# The Geometry of Soliton Moduli Spaces

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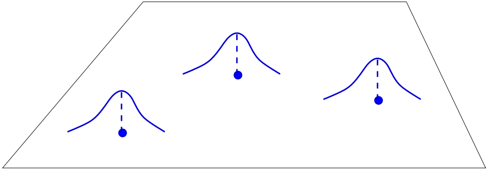
  hypothetical particles, resolve many theoretical puzzles in HEP
- . . . only **better!**
  exist in real world: magnetic flux tubes in superconductors, magnetic bubbles, optical pulses, crystal dislocations

# Soliton moduli spaces

- Interesting special case: static solitons exert no net force on each other



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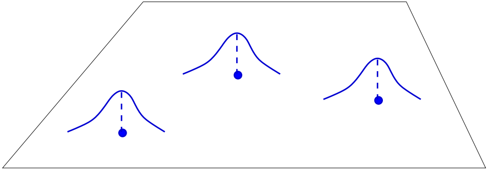

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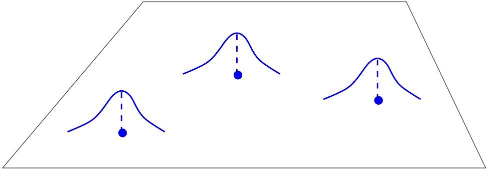

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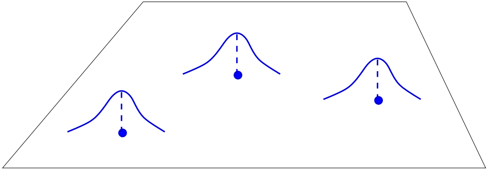

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- Low energy dynamics reduces to **geodesic** motion in $M_n$!
- Soliton dynamics $\longleftrightarrow$ Riemannian geometry

# Plan

- Planar antiferromagnets $\longrightarrow \mathbb{C}P^1$ model
- The Bogomol'nyi argument, $\mathsf{M}_n$
- The metric on $\mathsf{M}_n$, soliton scattering
- Other solitons
- Open problems

# Antiferromagnets

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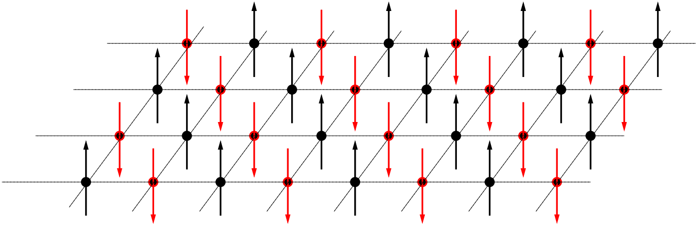

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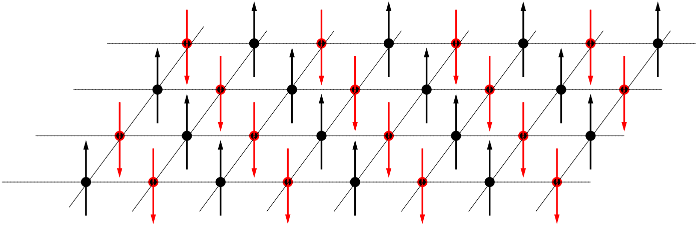

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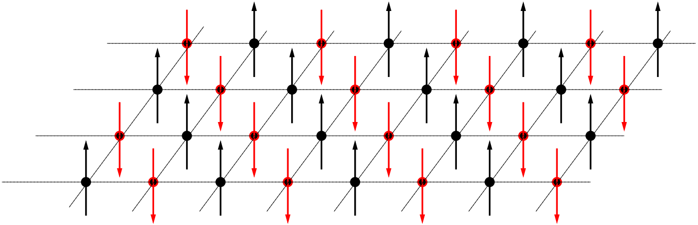

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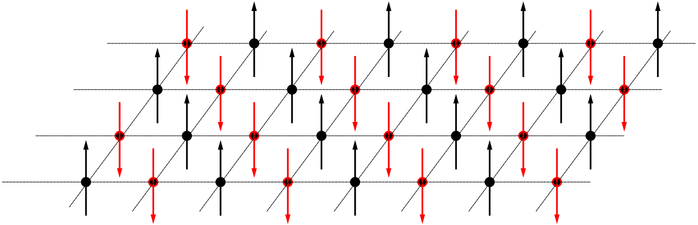

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- Continuum limit?

