Mass splitting in the Skyrme model

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LEEDS@Leeds, July 2018

Proton-neutron mass splitting

- $m_n = 939.56563$ MeV, $m_p = 938.27231$ MeV,

$$\frac{2(m_n - m_p)}{m_n + m_p} = 0.1377\%$$

- Skyrme model?

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$$U(t, r\mathbf{n}) = e^{-ivt\tau_3/2}[\cos f(r)\mathbb{I}_2 + i \sin f(r)\mathbf{n} \cdot \tau]e^{ivt\tau_3/2}$$

Quantize. Proton $I_3 = 1/2$, neutron $I_3 = -1/2$

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  Quantize. Proton $I_3 = 1/2$, neutron $I_3 = -1/2$
- Action must somehow distinguish between clockwise and anticlockwise isorotation in π_1-π_2 plane...
- ...and be Lorentz and parity invariant.
- Difficult.

Two solutions

- Holography (Bigazzi and Niro 2018): introduce explicit m_u-m_d difference in Sakai-Sugimoto model.
  - Also get $\pi^{\pm}$-π^0 mass difference
  - But it's not really the Skyrme model
  - U coupled to infinite tower of vector mesons
- Chiral perturbation theory

Chiral perturbation theory

PHYSICAL REVIEW D VOLUME 40, NUMBER 3 1 AUGUST 1989

Neutron-proton mass-splitting puzzle in Skyrme and chiral quark models

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$$
\begin{aligned}
\mathcal{L}_{\mathrm{SB}} = \mathrm{Tr}[(\alpha\lambda_3 + \alpha' T + \alpha'' S)(A_\mu^L U A_\mu^R + A_\mu^R U^\dagger A_\mu^L) \\
+ (\beta\lambda_3 + \beta' T + \beta'' S)(\partial_\mu U \partial_\mu U^\dagger U + U^\dagger \partial_\mu U \partial_\mu U^\dagger) \\
+ (\gamma\lambda_3 + \gamma' T + \gamma'' S)(F_{\mu\nu}^L U F_{\mu\nu}^R + F_{\mu\nu}^R U^\dagger F_{\mu\nu}^L) \\
+ (\delta\lambda_3 + \delta' T + \delta'' S)(U + U^\dagger - 2)]\,, \qquad (2.4)
\end{aligned}
$$

$$\mathcal{L}_0 = \frac{-F_\pi^2}{8}\mathrm{Tr}(\partial_\mu U \partial_\mu U^\dagger)$$

$$\mathcal{L}_1 = -\tfrac{1}{2}\,\mathrm{Tr}(F_{\mu\nu}^L F_{\mu\nu}^L + F_{\mu\nu}^R F_{\mu\nu}^R) + \gamma\,\mathrm{Tr}(F_{\mu\nu}^l U F_{\mu\nu}^R U^\dagger)$$

$$\mathcal{L}_2 = -m_0^2 \mathrm{Tr}(A_\mu^L A_\mu^L + A_\mu^R A_\mu^R) + B\,\mathrm{Tr}(A_\mu^L U A_\mu^R U^\dagger)$$

ω-meson Skyrme model

- Adkins and Nappi 1984:

$$\mathcal{L} = \frac{1}{16}\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{m^2}{8}\mathrm{tr}(U - \mathbb{I}_2) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}\omega_\mu\omega^\mu + \beta\omega_\mu B^\mu$$

- $m = m_\pi / m_\omega = 0.176$
- Coupling constant: $\beta_{AN} = 96.7$, $\beta_{Sutcliffe} = 34.7$.

ω-meson Skyrme model

- More geometric formulation: $M=\mathbb{R}^3$, $N=S^3\subset\mathbb{R}^4$,
  - $\phi: M\to N$,
  - $\omega_0\in C^\infty(M)$,
  - $\omega\in\Omega^1(M)$
  - $\Omega = vol_N/Vol(N)$ (so $B_0=\phi^*\Omega$)
- Static field equations

$$\begin{aligned}
\frac{1}{4}\tau(\phi)+\frac{m^2}{4}(\nabla\sigma)\circ\phi-\beta*(d\omega_0\wedge\Xi_\phi) &= 0\\
\Delta\omega_0+\omega_0+\beta*\phi^*\Omega &= 0\\
\delta d\omega+\omega &= 0
\end{aligned}$$

$$\text{where}\qquad \langle X,\Xi_\phi(Y_1,Y_2)\rangle = \Omega(X,d\phi(Y_1),d\phi(Y_2))$$

