2}\int_{\mathbb{C}\setminus S_\varepsilon} \overline{\eta}\wedge i\mathrm{d}\eta \end{aligned}$$

where we have used the PDE satisfied by η away from D, (5). Computing the integral is now easier using the following expansion

$$h(z) = 2\log|z - z_r|^2 + a_r + \frac{1}{2}b_r(z - z_r) + \frac{1}{2}(\overline{z} - \overline{z}_r) + \ldots$$

where a_r and b_r care respectively some real and complex valued functions of the vortex positions $\{z_s\}$. One finds T can be written in terms of b_r,

$$T = \frac{\pi}{2}\sum_{r=1}^{n}|\dot{z}_r|^2 - 2\sum_{r,s=1}^{n}\frac{\partial b_r}{\partial\overline{z}_s}\dot{\overline{z}}_s\dot{z}_r$$

from which one obtains the Riemannian metric in the moduli space whose geodesic flow we are so much interested in

$$g = \pi\sum_{r=1}^{n}\mathrm{d}\dot{z}_r\mathrm{d}\overline{\dot{z}_s} - \sum_{r,s=1}^{n}\frac{\partial b_r}{\partial\overline{z}_s}\mathrm{d}\dot{z}_r\mathrm{d}\overline{\dot{z}_s}.$$

One can check by the formula again that the metric is actually Kähler.

3 Statistical mechanics of vortices

We will now consider vortices on the sphere. Throughout this section $\Sigma = \mathbb{S}^2$ although nothing is assumed yet of the metric, which can be different from the round one. Let $L \twoheadrightarrow \Sigma$ be a smooth Hermitian line bundle over Σ of degree n, $\phi \in \Gamma(\Sigma, L)$ a section and $A \in \mathcal{A}(L)$ a unitary connection. We want to find the minimizers of the following functional

$$E(\phi, A) = \frac{1}{2}\left\|\mathrm{d}_A\phi\right\|_{L^2}^2 + \frac{1}{2}\left\|F_A\right\|_{L^2}^2 + \frac{1}{8}\left\|\tau - \phi\right\|_{L^2}^2.$$

Again this admits a Bogomol'nyi bound $E(\phi, A) \geq \pi\tau n$, with equality holding if and only if

$$\begin{aligned} \overline{\partial}_A\phi &= 0, \qquad (V1) \\ \star F_A &= \frac{1}{2}(\tau - |\phi|^2). \qquad (V2) \end{aligned}$$

These are the vortex equations. Here $\mathfrak{u}(1) = i\mathbb{R}$. Again there is a necessary condition for existence of solutions to the vortex equations, obtained by integrating (V2) over Σ:

$$2\pi n = \frac{1}{2}(\tau|\Sigma| - \|\phi\|_{L^2}^2)$$

and therefore $\|\phi\|_{L^2}^2 = \tau|\Sigma| - 4\pi n\tau =: \varepsilon \geq 0$. The moduli space of solutions is (using results from other courses in this conference, see Bradlow, García-Prada):

$$M_n = \begin{cases} \emptyset \text{ if } \varepsilon < 0 \\ \{*\} \text{ if } \varepsilon = 0 \\ \text{degree } n \text{ effective divisors on } \Sigma \text{ if } \varepsilon > 0 \end{cases}$$

Using the fact that $\Sigma = \mathbb{S}^2$, we can see degree n effective divisors first as $\mathrm{Sym}^n(\Sigma)$, and then parametrized by polynomials in one complex variable of degree at most n up to $\mathbb{C}^*$ action

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Identify the divisor D with the set of roots of $p(z)$, together with their multiplicities. If $\deg p < n$, complete the roots with the point at infinity ∞, since the polynomial is to be seen as a function in the sphere Σ via stereographic projection from the South Pole.

