The L^2 geometry of the space of $\mathbb{P}^1$ vortex-antivortex pairs

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Plan

- Gauged sigma model on Σ^2 with $\mathbb{P}^1$ target: two species of vortex
- Moduli space $=$ {pairs **disjoint** divisors}: noncompact even if Σ compact
- Natural Riemannian metric g_{L^2}. Complete? Volume?
- Consider $\Sigma = \mathbb{C}$, then S^2_R, focus on $(1,1)$ case
- (Almost) explicit formula for g_{L^2}, careful numerics
- Conjecture for g_{L^2} near coincidence: **incomplete**
- Conjectures for volumes (S^2)

The model

- Principal S^1 bundle $P \to \Sigma^2$, connexion A
- S^1 action on S^2, moment map μ
- Section $\mathbf{n}$ of $P \times_{S^1} S^2$

$$E = \frac{1}{2}\|\mathrm{d}_A \mathbf{n}\|^2 + \frac{1}{2}\|F_A\|^2 + \frac{1}{2}\|\mu \circ \mathbf{n}\|^2$$

- Primarily interested in $\Sigma^2 = \mathbb{R}^2$. $\mathbf{n} : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$, $D\mathbf{n} = d\mathbf{n} - A\mathbf{e} \times \mathbf{n}$

$$E = \frac{1}{2}\int_{\mathbb{R}^2} \left(|D\mathbf{n}|^2 + |B|^2 + (\mathbf{e} \cdot \mathbf{n})^2\right)$$

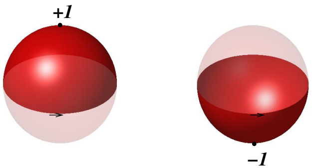
where $B = \mathrm{d}A$. Choose $\mathbf{e} = (0, 0, 1)$.

Flux quantization, vortices

- As $r \to \infty$, $\mathbf{e} \cdot \mathbf{n} \to 0$ and $D\mathbf{n} \to 0$

$$\int_{\mathbb{R}^2} B = \int_{S^1_\infty} A = 2\pi \deg(\mathbf{n}_\infty : S^1_\infty \to S^1_\mathbf{e})$$

- If $\deg \mathbf{n}_\infty = 1$, two ways to close off the cap:



- $n_\pm$ = number signed preimages of $\pm\mathbf{e}$
- $\int_{\mathbb{R}^2} B = 2\pi(n_+ - n_-)$

Bogomol'nyi argument

- Let $Q = (\mathbf{e}\cdot\mathbf{n})A$ and assume $Q \to 0$ as $r \to \infty$ suff. fast that $\int_{S^1_\infty} Q = 0$

$$(\mathbf{n} \times D\mathbf{n}) \cdot D\mathbf{n} = \mathbf{n}^*\omega + \mathrm{d}Q - (\mathbf{e}\cdot\mathbf{n})B$$

- For all such $(\mathbf{n}, A)$,

$$\begin{aligned} E &= \frac{1}{2}\int_{\mathbb{R}^2} \{|D_1\mathbf{n} + \mathbf{n}\times D_2\mathbf{n}|^2 + 2(\mathbf{n}\times D_1\mathbf{n})\cdot D_2\mathbf{n} + |B|^2 + (\mathbf{e}\cdot\mathbf{n})^2\} \\ &= \int_{\mathbb{R}^2} (\mathbf{n}^*\omega + \mathrm{d}Q) + \frac{1}{2}\|D_1\mathbf{n} + \mathbf{n}\times D_2\mathbf{n}\|^2 + \frac{1}{2}\| * B - \mathbf{e}\cdot\mathbf{n}\|^2 \\ &\geq \int_{\mathbb{R}^2} \mathbf{n}^*\omega = 2\pi(n_+ + n_-) \end{aligned}$$

with equality iff

$$D_1\mathbf{n} + \mathbf{n}\times D_2\mathbf{n} = 0, \qquad *B = \mathbf{e}\cdot\mathbf{n}$$

The Taubes equation

$$u = \frac{n_1 + i n_2}{1 + n_3}, \qquad h = \log |u|^2$$

- h finite except at $\pm$ vortices, $h = \pm\infty$. $h \to 0$ as $r \to \infty$.
- BOG1 $\Rightarrow A_{\bar{z}} = -i\frac{\partial_{\bar{z}} u}{u}$
- Eliminate A from BOG2

$$\nabla^2 h - 2\tanh\frac{h}{2} = 0$$

away from vortex positions

- $(+)$ vortices at z_r^+, $r = 1, \ldots, n_+$, $(-)$ vortices at z_r^-, $r = 1, \ldots, n_-$

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\sum_r \delta(z - z_r^+) - \sum_r \delta(z - z_r^-)\right)$$