- E.g. for a hedgehog $\phi(r\mathbf{n})=(\cos f(r),\sin f(r)\mathbf{n})$

$$\Xi_\phi = -\frac{\sin^2 f}{2\pi^2}\Omega_{S_1^2}(-\sin f,\cos f\mathbf{n})-\frac{f'\sin f}{2\pi^2}dr\wedge(0,\mathbf{n}\times d\mathbf{n})$$

ω-meson Skyrme model

- Coincides with constrained variational problem

$$\begin{aligned} E_\omega(\phi,\omega_0) &= \frac{1}{8}\|\mathrm{d}\phi\|^2+\frac{m^2}{4}\int_M(1-\sigma\circ\phi)+\frac{1}{2}\|\mathrm{d}\omega_0\|^2+\frac{1}{2}\|\omega_0\|^2 \\ (\Delta+1)\omega_0 &= -\beta*\phi^*\Omega \end{aligned}$$

- $\phi\Rightarrow\omega_0$, nonlocal functional $E_\omega(\phi)$
- Cf sextic model

$$E_6(\phi)=\frac{1}{8}\|\mathrm{d}\phi\|^2+\frac{m^2}{4}\int_M(1-\sigma\circ\phi)+\frac{\beta^2}{2}\|\phi^*\Omega\|^2$$

$$\frac{1}{4}\tau(\phi)+\frac{m^2}{4}(\nabla\sigma)\circ\phi-\beta^2*(\mathrm{d}*\phi^*\Omega)\wedge\Xi_\phi)=0$$

E_ω versus E_6

- Energy bounds: compact M

$$E_\omega(\phi),\, E_6(\phi) \geq \frac{C}{Vol(M)} B^2$$

- $M = \mathbb{R}^3$:

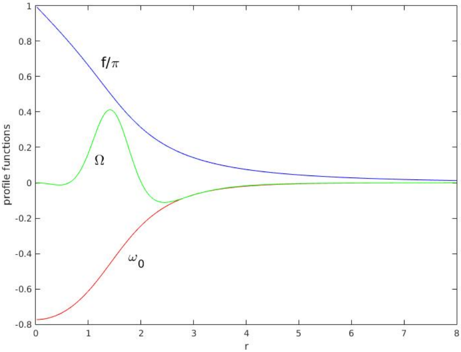
$$E_6(\phi) \geq C' B$$

- Unfortunately $E_\omega(\phi) \leq E_6(\phi)$, so doesn't imply bound on E_ω
- **Thm** For all $\beta^2 \geq 1/4$, $Id : N \to N$ is E_6 stable
- **Thm** If N is Einstein, there exists $\beta_0 \geq 0$ s.t. for all $\beta^2 \geq \beta_0^2$, $Id : N \to N$ is E_ω stable

The E_ω skyrmion

- Supports hedgehog solution $\phi(r\mathbf{n}) = (\cos f, \sin f\mathbf{n}), \omega_0(r)$

$$
\begin{aligned}
f'' + \frac{2}{r}f' - \frac{\sin 2f}{r^2} - m^2 \sin f + \frac{2\beta}{\pi^2 r^2}\omega_0' \sin^2 f &= 0 \\
\omega_0'' + \frac{2}{r}\omega_0' - \omega_0 + \frac{\beta}{2\pi^2 r^2} f' \sin^2 f &= 0
\end{aligned}
$$



The perturbation

$$\mathcal{L} = \{\cdots\} - \frac{\kappa}{4}\omega^{\mu\nu}\Pi_{\mu\nu}$$

where $\Pi_{\mu\nu} = \partial_\mu \pi_1 \partial_\nu \pi_2 - \partial_\nu \pi_1 \partial_\nu \pi_2$

- Has terms linear in ∂_t. E.g. for isospinning hedgehog

$$\kappa \partial_r \omega_0 (\partial_t \pi_1 \partial_r \pi_2 - \partial_t \pi_2 \partial_r \pi_1)$$

- Lorentz and parity invariant
- Static field equations

$$\begin{aligned} \frac{1}{4}\tau(\phi) + \frac{m^2}{4}(\nabla\sigma)\circ\phi - \beta * (d\omega_0 \wedge \Xi_\phi) & \\ + \frac{\kappa}{2} * (d\omega \wedge \phi^* d\pi_1 \nabla\pi_2 \circ \phi - d\omega \wedge \phi^* d\phi_2 \nabla\pi_1 \circ \phi) &= 0 \\ \Delta\omega_0 + \omega_0 + \beta^*\Omega &= 0 \\ \delta d\omega + \omega + \frac{\kappa}{2}\delta(\phi^* d\pi_1 \wedge d\pi_2) &= 0 \end{aligned}$$

- Bad news: no longer supports hedgehog ansatz. The $B = 1$ skyrmion is only axially symmetric. PDEs!!