Hence, we identify

$$M_n(\Sigma = \mathbb{S}^2) \leftrightarrow [a_0 : \cdots : a_n] \in \mathbb{P}^n$$

We now detail the construction of the L^2 metric on M_n. We have a projection $\mathcal{V}_n \twoheadrightarrow M_n$ from the space of solutions to the moduli space, by quotienting out the gauge group $\mathcal{G}$. Take a curve in M_n and now we need to lift it $(A(t), \phi(t))$. However, we need to choose the lift so that it is L^2 orthogonal to the gauge orbits. This is done, as shown in the previous lecture, by demanding Gauss' law to be satisfied at each instant

$$\delta A + (i\phi, \dot{\phi}) = 0$$

and then we can define the squared length

$$g_{L^2}(v, v) = \int_\Sigma |\dot{\phi}|^2 + |\dot{A}|^2$$

As it turns out, this metric is Kähler, and this is a nontrivial fact!

Once we have the metric, we can be interested in two things. First, in the low energy dynamics, with the hypothesis that the geodesic flow induced by this metric approximates well the dynamics. Secondly, we can consider the quantum dynamics!

Quantization and thermodynamics of vortices

In order to quantize this moduli space, we propose wavefunctions

$$\psi : \mathbb{R} \times M_n \to \mathbb{C}$$

demanding they satisfy the most naive Schrödinger equation one can think of in this context

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2}\Delta_{g_{L^2}}\psi.$$

Once we've done that we can write the partition function

$$Z(T) = \sum_{\lambda \in \mathrm{Spec}(\Delta_{g_{L^2}})} e^{-\frac{\hbar^2 \lambda}{2T}}.$$

Computing this seems hopeless as one needs to know the eigenvalues of the Laplacian of a metric which we really have no real firm grip on yet... However, for high temperatures, there is an asymptotic expansion that might give us some hope

$$Z(T) \simeq \left(4\pi\frac{\hbar^2}{2T}\right)^? \left(a_0 + a_1\left(\frac{\hbar^2}{2T}\right) + \mathcal{O}\left(\frac{\hbar^2}{2T}\right)\right)$$

As it turns out, the parameters a_0 and a_1 are geometric invariants of our moduli space with its L^2 metric.

$$a_0 = |M_n| = \mathrm{Vol}(M_n, g_{L^2}) \tag{6}$$

$$a_1 = \frac{1}{6}EH(g_{L^2}) = \frac{1}{6}\int_{M_n} S_{g_{L^2}} \tag{7}$$

where EH denotes the Einstein-Hilbert action of the metric g_{L^2}, that is, the integral of scalar curvature $S_{g_{L^2}}$ over M_n. The rest of this section is dedicated to calculating these coefficients. We begin by exactly computing the volume of the moduli space.

$$|M_n| = \int_{M_n} \frac{\omega^n}{n!}$$

since $\dim_{\mathbb{C}} M_n = n$, as it is the complex projective space of dimension n. However, even though we do not fully know ω, the integral defining $|M_n|$ depends only on its de Rham cohomology class $[\omega]$, and this we *can* compute exactly. Note that

$$H^2_{dR}(\mathbb{P}^n) = \mathbb{R} = \mathrm{Span}_{\mathbb{R}}(\omega_0)$$

for some normalized ω_0 such that $\int_{\mathbb{P}^1_0} \omega_0 = 1$. Here $\mathbb{P}^1_0$ is the generator of the second homology, or one can even see it as the generator of $\mathbb{P}^n$ as a CW-complex. This precise $\mathbb{P}^1_0$ is obtained in our vision of $\mathbb{P}^n$ as the projective class of the coefficients of polynomials as the class of $p(z) = a_0 + a_1 z$ as $[a_0, a_1]$ ranges over $\mathbb{P}^1$.

Our L^2 Kähler form must be proportional to ω_0, $\omega = \alpha\omega_0$, and we can compute

$$\alpha = \int_{\mathbb{P}^1_0} \omega$$

from which we would know the volume of the whole moduli space

$$|M_n| = \frac{\alpha^n}{n!}\cancelto{1}{\int_{M_n}\omega_0^n}$$

An idea by Nick Manton is to take another copy of $\mathbb{S}^2$ sitting inside of the moduli space, but a very special one, not homologous to the generating $\mathbb{P}^1_0$. We define

$$M_n^{co} \subset M_n$$

the space of cocentered vortex solutions, i.e., these vortex solutions such that there is only one zero of degree n of the Higgs field, $(\phi) = n\mathrm{pt}$. This is clearly topologically a sphere, and corresponds to (projective classes of) polynomials of the form

$$p(z) = (z-q)^n = z^n - nqz^{n-1} + \cdots + (-q)^n$$

where $q \in \mathbb{C} \cup \{\infty\}$. The homology relation is $M_n^{co} = n \cdot \mathbb{P}^1_0$, by counting the intersection numbers. In that case

$$\int_{M_n^{co}} \omega = n\int_{\mathbb{P}^1_0} \omega = n\alpha \implies \alpha = \frac{|M_n^{co}|}{n}$$