The Taubes equation

- **Theorem** (Yang, 1999): For each pair of disjoint divisors $[z_1^+, \ldots, z_{n_+}^+], [z_1^-, \ldots, z_{n_-}^-]$ there exists a unique solution of (TAUBES), and hence a unique gauge equivalence class of solutions of (BOG1), (BOG2).
- Moduli space of vortices: $\mathsf{M}_{n_+,n_1} \equiv (\mathbb{C}^{n_+} \times \mathbb{C}^{n_-}) \backslash \Delta_{n_+,n_-}$

Solving the (1,1) Taubes equation (numerically)

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta(z-\varepsilon) - \delta(z+\varepsilon)\right)$$

- Regularize: $h = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}$

$$\nabla^2\widehat{h} - 2\frac{|z-\varepsilon|^2 e^{\widehat{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\widehat{h}} + |z+\varepsilon|^2} = 0$$

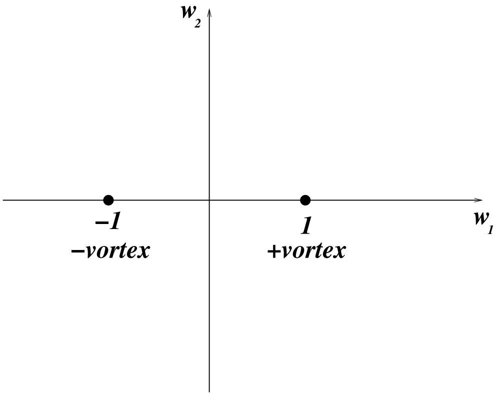
- Rescale: $z =: \varepsilon w$

$$\nabla_w^2\widehat{h} - 2\varepsilon^2\frac{|w-1|^2 e^{\widehat{h}} - |w+1|^2}{|w-1|^2 e^{\widehat{h}} + |w+1|^2} = 0$$

- Solve with b.c. $\widehat{h}(\infty) = 0$

Solving the (1,1) Taubes equation (numerically)

- Symmetry:



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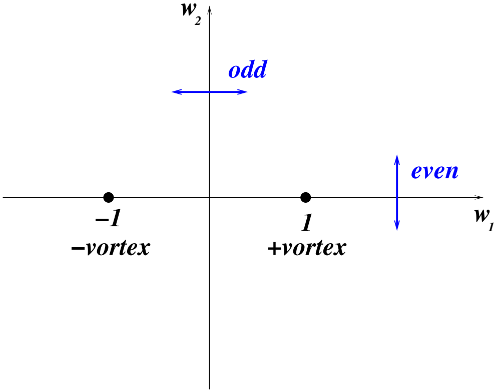

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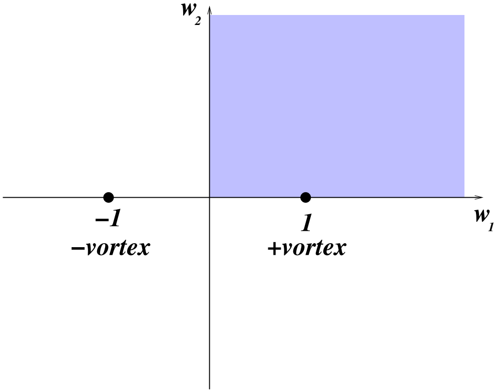

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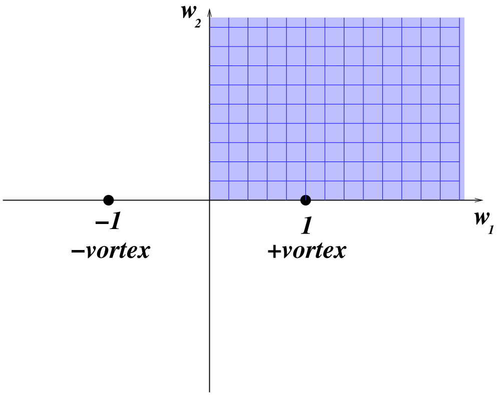

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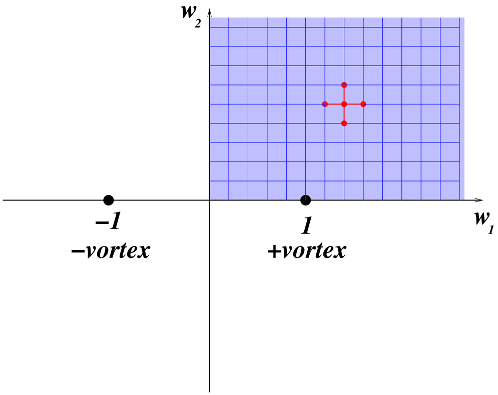

Solving the (1,1) Taubes equation (numerically)