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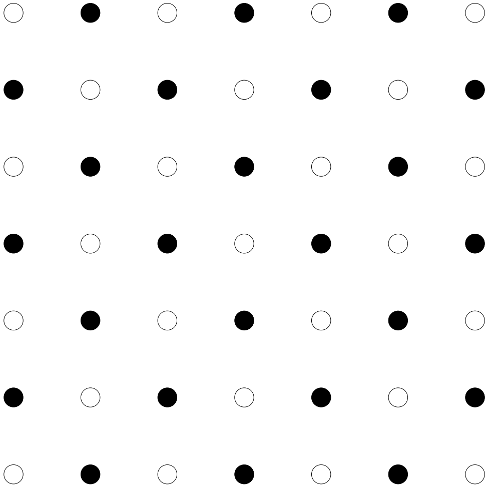

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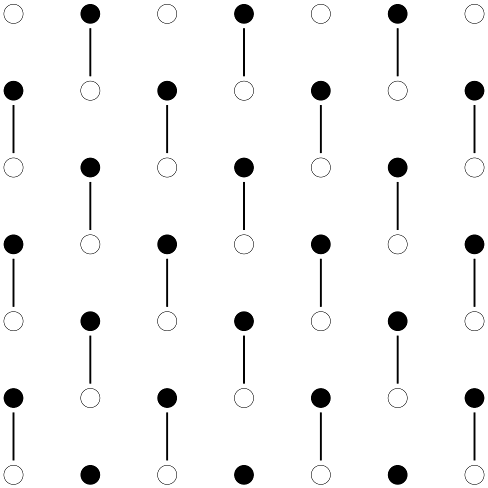

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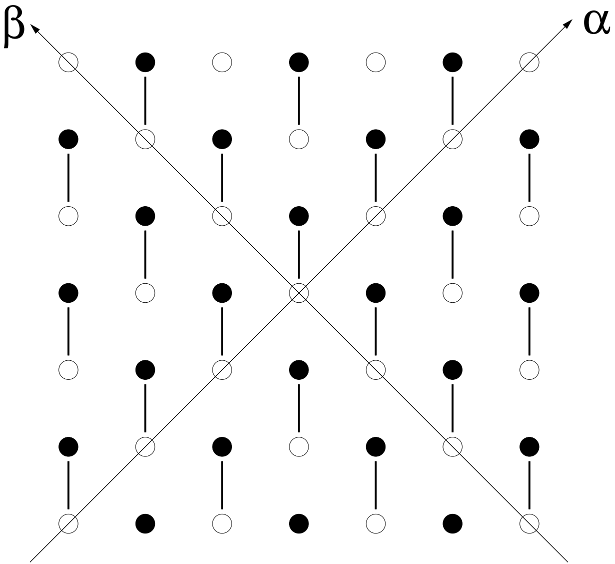