And to deduce this integral we will use the localization formula. This was obtained as the kinetic energy of vortex configurations by looking only around the zeroes of the Higgs field. The field $h = \log|\phi|^2$ has, around the coincident n-fold zero q, an expansion of the form

$$h = n\log|z-q|^2 + a + \frac{1}{2}b(z-q) + \frac{1}{2}\overline{b}(\overline{z}-\overline{q}) + \dots$$

where a and b are respectively some unknown real and complex functions of q. By the Strachan-Samols localization trick, one finds that

$$T = \frac{1}{2}n\tau\left(\tau\Omega(q,\overline{q}) + 2\frac{\partial b}{\partial \overline{q}}\right)|\dot{q}|^2$$

where Ω is the conformal factor of the metric on Σ (that is, $g_\Sigma = \Omega(z,\overline{z})\mathrm{d}z\mathrm{d}\overline{z}$). With this one can prove that the Kähler form on M_n^{co} is given by

$$\omega_{M_n^{co}} = \tau n\pi\omega_\Sigma - in\pi\overline{\partial}(\underbrace{b\mathrm{d}q}_{\beta}),$$

where we have pulled back via the diffeormorphism $M_n^{co} \simeq \mathbb{S}^2 = \Sigma$ the Kähler form on the base. Here β is a (1,0)-form which will play an important role. We can actually just use the differential and write

$$\omega_{M_n^{co}} = \tau n \pi \omega_\Sigma - i n \pi \mathrm{d}(\beta),$$

which is valid when using the stereographic coordinate q from the South Pole. One then needs to repeat this computation from the North Pole, using $\tilde{z} = 1/z, \tilde{q} = 1/q$. Again

$$h = \log |\phi|^2 = n \log |1/z - 1/q| + \tilde{a} + \frac{1}{2}\tilde{b}\left(\frac{1}{z} - \frac{1}{q}\right) + \text{ complex conjugate } + \dots$$

and after expansion and comparison one deduces that $\tilde{b} = 2nq - q^2 b$, from which

$$\tilde{\beta} = b\mathrm{d}(1/q) = \beta + \frac{2n}{q}\mathrm{d}q$$

which if you look carefully is the transformation law of a connection on a degree $2n$ bundle on $\mathbb{S}^2$. Therefore this β defines a (namesake) connection β. Note that $\mathrm{d}\beta$ is precisely the curvature of this connection, well defined on the entire sphere. We obtain the result

$$\omega_{L^2}|_{M_n^{co}} = \tau \pi n \omega_\Sigma - n\pi i F_\beta.$$

Note how the Chern–Weil theory already tells us all we need to know about the cohomology class of F_β. Thus, we integrate this over our 2-sphere M_n^{co} to obtain

$$|M_n^{co}| = \pi n \tau |\Sigma| - n\pi 2\pi(2n) = n\pi(\tau|\Sigma| - 4\pi).$$

Note the appearance of the *dissolution* parameter $\varepsilon = \tau|\Sigma| - 4\pi$. We obtain thus that $\alpha = \pi(\tau|\Sigma| - 4\pi)$. We can already use this to obtain the volume of the entire moduli space

$$|M_n|_{L^2} = \frac{\pi^n(\tau|\Sigma| - 4\pi)^n}{n!}.$$

This concludes the computation of the first coefficient in the asymptotic expansion of the partition function a_0. The other one is actually simpler. Recall that the Einstein–Hilbert action is, for Kähler manifolds, more manageable

$$EH(g) = \int_{M_n} S_g = \langle \rho, \omega \rangle_{L^2} = \int_{M_n} \rho \wedge \frac{\omega^{n-1}}{(n-1)!}$$

where ρ is the Ricci form. So we only need compute the (de Rham class of the) Ricci form for the L^2 metric in the moduli space of vortices M_n.