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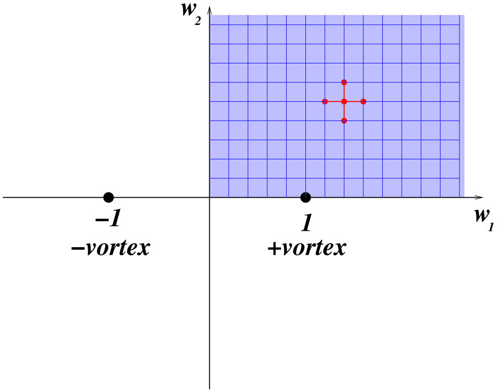
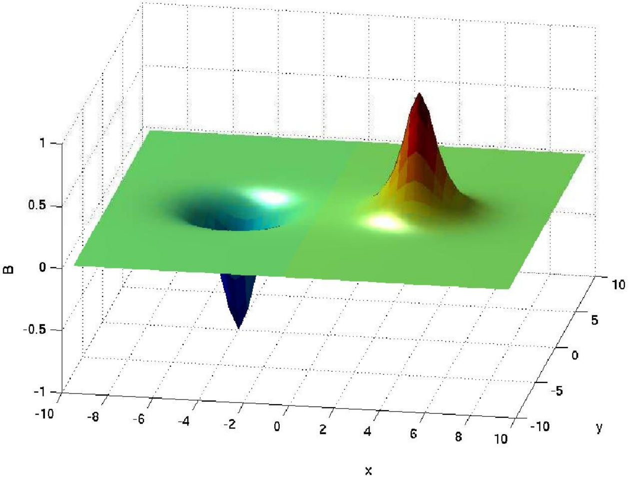


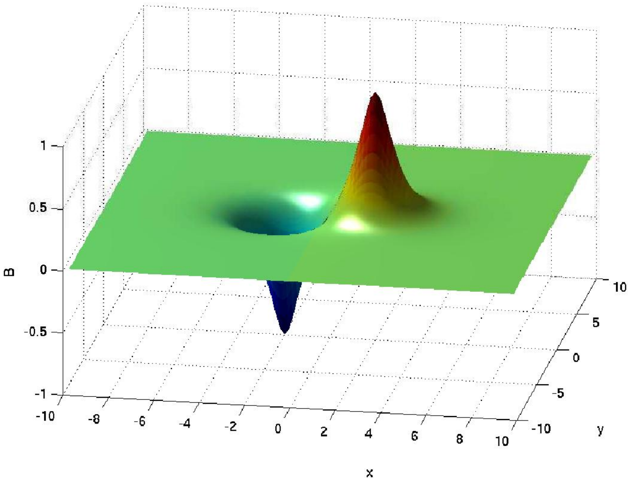
- $F(\widehat{h_{ij}}) = 0$, solve with Newton-Raphson

