Solitons and point particles

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What are topological solitons?

- Smooth, spatially localized solutions of nonlinear relativistic classical field theories
- Stable; in fact *topologically* stable
- Have relativistic kinematics: $E(v) = \dfrac{E(0)}{\sqrt{1 - v^2}}$
- Have antisolitons

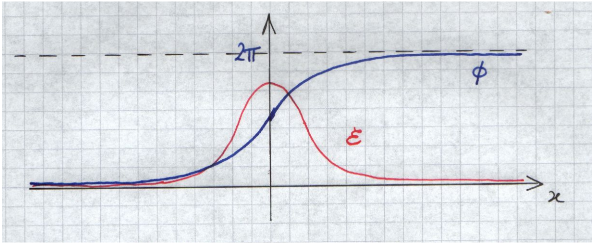
Rather like classical point particles. In particular, fields away from soliton core often look like fields induced in linear field theories by point particle sources.
Idea: forces between widely separated solitons should be the same as forces between appropriate point particles interacting via linear field theory. (Originally applied by Manton to BPS monopoles.)

Sine-Gordon Kinks

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - (1 - \cos\phi)$$

$$\partial_\mu\partial^\mu\phi + \sin\phi = 0$$

- Vacua: $\phi(x) = 2n\pi$
- Kink: $\phi(x) = 4\tan^{-1} e^{x-x_0} \sim \begin{cases} 4e^{x-x_0} & x \to -\infty \\ 2\pi - 4e^{-(x-x_0)} & x \to \infty \end{cases}$



Linearized model

- Linearize about vacuum: $\phi = 2n\pi + \psi$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}\psi^2 + \cdots$$

- Klein-Gordon theory

$$\partial_\mu\partial^\mu\psi + \psi = 0$$

- General static soln: $\psi(x) = c_1 e^x + c_2 e^{-x}$

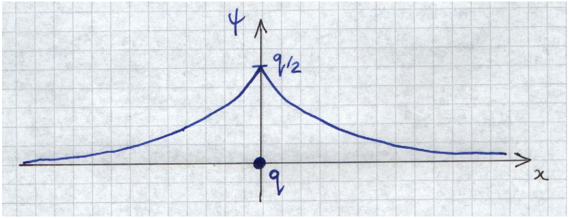
Linearized model

- Add sources:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}\psi^2 + \kappa\psi$$

$$\partial_\mu\partial^\mu\psi + \psi = \kappa$$

- Scalar monopole $\kappa(x) = q\delta(x)$ induces $\psi(x) = \frac{q}{2}e^{-|x|}$



Linearized model

- Interaction between a pair of sources $\kappa_1(x,t), \kappa_2(x,t)$ inducing fields $\psi_1(x,t), \psi_2(x,t)$:

$$
\begin{aligned}
S &= \int \{\frac{1}{2}\partial_\mu(\psi_1+\psi_2)\partial^\mu(\psi_1+\psi_2) - \frac{1}{2}(\psi_1+\psi_2)^2 \\
&\qquad +(\kappa_1+\kappa_2)(\psi_1+\psi_2)\} dt\, dx \\
&= S_1 + S_2 + \int \{\partial_\mu\psi_1\partial^\mu\psi_2 - \psi_1\psi_2 + \kappa_1\psi_2 + \kappa_2\psi_1\}\, dt\, dx \\
&= S_1 + S_2 + \int \{\psi_1(-\partial_\mu\partial^\mu\psi_2 - \psi_2 + \kappa_2) + \kappa_1\psi_2\}\, dt\, dx \\
&= S_1 + S_2 + \int \kappa_1\psi_2\, dt\, dx \\
L_{\mathrm{int}} &= \int \kappa_1\psi_2\, dx
\end{aligned}
$$

Linearized model

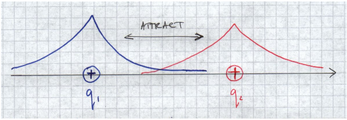
- Two scalar monopoles: $\kappa_1 = q_1 \delta(x - x_1), \kappa_2 = q_2 \delta(x - x_2)$

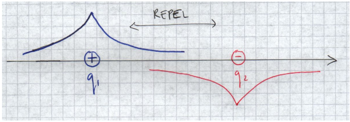
$$L_{\mathrm{int}} = \int_{-\infty}^{\infty} q_1 \delta(x - x_1) \frac{q_2}{2} e^{-|x - x_2|} dx = \frac{1}{2} q_1 q_2 e^{-|x_1 - x_2|}$$

- Interaction potential

$$V_{\mathrm{int}} = -L_{\mathrm{int}} = -\frac{1}{2} q_1 q_2 e^{-|x_1 - x_2|}$$

Like charges attract, unlike repel!