Recall that the Ricci form is also the curvature of the connection on the line bundle $K^{-1} = \Lambda^n T^{1,0} M_n$ induced by the Levi–Civita connection. Note that by Chern–Weil theory again we have that $[\rho] = c_1(K^{-1})$. And we precisely know who the anticanonical bundle is for $M_n = \mathbb{P}^n$, namely $K^{-1} = \mathcal{O}(n+1)$. Therefore

$$\int_{\mathbb{P}^1_0} \rho = 2\pi(n+1) \implies [\rho] = 2\pi(n+1)[\omega_0]$$

This immediately gives us the Einstein–Hilbert action.

$$EH(g_{L^2}) = 2\pi(n+1)\pi^{n-1}\frac{(\tau|\Sigma| - 4\pi)^n}{(n-1)!}\cancelto{1}{\int_{M_n} \omega_0}$$

One is now well prepared to tackle the quantum thermodynamics of vortices, through the geometry on the (quantized) moduli space.

4 $\mathbb{P}^1$-vortices

We will now come back to noncompact moduli spaces. Consider vortices with target space $\mathbb{P}^1$. In particular, let $\phi : \Sigma = \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, fix a unit vector $e = (0,0,1) \in \mathbb{S}^2$ and let's gauge $\mathbb{S}^1$ rotations about the axis $\mathbb{R} \cdot e$. Recall the usual Hamiltonian action of $SO(3)$ on $\mathbb{S}^2$ which has a moment map $\mu(\phi) = e \cdot \phi$.

In this case a connection is simply $A \in \Omega^1(\Sigma)$, and its covariant derivative is given by

$$\mathrm{d}_A \phi = \mathrm{d}\phi - A(e \times \phi)$$

Consider the energy functional

$$E(\phi) = \frac{1}{2}\int_{\mathbb{R}^2} |\mathrm{d}_A\phi|^2 + |F_A|^2 + \underbrace{(e\cdot\phi)^2}_{\mu}.$$

Let us now impose finite energy. This requires ϕ to localize at the equator of the target $\mathbb{S}^2$, as we approach spatial infinity, so that $e\cdot\phi \to 0$. This means that at a circumference at spatial infinity $S^1_\infty \subset \Sigma$ we have a map $\phi : \mathbb{S}^1_\infty \to \mathbb{S}^1_{eq}$, of some degree $n \in \mathbb{Z}$.

On the other hand, it is required that ϕ be covariantly constant at spatial infinity $|\mathrm{d}_A\phi| \to 0$.

Note that the magnetic flux

$$\Phi = \int_{\mathbb{R}^2} \mathrm{d}A \overset{\text{Stokes}}{=} \oint_{\mathbb{S}^1_\infty} A = 2\pi n$$

is necessarily quantized. There are, however, extra topological charges in this theory. In bundle theory language ϕ is a section of a $\mathbb{S}^2$-bundle over $\mathbb{R}^2$ which by $\mathbb{R}^2$ being contractible is necessarily trivial $\mathbb{R}^2 \times \mathbb{S}^2$. There are distinguished sections of this bundle, the constant North Pole and South Pole sections, denoted by e and $-e$. Think of the image $\phi(\mathbb{R}^2)$ sitting inside the total space and we have therefore a two-dimensional manifold inside a four-dimensional one with other distinguished two-dimensional surfaces given by the images of e and $-e$, and thus we can consider the relative intersection numbers.

The possible intersection numbers are thus $k_+ = \#(\phi(\Sigma), e)$ and $k_- = \#(\phi(\Sigma), -e)$. However, these are constrained by $n = k_+ - k_-$. For $(k_+, k_-) = (1, 0)$ we have "North vortices" with flux $\Phi = 2\pi$. For $(k_+, k_-) = (0, 1)$ we have "South antivortices" with flux $\Phi = -2\pi$. Unlike the usual vortices and antivortices these do not annihilate, but can coexist in equilibrium.