(1,1) vortices



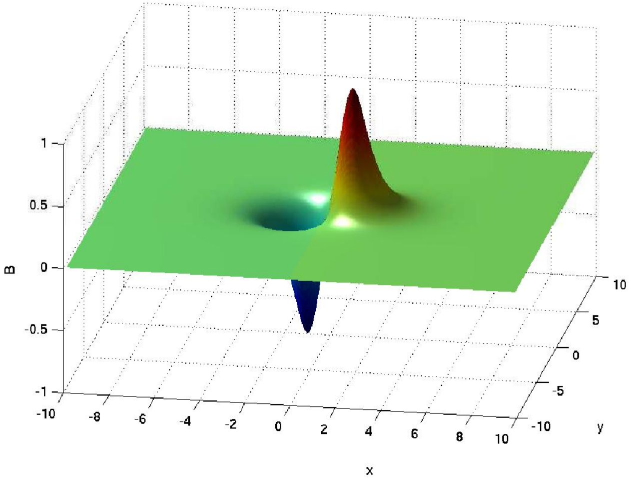
$\varepsilon = 4$

(1,1) vortices



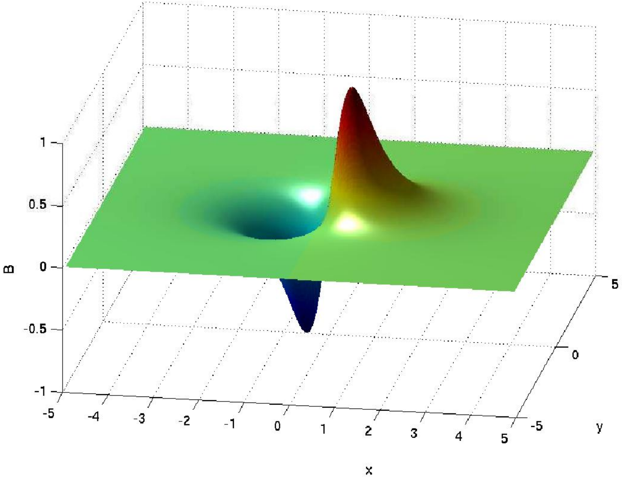
$\varepsilon = 2$

(1,1) vortices



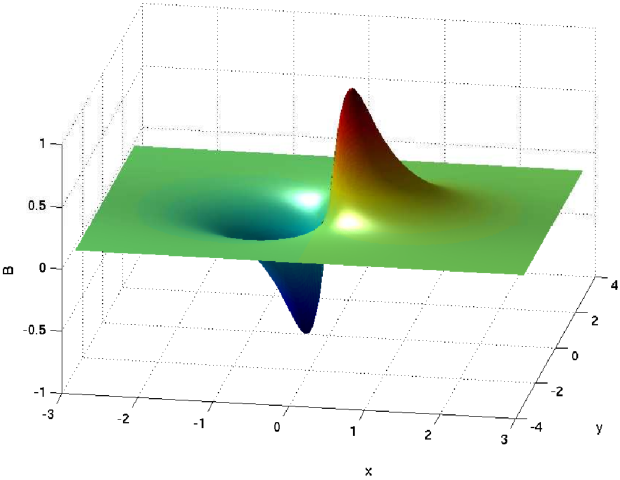
$\varepsilon = 1$

(1,1) vortices



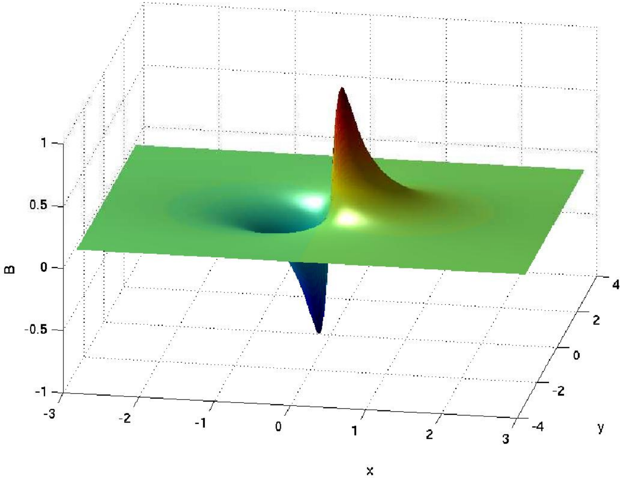
$\varepsilon = 0.5$

(1,1) vortices



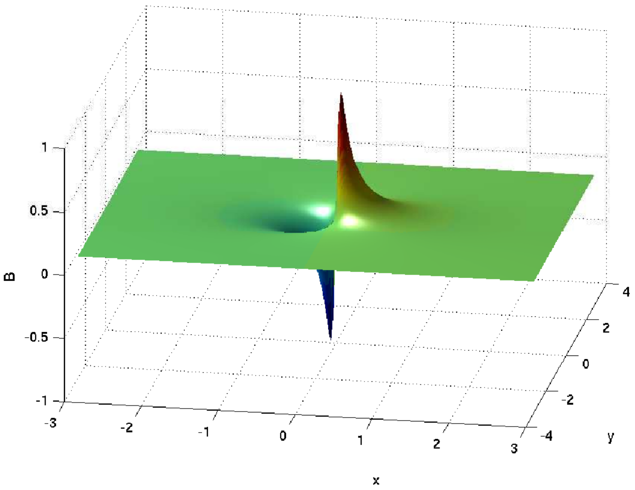
$\varepsilon = 0.3$

(1,1) vortices



$\varepsilon = 0.15$

(1,1) vortices



$\varepsilon = 0.06$

The L^2 metric on M_{n_+,n_-}

- Consider a curve $(\mathbf{n}(t), A(t))$ of vortex solutions
- Demand $(\dot{\mathbf{n}}, \dot{A})$ is L^2 orthogonal to all infinitesimal gauge transformations:

$$-\delta\dot{A} = \dot{\mathbf{n}} \cdot (\mathbf{e} \times \mathbf{n})$$

  Gauss's Law
- Kinetic energy

$$T = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\dot{\mathbf{n}}|^2 + |\dot{A}|^2\right)$$

  defines a Riemannian metric on M_{n_+,n_-}

Strachan-Samols localization

- Consider a curve in M_{n_+,n_-} along which all vortex positions $z_r^{\pm}(t)$ remain distinct
- Let $u =: exp(\frac{1}{2}h + i\chi)$ and $\dot{u} =: u\eta$, so $\eta = \frac{1}{2}\dot{h} + i\dot{\chi}$
- $\dot{h}$ satisfies linearized (TAUBES)
- $\dot{\chi}$ determined by (GAUSS)