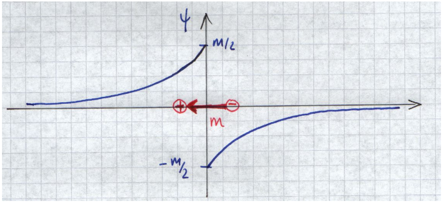
"Point kinks"

- Compare with kink asymptotics:

$$\psi_K(x) \sim \begin{cases} 4e^{-|x|} & x < 0 \\ -4e^{-|x|} & x > 0 \end{cases}$$

- Looks like a monopole of charge $q = 8$ for $x < 0$, charge $q = -8$ for $x > 0$. Dipole?
- Yes! Dipole source $\kappa(x) = m\delta'(x)$ with $m = 8$ induces field

$$\psi(x) = \frac{d}{dx}\frac{m}{2}e^{-|x|} = \psi_K(x)$$



Kink interactions

- Kink-Kink interaction at long range should be the same as interaction between a pair of equal dipoles (kink-antikink: opposite dipoles)
- Interaction Lagrangian $\kappa_1 = m_1\delta'(x - x_1)$, $\kappa_2 = m_2\delta'(x - x_2)$:

$$
\begin{aligned}
L_{\text{int}} &= \int_{-\infty}^{\infty} m_1\delta'(x - x_1)\psi_2\,dx = -m_1 \left.\frac{d\psi_2}{dx}\right|_{x=x_1} \\
&= -\frac{1}{2} m_1 m_2 e^{-|x_1 - x_2|}
\end{aligned}
$$

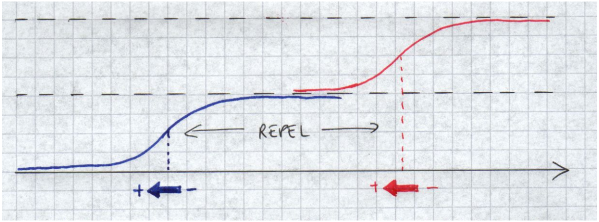
- Interaction potential

$$
V_{\text{int}} = \frac{1}{2} m_1 m_2 e^{-|x_1 - x_2|}
$$

Kink interactions

- Kink-Kink: $m_1 = m_2 = 8$

$$V_{\text{int}} = 32e^{-|x_1 - x_2|} \qquad \text{repulsive}$$



- Kink-Antikink: $m_1 = 8$, $m_2 = -8$

$$V_{\text{int}} = -32e^{-|x_1 - x_2|} \qquad \text{attractive}$$

- Turns out to be correct! Perring and Skyrme (1962), Rajaraman (1977)

Vortices in (2+1)D

- Abelian Higgs model: $D_\mu = \partial_\mu + iA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L} = \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\mu^2}{8}(1 - |\phi|^2)^2$$

- Gauge invariant: $\phi \mapsto e^{i\Lambda}\phi$, $A_\mu \mapsto A_\mu - \partial_\mu \Lambda$
- Magnetic flux quantization:
  finite energy $\Rightarrow |\phi| \to 1$, $\mathbf{D}\phi \to 0$ as $r \to 0$. So at large r

$$\phi \sim e^{i\chi(\theta)}, \qquad \mathbf{A} \sim -i \frac{\nabla \phi}{\phi} \sim \nabla \chi(\theta)$$

Hence, by Stokes,

$$\int_{\mathbb{R}^2} B = \int_{S^1_\infty} \mathbf{A} \cdot d\ell = 2\pi(\chi(2\pi) - \chi(0)) = 2\pi n$$

where $n =$ winding of ϕ around unit circle.