Again the Bogomol'nyi bound is obtained $E \geq 2\pi(k_+ + k_-)$ and equality holds if and only if

$$\begin{cases}\overline{\partial}_A\phi = 0, \star F_A = \phi\cdot e.\end{cases} \tag{8}$$

If Σ is compact, then one would obtain again a Bradlow constraint

$$-|\Sigma| \leq 2\pi(k_+ - k_-) = \int_\Sigma \underbrace{\phi\cdot e}_{\in[-1,1]} \leq |\Sigma|,$$

which is a bound on charge *asymmetry*. That is, it allows both k_+ and k_- to be arbitrarily large, provided they are not very different from one another.

The moduli space can be seen as a symplectic quotient. The moduli space is the space of pairs of *disjoint* divisors

$$M_{k_+,k_-}(\Sigma) \simeq Sym^{k_+}(\Sigma) \times Sym^{k_-}(\Sigma) \setminus \Delta$$

where Δ is the big diagonal of non-disjoint divisors, i.e., pairs of divisors which share some point in common. On $\Sigma = \mathbb{C} = \mathbb{R}^2$ the symplectic quotient construction is purely formal, but it is feasible to prove that this is indeed still the moduli space by direct analytic methods.

Theorem 4.1 (Yang)**.** *Let P, Q be effective divisors of degrees k_+, k_- on $\mathbb{C}$ with $P \cap Q = \emptyset$. Then there exists a smooth solution to equations* (8) *with*

- $\phi^{-1}(e) = P$
- $\phi^{-1}(-e) = Q$
- *It is unique up to gauge.*
- *It is exponentially localized*

Proof. We give here a sketch of the proof, which has the same key idea as in Taubes's theorem for regular vortices. We will trade the vortex equations in for a second order PDE for a single function. Define

$$h = \log\frac{1 - e\cdot\phi}{1 + e\cdot\phi} = \begin{cases}-\infty \text{ on } P,\\ \infty \text{ on } Q.\end{cases}$$

The equation rewritten for h reduces to (including distributional data for the zeroes of the divisors)

$$\nabla^2 h = 2\frac{e^h - 1}{e^h + 1} + 4\pi\sum_{x\in P}\delta_x - 4\pi\sum_{y\in Q}\delta_y \tag{9}$$

which we can compare to Taubes' equation

$$\nabla^2 h = e^h - 1 + 4\pi \sum_{x \in D} \delta_x \tag{10}$$

Note that (10) has a unique solution h_D depending on the prescribed effective divisor D, smooth away from D and exponentially localized, $h_D < 0$.

We aim at construction of sub and supersolutions to (9). Define $w_0 := -h_Q$, i.e., minus the solution to Taubes equation (10) for the divisor Q. So in particular $w_0 > 0$. Furthermore

$$\nabla^2 w_0 = \underbrace{1 - e^{-w_0}}_{>0} - 4\pi \sum_{y \in Q} \delta_x < \frac{2}{1 + e^{-w_0}}(1 - e^{-w_0}) - 4\pi \sum_{y \in Q} \delta_y + 4\pi \sum_{x \in P} \delta_x,$$

and so this is a supersolution to (9). Let us now take the solution to the Taubes equation for the divisor P, $w_\infty = h_P < 0$. It is a subsolution to (9):

$$\nabla^2 w_\infty = \underbrace{e^{w_\infty} - 1}_{<0} + 4\pi \sum_{x \in P} \delta_x > \frac{2}{e^{w_\infty} + 1}(e^{w_\infty} - 1) + 4\pi \sum_{x \in P} \delta_x - 4\pi \sum_{y \in Q} \delta_y$$

By general analytic arguments there must exist a solution of (9) trapped between the super and the subsolution. One sets up an iteration scheme as follows. Consider $F(s) = 2\tanh(s/2)$, which has the property $F'(s) < 1 \ \forall s \in \mathbb{R}$. Let $C_0 > 1$ and $w_0 = -h_Q$, the supersolution. Define iteratively w_{n+1} as the solution to the PDE

$$\nabla^2 w_{n+1} - c_0 w_{n+1} = F(w_n) - c_0 w_n + 4\pi \sum_{x \in P} \delta_x - 4\pi \sum_{y \in Q} \delta_y,$$

which being linear has a guaranteed solution. We aim at showing that (w_n) converges. We claim that the sequence is monotonically decreasing