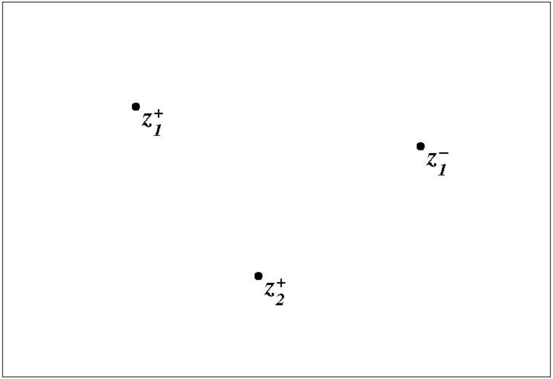
$$\nabla^2\eta - \text{sech}^2\frac{h}{2}\eta = 4\pi\left(\sum_r \dot{z}_r^+\delta(z - z_r^+) - \sum_r \dot{z}_r^-\delta(z - z_r^-)\right)$$

whence

$$\eta = \sum_r \dot{z}_r^+\frac{\partial h}{\partial z_r^+} + \sum_r \dot{z}_r^-\frac{\partial h}{\partial z_r^-}$$

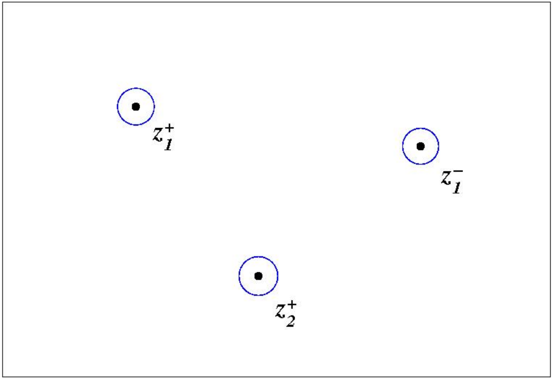
- η is a very good way to characterize $(\dot{\mathbf{n}}, \dot{A})$. Why?

Strachan-Samols localization



$$T = \frac{1}{2}\int_{\mathbb{R}^2}\left(4\partial_z\bar{\eta}\partial_{\bar{z}}\eta + \text{sech}^2\frac{h}{2}\bar{\eta}\eta\right)$$

Strachan-Samols localization



$$T = \lim_{\varepsilon \to 0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus D_\varepsilon} \left(4 \partial_z \bar{\eta} \partial_{\bar{z}} \eta + \operatorname{sech}^2 \frac{h}{2} \bar{\eta} \eta \right)$$

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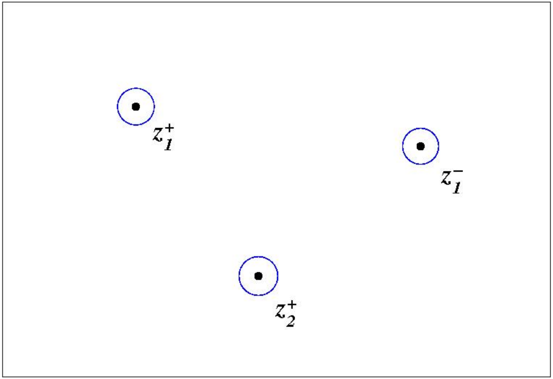

$$T = \lim_{\varepsilon \to 0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus D_\varepsilon} \left(4\partial_z(\bar{\eta}\partial_{\bar{z}}\eta) - \bar{\eta}(\nabla^2 - \text{sech}^2 \frac{h}{2})\eta \right)$$

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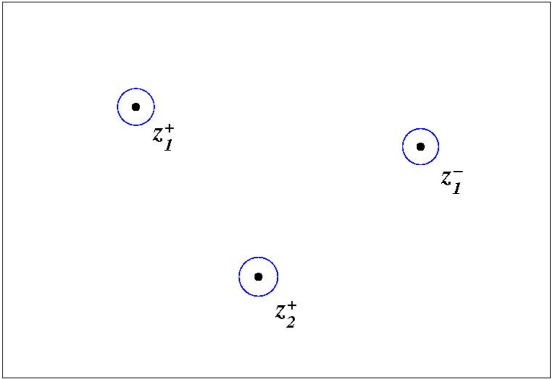
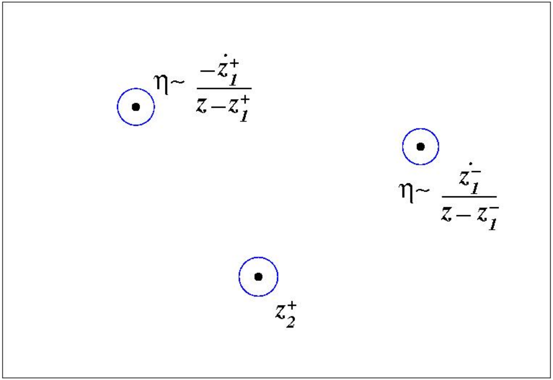


$$T = \lim_{\varepsilon \to 0} i \sum_r \oint_{C_r} \bar{\eta} \partial_{\bar{z}} \eta \, d\bar{z}$$