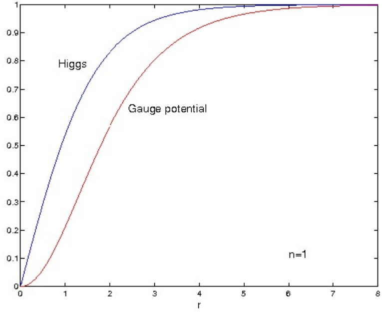
Vortices

- Vortex ansatz:

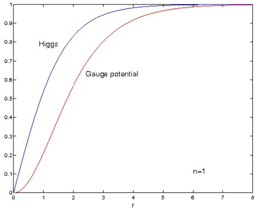
$$\phi = \sigma(r)e^{i\theta}; \qquad A^0 = 0, \qquad \mathbf{A} = \frac{a(r)}{r^2}(-y, x)$$

$\sigma(r)$, $a(r)$ profile functions, both $\to 1$ as $r \to \infty$.
Field equations $\Rightarrow$ coupled nonlinear ODEs for σ, a.

- Exact solutions not known. Numerics:



Vortex asymptotics



- Asymptotics: for $\mu \leq 2$,

$$\sigma(r) \sim 1 + \frac{q}{2\pi} K_0(\mu r)$$
$$a(r) \sim 1 - \frac{m}{2\pi} r K_0'(r)$$

  where K_0 = modified Bessel function of the second kind, q, m are unknown constants.

- Note $K_0(r) \sim \sqrt{\frac{\pi}{2r}} e^{-r}$

Linearized model

- Try same trick: replicate far-field of vortex in linear theory by introducing appropriate point sources.
- Linearize AHM: choose real gauge, $\phi = 1 + \psi$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{\mu^2}{2}\psi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}A_\mu A^\mu + \kappa\psi - j_\mu A^\mu$$

- Field equations

$$(\partial_\mu\partial^\mu + \mu^2)\psi = \kappa, \qquad (\partial_\mu\partial^\mu + 1)A^\nu = j^\nu$$

Klein-Gordon-Proca theory: ψ scalar boson (Higgs) of mass μ, A^μ vector boson (photon) of mass 1.

Point vortices

- Unwound vortex (after gauge transform $\phi \mapsto e^{-i\theta}\phi$)

$$\psi = \frac{q}{2\pi} K_0(\mu r), \qquad \mathbf{A} = -\frac{m}{2\pi}\mathbf{k} \times \nabla K_0(r)$$

- Green's function for Helmholtz equation:

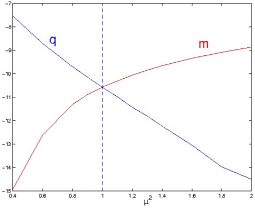
$$(-\nabla^2 + \mu^2) K_0(\mu r) = 2\pi\delta(\mathbf{x})$$

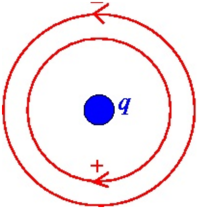
- So asymptotic vortex fields induced by

$$\begin{aligned} \kappa &= q\delta(\mathbf{x}) && \text{scalar monopole, charge } q \\ \mathbf{j} &= -m\mathbf{k} \times \nabla\delta(\mathbf{x}) && \text{magnetic dipole of moment } m\mathbf{k} \end{aligned}$$

Composite point source, "point vortex"

Point vortices





- $m < 0$: *clockwise* current loop!
- At $\mu = 1$, $q = m$. Not a coincidence!