$$w_\infty < \ldots < w_n < w_{n-1} < \cdots < w_1 < w_0$$

To prove this, assume by induction that indeed $w_n < w_{n-1}$, i.e. $w_n - w_{n-1} < 0$. Then

$$\begin{aligned}
\nabla^2(w_{n+1} - w_n) - c_0(w_{n+1} - w_n) &= F(w_n) - F(w_{n-1}) - c_0(w_n - w_{n-1}) \\
&= \underbrace{F'(s_n)}_{<1}(w_n - w_{n-1}) - c_0 \underbrace{(w_n - w_{n-1})}_{<0} \\
&> \underbrace{(1 - c_0)}_{<0}(w_n - w_{n-1}) > 0
\end{aligned}$$

By the maximum principle $w_{n+1} - w_n$ must be negative. Let us see this, by contradiction. Assume it is not negative. Then $w_{n+1} - w_n$ attains a positive maximum at some $z_* \in \mathbb{C}$. But $\nabla^2(w_{n+1} - w_n)|_{z_*} > 0$ by the above computation. However, ∇^2 is the trace of the Hessian, which must be negative definite at the maximum z_*, and thus it is impossible for $\nabla^2(w_{n+1} - w_n)$ to be positive.

One then proves the inequalities for the initial step $w_0 > w_1$ and and by monotonicity the sequence converges. The limit provides a smooth solution to (9), which is exponentially localized thanks to the monotonicity of the above sequence. □

The above theorem proves that the moduli space of $\mathbb{P}^1$-vortices is given by

$$M_{k_+,k_-}(\Sigma) \simeq Sym^{k_+}(\Sigma) \times Sym^{k_-}(\Sigma) \setminus \Delta$$

We now want to obtain a localization formula in order to obtain an expression for the L^2-metric on this moduli space. Let P, Q be a pair of effective divisors in the above moduli space. Define a sign function on all $x \in P \cup Q$:

$$s(x) \begin{cases} +1 & x \in P \\ -1 & x \in Q \end{cases}$$

Near some element $z_j \in P \cup Q$ there is an expansion for the function h solution to equation (9) close to z_j given by

$$s(z_j) h = \log|z - z_j|^2 + a_j + \frac{1}{2} b_j (z - z_j) + \frac{1}{2}\overline{b_j}(\overline{z} - \overline{z_j}) + \ldots$$

One then defines from the coefficients in the above expansion a local $(1,0)$ form

$$b = \sum_{z_j \in Q \cup P} b_j \mathrm{d}z_j.$$

Then the localization formula (Speight, Romão) asserts that the Kähler form of the L^2 metric is

$$\omega_{L^2} = i\pi \left(\sum_j \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j} - \overline{\partial} b \right)$$

There is a simple case in which interesting conclusions can be drawn. Consider $k_+ = k_- = 1$. The moduli space is given by

$$M_{k_+,k_-}(\Sigma) = \Sigma \times \Sigma \setminus \Delta$$

and b globalizes as a connection on $K^{-1}_{M_{k_+,k_-}}$. For our studied case of $\Sigma = \mathbb{C}$, there is an alternative parametrization of the moduli space via center of mass in $\mathbb{C}$ and relative position of the two vortices via

$$\left(\underbrace{\frac{z_1 + z_2}{2}}_{Z}, \underbrace{\frac{z_1 - z_2}{2} = \varepsilon}_{\varepsilon} \right) \in \mathbb{C} \times \mathbb{C}^*$$

The metric is rewritten in terms of these coordinates, taking into account the symmetry, as

$$g_{L^2} = 4\pi \mathrm{d}Z \mathrm{d}\overline{Z} + F(|\varepsilon|) \mathrm{d}\varepsilon \overline{\varepsilon}$$

One can extract the F factor numerically and there is an analytical conjecture for its small $|\varepsilon|$ asymptotics, as $F(|\varepsilon|) \sim -8\pi \log |\varepsilon|$

If that is true (and it has very good numerical backing) then it happens that $M_{1,1}(\mathbb{C})$ is incomplete, and the scalar curvature diverges as $\varepsilon \to 0$. This implies that the moduli space cannot be isometrically embedded in any compactification of itself.