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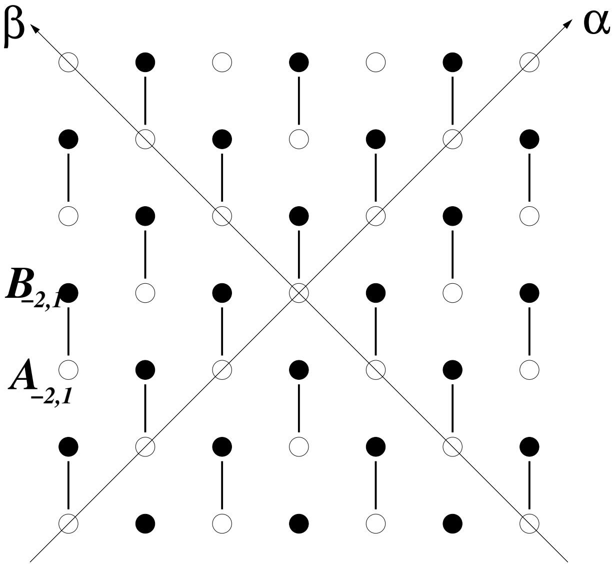

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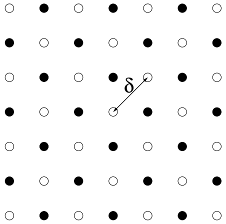


- $x = \alpha\delta$, $y = \beta\delta$, $t = 2\tau\delta$
- Assumption:

$$\left.\begin{array}{l}\mathbf{A}_{\alpha,\beta}\\ \mathbf{B}_{\alpha,\beta}\end{array}\right\} \xrightarrow{\delta\to 0} \left\{\begin{array}{l}A(x,y)\\ B(x,y)\end{array}\right.$$

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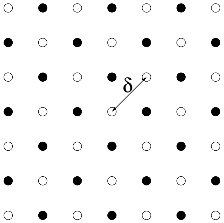


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- Replace $\mathbf{A}_{\alpha+1,\beta}$ by $A + \delta A_x + \frac{1}{2}\delta^2 A_{xx} + \cdots$ etc
- Work to order $\delta^2$

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2\delta A_t &= -A \times [4B - 2\delta(B_x + B_y) + \delta^2(B_{xx} + B_{yy} + B_{xy})] \\
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\mathbf{m}_t &= -(\partial_x + \partial_y)[\mathbf{m} \times \varphi] + \frac{\delta}{4}[2\varphi \times (\varphi_{xx} + \varphi_{yy} + \varphi_{xy}) && (1) \\
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- Subst in (1): $\varphi \times \varphi_{tt} = \varphi \times (\varphi_{xx} + \varphi_{yy}) + O(\delta)$

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  Nonlinear wave equation! Lorentz invariant!

- Variational formulation: action of field $\varphi : \mathbb{R} \times \Sigma \to S^2$

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- Physicists call this the $\mathbb{C}P^1$ model

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- Topological lower energy bound:

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$$\begin{aligned} 0 &\leq \frac{1}{2}\int_\Sigma |\varphi_x + \varphi\times\varphi_y|^2 = E - \int_\Sigma \varphi\cdot(\varphi_x\times\varphi_y) = E - 4\pi n \\ E &\geq 4\pi n \\ E &= 4\pi n \quad \Leftrightarrow \quad \varphi_x + \varphi\times\varphi_y = 0 \qquad \textbf{1st order PDE!} \end{aligned}$$

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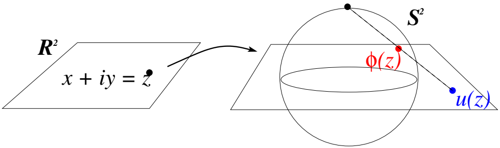
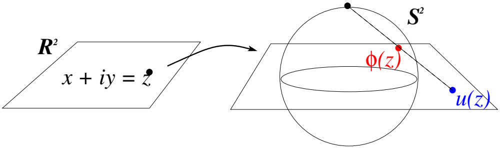
- $\Sigma = \mathbb{R}^2 \cong \mathbb{C}$ is also a complex manifold.
  $J_\Sigma \partial_x = \partial_y, \qquad J_\Sigma \partial_y = -\partial_x$
- Bogomol'nyi equation equivalent to $\mathrm{d}\varphi \circ J_\Sigma = J \circ \mathrm{d}\varphi$
  that is, $\varphi : \Sigma \to S^2$ is **holomorphic**