Point vortex interactions

- Interaction Lagrangian

$$L_{\mathrm{int}} = \int_{\mathbb{R}^2} \{\kappa_{(1)}\psi_{(2)} - j^{\mu}_{(1)} A^{(2)}_{\mu}\} = L_{\psi} + L_A$$

- Two point vortices at rest at $\mathbf{y}$, $\mathbf{z}$

$$\begin{aligned} L_{\psi} &= \int q\delta(\mathbf{x}-\mathbf{y})\frac{q}{2\pi}K_0(\mu|\mathbf{x}-\mathbf{z}|)\, d^2\mathbf{x} = \frac{q^2}{2\pi}K_0(\mu|\mathbf{y}-\mathbf{z}|) \\ L_A &= \int [-m\mathbf{k}\times\nabla\delta(\mathbf{x}-\mathbf{y})]\cdot[-\frac{m}{2\pi}\mathbf{k}\times\nabla K_0(|\mathbf{x}-\mathbf{z}|)]\, d^2\mathbf{x} \\ &= -\frac{m^2}{2\pi}\nabla^2_y K_0(|\mathbf{y}-\mathbf{z}|) = -\frac{m^2}{2\pi}K_0(|\mathbf{y}-\mathbf{z}|) \end{aligned}$$

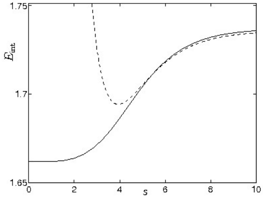
- Interaction potential

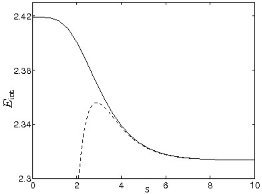
$$V_{\mathrm{int}} = \frac{1}{2\pi}[m^2 K_0(|\mathbf{y}-\mathbf{z}|) - q^2 K_0(\mu|\mathbf{y}-\mathbf{z}|)]$$

Vortex interaction potential

$$V_{\rm int} = \frac{1}{2\pi}[m^2 K_0(s) - q^2 K_0(\mu s)]$$

- $\mu < 1$ attractive - type I
  $\mu > 1$ repulsive - type II
  $\mu = 1$ cancel, $V_{\rm int} = 0$. Not a coincidence!
- Cf constrained minimization:





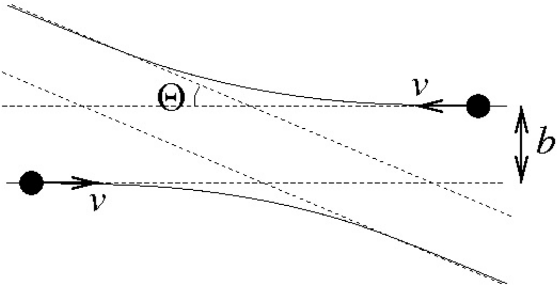
Vortex-antivortex interaction

- Obtain antivortex (winding $n = -1$) by $\phi \mapsto \overline{\phi}$, $\mathbf{A} \mapsto -\mathbf{A}$
- $\bar{q} = q$, $\bar{m} = -m$
- Similar calculation

$$V_{\rm int} = \frac{1}{2\pi}[-m^2 K_0(s) - q^2 K_0(\mu s)]$$

- Always attractive

- Vortex scattering

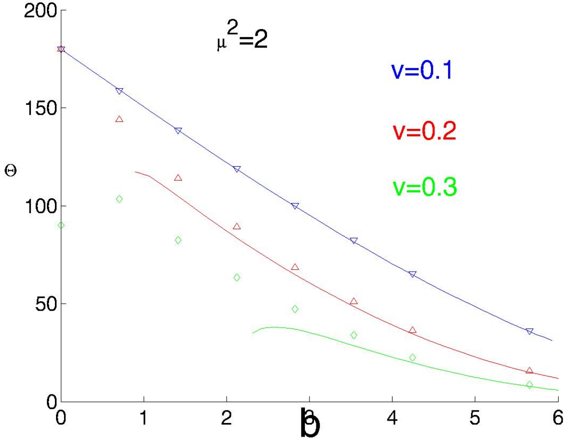


- Simple model:

$$L = \frac{1}{2} M(\mu)(|\dot{\mathbf{y}}|^2 + |\dot{\mathbf{z}}|^2) - V_{\rm int}(|\mathbf{y} - \mathbf{z}|)$$

Vortex scattering

- Cf Myers, Rebbi, Strilka (1992)