Strachan-Samols localization



$$T = \lim_{\varepsilon \to 0} i \sum_r \oint_{C_r} \bar{\eta} \partial_{\bar{z}} \eta \, d\bar{z}$$

Strachan-Samols localization

$$T = \pi\Bigg\{\sum_r |\dot{z}_r^+|^2 + \sum_r |\dot{z}_r^-|^2 + \sum_{r,s} \frac{\partial b_s^+}{\partial z_r^+}\dot{z}_r^+\dot{\bar{z}}_s^+ + \sum_{r,s} \frac{\partial b_s^-}{\partial z_r^-}\dot{z}_r^-\dot{\bar{z}}_s^- + \sum_{r,s} \frac{\partial b_s^-}{\partial z_r^+}\dot{z}_r^+\dot{\bar{z}}_s^- + \sum_{r,s} \frac{\partial b_s^+}{\partial z_r^-}\dot{z}_r^-\dot{\bar{z}}_s^+\Bigg\}$$

where, in a nbhd of z_s^+,

$$h = \log|z - z_s^+|^2 + a_s^+ + \frac{1}{2}\bar{b}_s^+(z - z_s^+) + \frac{1}{2}b_s^+(\bar{z} - \bar{z}_s^+) + \cdots$$

Strachan-Samols localization

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\begin{aligned}
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\end{aligned}
$$

where, in a nbhd of $z_s^\pm$

$$
h = \pm\log|z - z_s^\pm|^2 \pm a_s^\pm \pm \frac{1}{2}\bar{b}_s^\pm(z - z_s^\pm) \pm \frac{1}{2}b_s^\pm(\bar{z} - \bar{z}_s^\pm) + \cdots
$$

Strachan-Samols localization

$$
\begin{aligned}
g &= 2\pi\Big\{\sum_r |dz_r^+|^2 + \sum_r |dz_r^-|^2 + \sum_{r,s}\frac{\partial b_s^+}{\partial z_r^+} dz_r^+ d\bar{z}_s^+ + \sum_{r,s}\frac{\partial b_s^-}{\partial z_r^-} dz_r^- d\bar{z}_s^- \\
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$$

- $\bar{T} = T \Rightarrow g$ hermitian $\Rightarrow \frac{\partial b_s^+}{\partial z_r^+} = \frac{\partial \bar{b}_r^+}{\partial \bar{z}_s^+}$ etc

Strachan-Samols localization

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g &= 2\pi\Big\{\sum_r |dz_r^+|^2 + \sum_r |dz_r^-|^2 + \sum_{r,s}\frac{\partial b_s^+}{\partial z_r^+}dz_r^+ d\bar{z}_s^+ + \sum_{r,s}\frac{\partial b_s^-}{\partial z_r^-}dz_r^- d\bar{z}_s^- \\
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- $\bar{T} = T \Rightarrow g$ hermitian $\Rightarrow \frac{\partial b_s^+}{\partial z_r^+} = \frac{\partial \bar{b}_r^+}{\partial \bar{z}_s^+}$ etc $\Rightarrow g$ kähler

Strachan-Samols localization

$$g = 2\pi \Bigg\{ \sum_r |dz_r^+|^2 + \sum_r |dz_r^-|^2 + \sum_{r,s} \frac{\partial b_s^+}{\partial z_r^+} dz_r^+ d\bar{z}_s^+ + \sum_{r,s} \frac{\partial b_s^-}{\partial z_r^-} dz_r^- d\bar{z}_s^- + \sum_{r,s} \frac{\partial b_s^-}{\partial z_r^+} dz_r^+ d\bar{z}_s^- + \sum_{r,s} \frac{\partial b_s^+}{\partial z_r^-} dz_r^- d\bar{z}_s^+ \Bigg\}$$

where, in a nbhd of $z_s^\pm$

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- $\bar{T} = T \Rightarrow g$ hermitian $\Rightarrow \frac{\partial b_s^+}{\partial z_r^+} = \frac{\partial \bar{b}_r^+}{\partial \bar{z}_s^+}$ etc $\Rightarrow g$ kähler
- Can compute g if we know $b_r(z_1^+, \ldots, z_{n_-}^-)$