# Moduli space




$$u(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_n z^n}$$

# Moduli space

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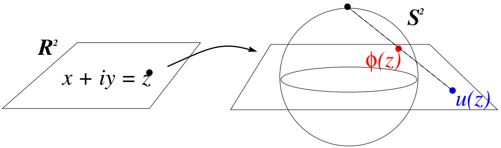


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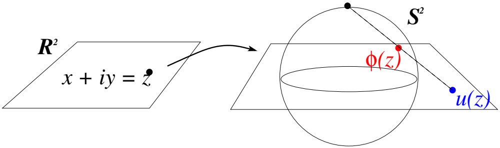


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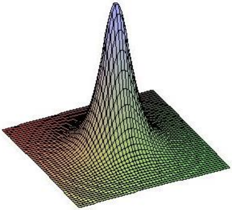
# Moduli space

- $M_1 = \mathbb{C} \times \mathbb{C}^{\times}$

$$u(z) = \frac{a_0 + z}{b_0}$$

- Position $-a_0$, width $|b_0|$, orientation $arg(b_0)$



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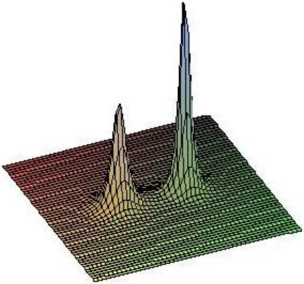

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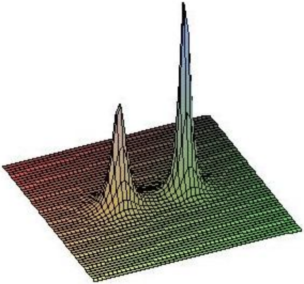
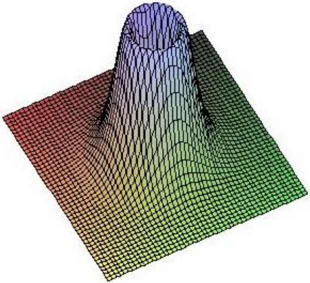
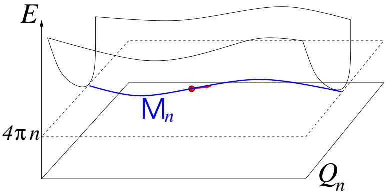
- Lose identity when close, e.g. $u = z^2$





# Geodesic approximation (Ward, after Manton)




$$E = \frac{1}{2}\int_{\Sigma} |\varphi_x|^2 + |\varphi_y|^2$$

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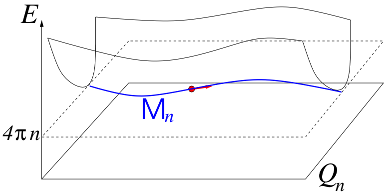


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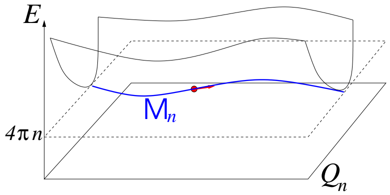


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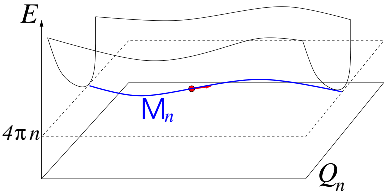


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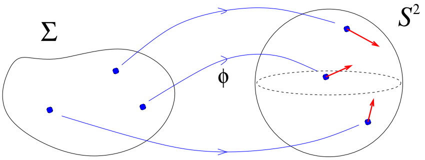
**Geodesic** motion on $(\mathsf{M}_n, \gamma)$ where $\gamma = L^2$ metric.