There are some rigorous results for compact surfaces, obtained by elliptic estimates on the solutions h of Yang's equation which, after suitable regularization, is a smooth semilinear elliptic PDE on Σ.

1. For $\Sigma = \mathbb{S}^2$, $M_{1,1}(\mathbb{S}^2)$ is incomplete. Its volume is given by $|M_{1,1}(\mathbb{S}^2)| = \left(2\pi|\mathbb{S}^2|\right)^2$. (Speight, Romaão).
2. For Σ compact oriented surface of genus g or $\Sigma = \mathbb{C}$, $M_{1,1}(\Sigma)$ is incomplete. (Garcia Lara)
3. For the flat 2-torus, the volume is $|M_{1,1}(\Sigma_1)| = (2\pi|\Sigma|)^2 + 16\pi^2|\Sigma|$. (Garcia Lara). This last result is remarkable: recall that b globalizes as a connexion on $K^{-1}(M_{1,1})$ which, in the case $\Sigma = T^2$ is a trivial bundle. Hence, its curvature F_b is cohomologically trivial, so one might expect that

$$\int_{M_{1,1}} F_b = 0,$$

which would imply that $|M_{1,1}| = (2\pi|\Sigma|)^2$ as for $\Sigma = \mathbb{S}^2$. But this ignores the fact that $M_{1,1}$ is noncompact: there is an extra term originating from the integral of b over the removed diagonal Δ and this contributes the extra volume of $4|\Sigma|(2\pi)^2$.

Some paths into the literature

The rigorous result on the adiabatic limit of wave-map flow is proved in

> J.M. Speight, "The adiabatic limit of wave map flow on a two torus" *Trans. Am. Math. Soc.* 367 (2015) 12, 8997-9026 .

Taubes's original existence proof for n-vortices in the plane is described in the book

> *Vortices and Monopoles* by A. Jaffe and C. Taubes.

A comprehensive treatment of the L^2 metric on the moduli space of vortices is given in chapter 3 of the book

> *Topological Solitons* by N.S. Manton and P.M. Sutcliffe.

This book also has extensive discussion of the geodesic approximation to soliton dynamics.

The Strachan-Samols localization trick first appeared in

> I.A.B. Strachan, I. A. B., "Low-velocity scattering of vortices in a modified abelian Higgs model", *J. Math. Phys.* **33** (1992) 102–110,

in the (very) special case that the domain is the hyperbolic plane. Samols shortly afterwards converted it into a more systematic method in

> T.M. Samols, "Vortex scattering" *Commun. Math. Phys.* **145** (1992) 149–179.

The cohomological trick for computing the volume of M_n for linear vortices on a compact domain appears in

> N.S. Manton and S.M. Nasir, "Volume of vortex moduli spaces" *Commun. Math. Phys.* **199** (1999) 591–604.

The Bogomol'nyi bound for the $\mathbb{P}^1$ model was discovered by Bernd Schroers

> B.J. Schroers, "Bogomol'nyi solitons in a gauged O(3) sigma model" *Phys. Lett. B* **356** (1995) 291–296.

Yang's original existence proof for $\mathbb{P}^1$ vortices on the plane is described in chapter 11 of the book

> *Solitons in Field Theory and Nonlinera Analysis* by Yisong Yang.

A similar result but on compact domains is obtained in

> L. Sibner, R. Sibner and Y. Yang, "Abelian Gauge Theory on Riemann Surfaces and New Topological Invariants" *Proc. R. Soc. Lond. A* **456** (2000) 593–613.

The localization formula for the metric on the moduli space of $\mathbb{P}^1$ vortices appears in

> N.M. Romão and J.M. Speight "The geometry of the space of BPS vortex–antivortex pairs" *Commun. Math. Phys.* **379** (2020) 723–772.

This also presents numerical results on the geometry of $M_{1,1}(\mathbb{C})$ and rigorous results on incompleteness and volume of $M_{1,1}$ for the round two-sphere. Incompleteness for $\Sigma = \mathbb{C}$ and Σ compact but arbitrary were proved in the PhD thesis of Rene Garcia Lara. This also contains a rigorous computation of the volume $|M_{1,1}(T^2)|$.