The metric on $M_{1,1}$

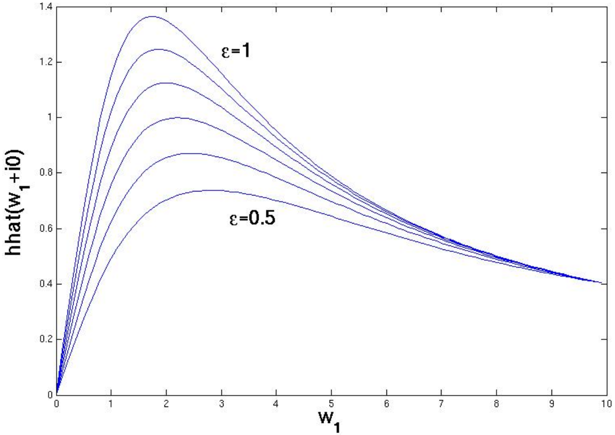
- $M_{1,1} = (\mathbb{C} \times \mathbb{C}) \backslash \Delta = \mathbb{C}_{com} \times \mathbb{C}^{\times}$
- $M^0_{1,1} = \mathbb{C}^{\times}$

$$g^0 = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d}{d\varepsilon} (\varepsilon b(\varepsilon)) \right) (d\varepsilon^2 + \varepsilon^2 d\psi^2)$$

where $b(\varepsilon) = b_+(\varepsilon, -\varepsilon)$

- $\varepsilon b(\varepsilon) = \frac{\partial \widehat{h}}{\partial w_1}\Big|_{w=1} - 1$
- Can easily extract this from our numerics

The metric on $M_{1,1}$



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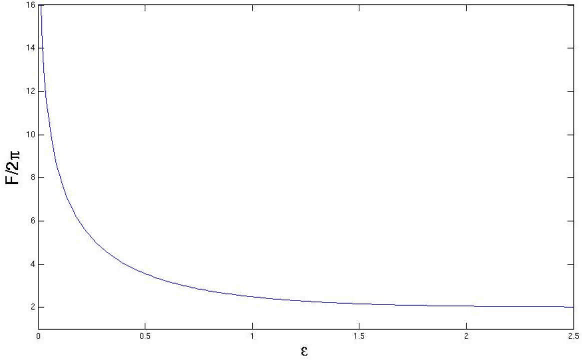
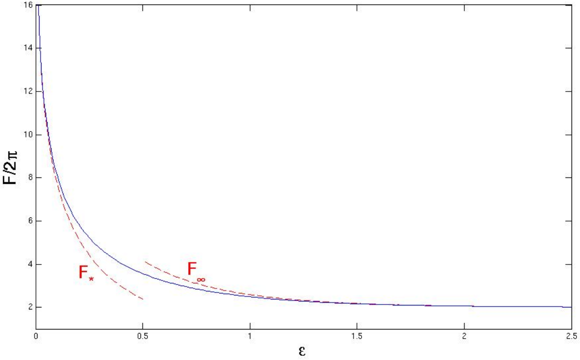
The metric on $M_{1,1}$



$$F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d(\varepsilon b(\varepsilon))}{d\varepsilon} \right)$$

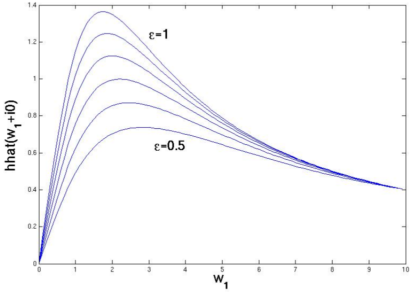
The metric on $M_{1,1}$: conjectured asymptotics



$$F_*(\varepsilon) = 2\pi\left(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)\right) \sim -8\pi \log \varepsilon$$

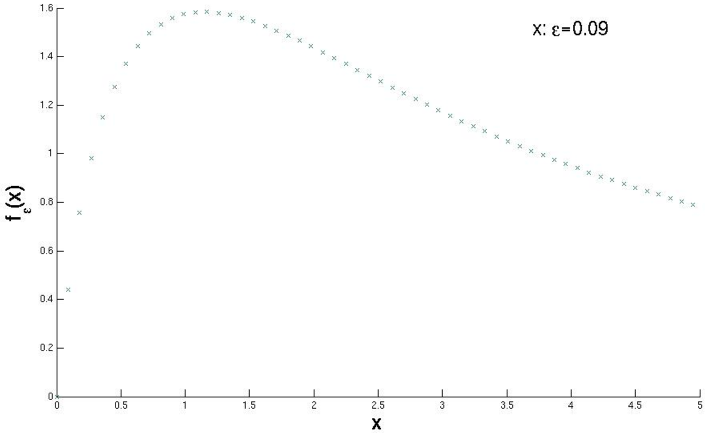
$$F_\infty(\varepsilon) = 2\pi\left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon)\right)$$

Self similarity as $\varepsilon \longrightarrow 0$



- Suggests $\widehat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w)$ for small ε, where f_* is fixed?
- Define $f_\varepsilon(z) := \varepsilon^{-1}\widehat{h}_\varepsilon(\varepsilon^{-1} z)$