# $L^2$ metric

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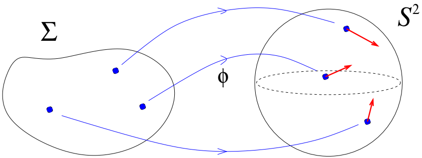


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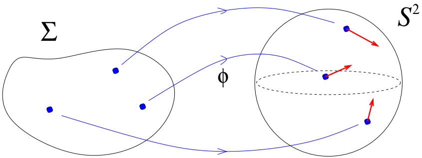


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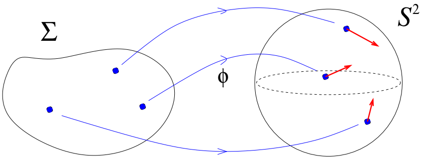


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- Geodesic motion is constant speed motion along "straightest possible" curve

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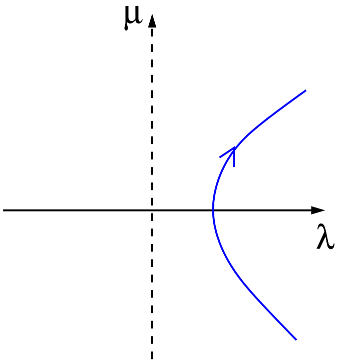


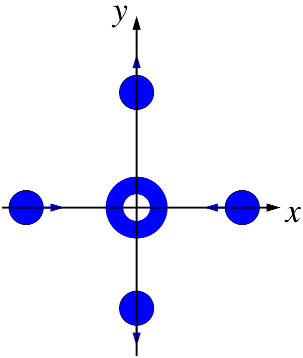

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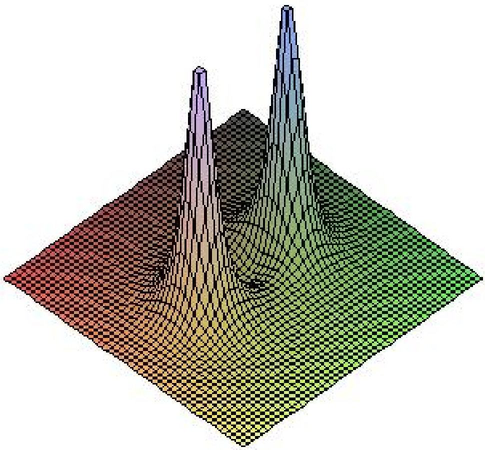

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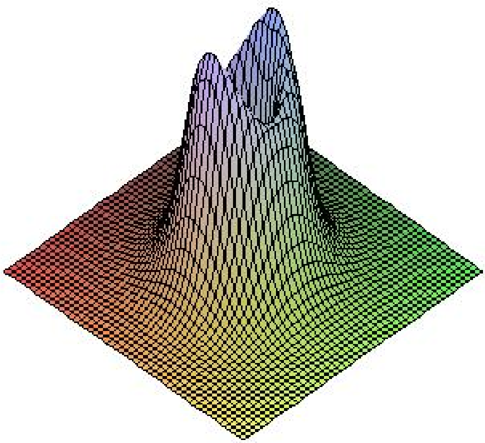

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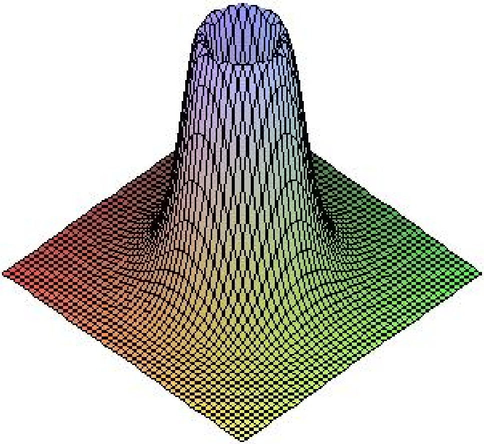