The perturbation

- Perturbative calculation: $\omega = O(\kappa)$
- ϕ, ω_0 unperturbed to leading order
- Hedgehog has

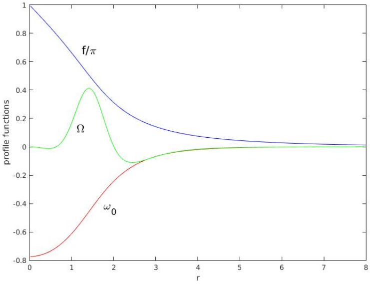
$$\delta(\phi^* \mathrm{d}\pi_1 \wedge \mathrm{d}\pi_2) = P_f(r) \sin^2\theta \mathrm{d}\phi$$

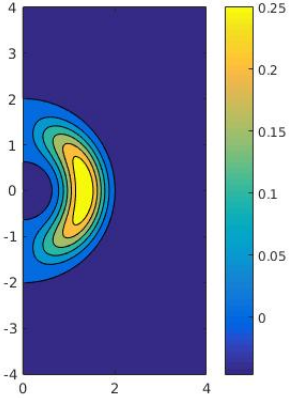
Nice fact: $\delta \mathrm{d}(P(r)\sin^2\theta \mathrm{d}\phi) = [-P''(r) + 2r^{-1}P(r)]\sin^2\theta \mathrm{d}\phi$

- To leading order, still an ODE problem! $\omega = \kappa W(r)\sin^2\theta \mathrm{d}\phi$,

$$\begin{aligned} f'' + \frac{2}{r}f' - \frac{\sin 2f}{r^2} - m^2 \sin f + \frac{2\beta}{\pi^2 r^2}\omega_0' \sin^2 f &= 0 \\ \omega_0'' + \frac{2}{r}\omega_0' - \omega_0 + \frac{\beta}{2\pi^2 r^2} f' \sin^2 f &= 0 \\ W'' - \left(1 + \frac{2}{r^2}\right) W - \frac{1}{8}\left(F'' - \frac{1}{r^2}F + \frac{1}{r^2}\right) &= 0, \quad F := \cos 2f \end{aligned}$$

The perturbed skyrmion





Rigid body quantization

- Classical static solution $(\phi_H, \omega_0, \omega)$
- Spin-isospin symmetry group: $G = SU(2) \times U(1)$

$$(g,\lambda) : (\phi_H, \omega_0, \omega) \mapsto (h(\lambda)(\phi_H \circ \mathscr{R}_{g^{-1}})h(\lambda)^\dagger, \omega_0 \circ \mathscr{R}_{g^{-1}}, \mathscr{R}^*_{g^{-1}}\omega)$$

where $h(\lambda) = \mathrm{diag}(\lambda, \bar{\lambda})$ and $\mathscr{R}_g : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the orthogonal linear map defined so that

$$(\mathscr{R}_g \mathbf{x}) \cdot i\tau = g(\mathbf{x} \cdot i\tau)g^{-1}$$

- Isotropy group $H = \{(\pm h(\lambda), \lambda) : \lambda \in U(1)\} = U(1) \times \mathbb{Z}_2$
- Orbit of static solution $\mathscr{M} \equiv G/H \equiv SU(2)/\mathbb{Z}_2$

$$\{\pm g\} \mapsto (U_H, \omega_0, \omega)_{(g,1)}$$

Induced action of G on $SU(2)/\mathbb{Z}_2$

$$(g,\lambda) : \{\pm g'\} \mapsto \{\pm h(\lambda)^\dagger g' g\}$$

Rigid body quantization

- Restrict field theory Lagrangian $L = \int_M \mathcal{L}$ to $\mathscr{M}$, i.e. compute for $(U_H, \omega_0, \omega)_{(g(t),1)}$
- $\mathcal{L}$ a quadratic polynomial in time derivatives

$$L(g, \dot{g}) = \frac{1}{2}\gamma(\dot{g}, \dot{g}) + A(\dot{g}) - M_0$$

  where γ, A, M_0 are a metric, one form and function on $\mathscr{M}$
- L invariant under induced G action:

$$\gamma = \Lambda_1(\Sigma_1^2 + \Sigma_2^2) + \Lambda_3\Sigma_3^2, \quad A = C\Sigma_3, \quad M_0 = const$$

  where Σ_a are **right** invariant one forms on $SU(2)$ dual to $-i\tau_a/2$
- Up to here: true for exact axial solution also