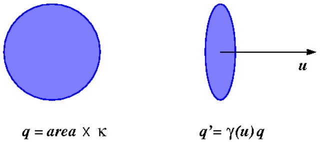
Critical coupling

- $\mu = 1$: Higgs and photon have same mass, $q \equiv m$, so $V_{\text{int}} \equiv 0$
  [Aside: David Tong argues that $q = -2\pi 8^{\frac{1}{4}}$]
  For static (point) vortices, scalar attraction exactly balances magnetic repulsion
- Vortex scattering trivial? Certainly not.
- Scalar attraction mediated by scalar field ψ, magnetic repulsion mediated by vector field A
  Different tranformation properties under Lorentz boosts
  Do not balance for (point) vortices in relative motion
- Can compute L_{int} for point vortices moving along arbitrary trajectories $\mathbf{y}(t)$ and $\mathbf{z}(t)$, as an expansion in time derivatives

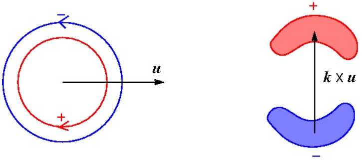
Moving point vortex

Point vortex moving along $\mathbf{y}(t)$ at constant velocity has

- $\kappa(\mathbf{x}, t) = q\left(1 - \frac{1}{2}|\dot{\mathbf{y}}|^2\right)\delta(\mathbf{x} - \mathbf{y}(t))$



- $j^\mu = q(\mathbf{k} \times \dot{\mathbf{y}} \cdot \nabla, -\mathbf{k} \times \nabla + (\mathbf{k} \times \dot{\mathbf{y}})\dot{\mathbf{y}} \cdot \nabla)\delta(\mathbf{x} - \mathbf{y})$



Interaction Lagrangian

$$L_{\text{int}} = \int_{\mathbb{R}^2} \{\kappa_{(1)}\psi_{(2)} - j^{\mu}_{(1)} A^{(2)}_{\mu}\}$$

- Need fields induced by moving point vortex 2
- If linear theory were massless, would use retarded potentials (standard problem)
- Use formal temporal Fourier tranform trick, expansion in time derivatives
- Actually, need to work up to terms linear in $\ddot{\mathbf{y}}, \ddot{\mathbf{z}}$

$$L_{\text{int}} = -\frac{q^2}{4\pi}|\dot{\mathbf{y}} - \dot{\mathbf{z}}|^2 K_0(|\mathbf{y} - \mathbf{z}|)$$

Two vortex dynamics at critical coupling

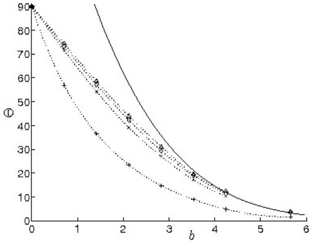
$$L = \frac{\pi}{2}(|\dot{\mathbf{y}}|^2 + |\dot{\mathbf{z}}|^2) - \frac{q^2}{4\pi} K_0(|\mathbf{y} - \mathbf{z}|)|\dot{\mathbf{y}} - \dot{\mathbf{z}}|^2$$

- Geodesic motion on $\mathbb{R}^2 \times \mathbb{R}^2$ wrt to non-Euclidean metric

$$g = \pi \left(1 - \frac{q^2}{2\pi^2} K_0(|\mathbf{y} - \mathbf{z}|)\right)(d\mathbf{y}\cdot d\mathbf{y} + d\mathbf{z}\cdot d\mathbf{z}) + \frac{q^2}{\pi} K_0(|\mathbf{y} - \mathbf{z}|) d\mathbf{y}\cdot d\mathbf{z}$$

  Asymptotic to the Samols metric

- Critical vortex scattering



Concluding remarks

- Point soliton approx. now a standard tool for analyzing long range intersoliton forces
- Formal, but often confirmed by (more or less) rigorous analysis
- BPS monopoles: Manton (1985), Bielawski (1998)
- Static vortices: Speight (1997); moving critical vortices: Manton and Speight (2003)
- Skyrmions and baby Skyrmions: Schroers (1993)
- An aid to numerics e.g. Faddeev-Hopf solitons: Ward (2000); Skyrme sphalerons: Krusch and Sutcliffe (2004)