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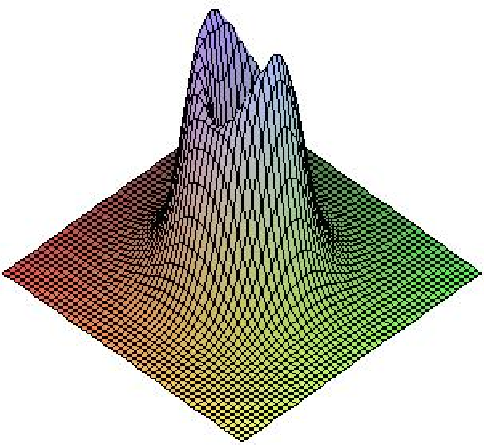

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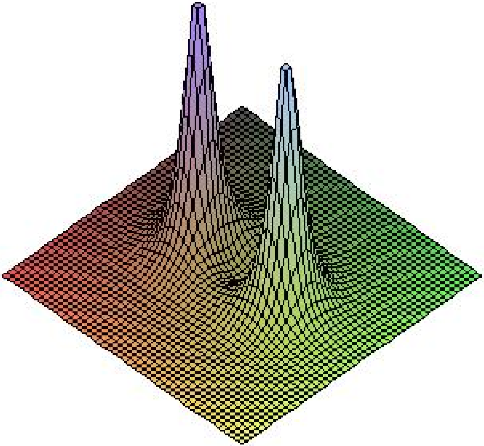

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- Consequence: centre of mass motion decouples, $\mathsf{M}_n = \mathbb{C} \times \mathsf{M}_n^{\mathrm{red}}$

  $$\gamma = 4\pi n \gamma_{\mathrm{Euc}} + \gamma_{\mathrm{red}}$$

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- $\Sigma, N$ Kähler, keeps key features: $n = [\varphi^*\omega]$
  $M_n = \mathrm{hol}_n(\Sigma, N)$, Kähler

# Other solitons: Gauge theory

- $\varphi$ a section of a complex vector bundle $W$ over $\Sigma$
- $\nabla =$ unitary connexion on $W$, curvature $F$

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- Calorons: $\Sigma^4 = \mathbf{S}^1 \times \mathbb{R}^3, T^2 \times \mathbb{R}^2, \ldots$

# Other solitons: Gauge theory

- Instantons: $\Sigma^4 = S^4, \mathbb{R}^4$, $W = C^2$ bundle, no $\varphi$,
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- Calorons: $\Sigma^4 = \mathbf{S}^1 \times \mathbb{R}^3, T^2 \times \mathbb{R}^2, \ldots$
- Monopoles = translation invariant instantons
- Vortices = $SO(3)$ invariant instantons

# Open questions: Geometry

- Volume, diameter of $M_n$?
- Curvature properties?
- Periodic geodesics?
- Ergodicity?
- Symplectic geometry of $(M_n, \omega)$?

# Open questions: Quantization

$$
\begin{tikzcd}
\text{Classical Field Theory} \arrow[r, "\text{quantize}"] \arrow[d, "\text{truncate}"'] & \text{Quantum Field Theory} \arrow[d, "\text{truncate}"'] \\
\text{Geodesic Approximation} \arrow[r, "\text{quantize}"] & ???
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- Wavefunction $\psi : \mathbb{R} \times \mathsf{M}_n \to \mathbb{C}$

$$i\frac{\partial \psi}{\partial t} = \frac{1}{2}\Delta\psi$$

Spectral geometry of $\mathsf{M}_n$

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$$\varphi_\varepsilon(\tau) = \varphi(\tau/\varepsilon).$$

Then there exists $T > 0$ such that $\varphi_\varepsilon : [0, T] \times \Sigma \to N$ converges uniformly, as $\varepsilon \to 0$, to $\psi(\tau)$, the geodesic in $\mathsf{M}_n$ with initial data $(\varphi_0, \varphi_1)$.

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- Proved for vortices, monopoles (Stuart), $\mathbb{C}P^1$ lumps on $T^2$ (JMS)

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- Further reading: “Topological Solitons” Manton and Sutcliffe