Rigid body quantization

Explicit computation (perturbed hedgehog)

$$
\begin{aligned}
\Lambda_1 &= \frac{2\pi}{3}\int_0^\infty r^2 \sin^2 f\, dr + O(\kappa^2) \\
\Lambda_3 &= \Lambda_1 + O(\kappa^2) \\
C &= \kappa C_* + O(\kappa^2) \\
C_* &= \frac{4}{3}\int_0^\infty \mathit{complicated}(f, \omega_0, W)\, dr + O(\kappa) \\
M_0 &= \int_0^\infty \mathit{complicated}(f, \omega_0)\, dr + O(\kappa^2).
\end{aligned}
$$

$M_0 =$ static energy of hedgehog

Rigid body quantization

$$L = \frac{1}{2}\gamma(\dot{g},\dot{g}) + A(\dot{g}) - M_0$$

- Fermionic quantization: lift dynamics to double cover $\widetilde{\mathscr{M}} \equiv SU(2)$

$$\psi : SU(2) \to \mathbb{C}, \qquad \psi(-g) = -\psi(g)$$

- Unit mass particle moving on mfd $(SU(2),\gamma)$ under influence of "magnetic field" $B = \mathrm{d}A$:

$$H\psi = -\frac{1}{2} * \mathrm{d}_A * \mathrm{d}_A\psi + M_0\psi$$

$\mathrm{d}_A = \mathrm{d} - iA$.

$$H\psi = -\frac{1}{2\Lambda_1}(\Theta_1^2 + \Theta_2^2 + \Theta_3^2 - 2i\kappa C_*\Theta_3)\psi + M_0\psi + O(\kappa^2)$$

Rigid body quantization

- Reexpress in terms of angular momentum operators.
  - Spatial rotation about j axis $\leftrightarrow$ **right** multiplication by $\exp(i\alpha\tau_j/2)$, generated by **left** invariant vector field $-\theta_j$.

$$S_j = -i(-\theta_j)$$

  - Isorotation about 3 axis $\leftrightarrow$ **left** multiplication by $\exp(-i\alpha\tau_3/2)$, generated by **right** invariant vector field Θ_3.

$$I_3 = -i\Theta_3$$

- $\Theta_1^2 + \Theta_2^2 + \Theta_3^2 = \theta_1^2 + \theta_2^2 + \theta_3^2$, so

Rigid body quantization

$$H = \frac{1}{2\Lambda_1}|\mathbf{S}|^2 - \frac{\kappa C_*}{\Lambda_1} I_3 + M_0$$

- Spectrum $H|S, I_3\rangle = \left(\frac{S(S+1)}{2\Lambda_1} - \frac{\kappa C_* I_3}{\Lambda_1} + M_0\right)|S, I_3\rangle$
  - Proton $|1/2, 1/2\rangle$,

$$m_p = M_0 + \frac{1}{2\Lambda_1}\left(\frac{3}{4} - \kappa C_*\right)$$

  - Neutron $|1/2, -1/2\rangle$,

$$m_p = M_0 + \frac{1}{2\Lambda_1}\left(\frac{3}{4} + \kappa C_*\right)$$

- For $\beta_{Sutcliffe}$, $\kappa = -0.08075$ gives correct mass splitting

Electric charge density

- Noether current associated to isospin symmetry

$$J^{\mu} \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\pi_a)}\Delta\pi_a, \qquad \Delta\pi = (\pi_2, -\pi_1, 0)$$

- Electric charge density?

$$\rho_e(\mathbf{x}) = \frac{1}{2}B_0(\mathbf{x}) + I_3\frac{J_0(\mathbf{x})}{\int_{\mathbb{R}^3} J_0}$$

 where J_0 is the Noether charge density of a classical isospinning hedgehog.

- Problem: $\exp(-iv\tau_3/2)U_H\exp(iv\tau_3/2)$ has

$$J_0(\mathbf{x}) = \left(\frac{v}{4}\sin^2 f + \frac{\beta\kappa}{2\pi^2 r^2}Wf'\sin f + \frac{\kappa}{4}\omega_0'(\sin 2f)'\right)\sin^2\theta$$

 Not homogeneous in v! How should v be chosen?

Electric charge density

- $J_0 = v\rho_1 + \rho_2$

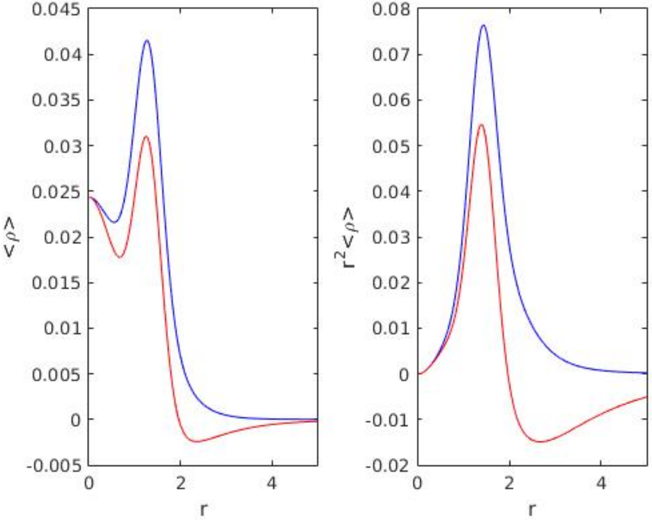
$$\rho_e(\mathbf{x}) = \frac{1}{2}B_0(\mathbf{x}) + [v\rho_1(\mathbf{x}) + \rho_2(\mathbf{x})]$$

Choose v such that $\int \rho_e = 1$ (proton) or $\int \rho_e = 0$ (neutron)

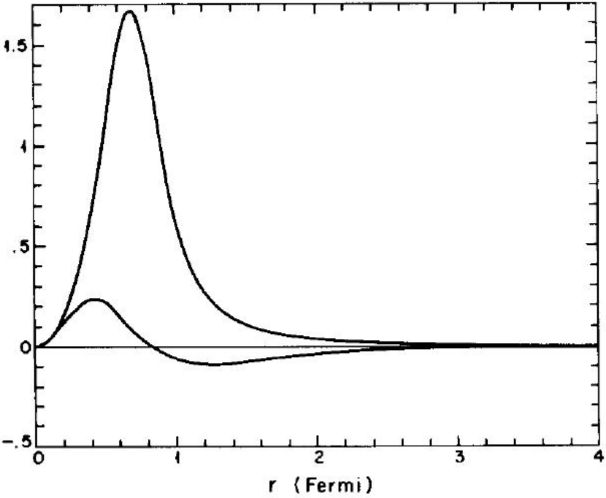
- $\int \rho_1 = \Lambda_1$, $\int \rho_2 = \kappa C_*$ (of course)

$$v_p = \frac{1}{\Lambda_1}\left(\frac{1}{2} - \kappa C_*\right)$$
$$v_n = -\frac{1}{\Lambda_1}\left(\frac{1}{2} + \kappa C_*\right)$$

Electric charge density $\beta_{Sutcliffe}$, κ_{tuned}



Electric charge density β_{AN}, $\kappa = 0$



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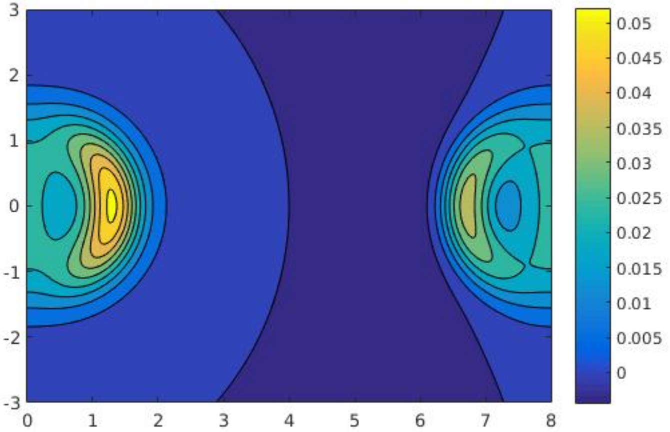

Conclusion

- Very simple perturbation of $\mathcal{L}_\omega$ can produce p-n mass splitting
- Beyond perturbative calculation: (U, ω_0, ω) axially symmetric, $\Lambda_3 \neq \Lambda_1$

$$E(s, I_3) = \frac{s(s+1)}{2\Lambda_1} - \frac{CI_3}{\Lambda_3} + M_0 + \frac{C^2}{2\Lambda_3} + \frac{\Lambda_1 - \Lambda_3}{2\Lambda_1\Lambda_3} I_3^2.$$

- Isospin only softly broken: quantize motion on whole $G' = SU(2) \times SU(2)$ orbit.
  - $\mathscr{M}'$ 5 dimensional
  - γ,A, M_0 only G invariant (M_0 =potential)
- Higher B: would like proper solutions of *unperturbed* model...