Self similarity as $\varepsilon \longrightarrow 0$



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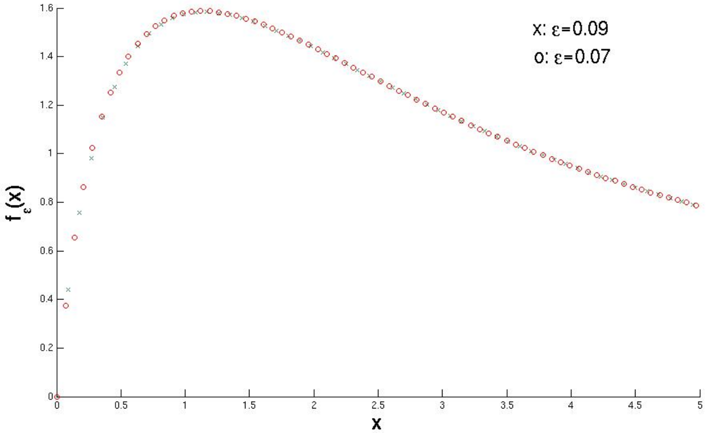

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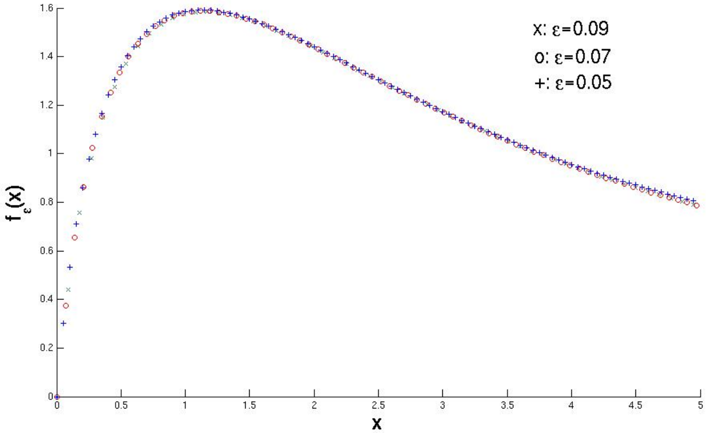

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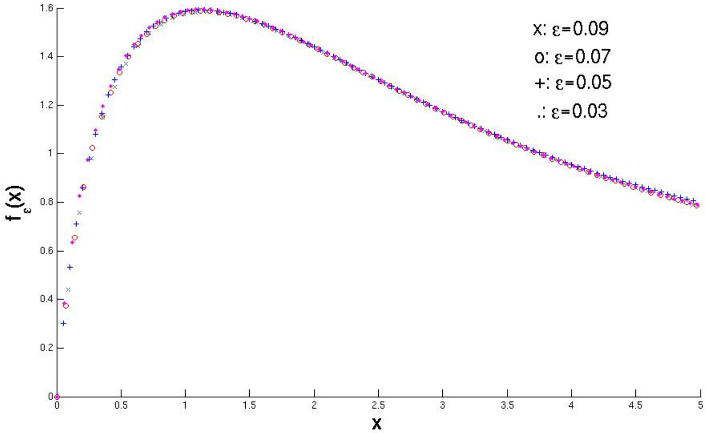

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 \widehat{h})(w) = 2\varepsilon^2 \frac{|w-1|^2 e^{\widehat{h}(w)} - |w+1|^2}{|w-1|^2 e^{\widehat{h}(w)} + |w+1|^2}$$

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 \widehat{h})(w) = 2\varepsilon^2 \frac{|w-1|^2 e^{\widehat{h}(w)} - |w+1|^2}{|w-1|^2 e^{\widehat{h}(w)} + |w+1|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 f_\varepsilon)(z) = \frac{2}{\varepsilon} \frac{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} + |z+\varepsilon|^2}$$

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- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \longrightarrow 0$

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
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Self similarity as $\varepsilon \longrightarrow 0$

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- Take formal limit $\varepsilon \to 0$
- Screened inhomogeneous Poisson equation, source $-4\cos\theta/r$

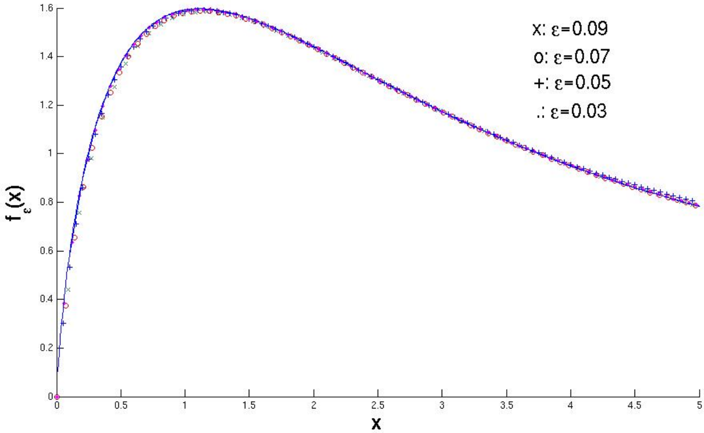
Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \longrightarrow 0$
- Screened inhomogeneous Poisson equation, source $-4\cos\theta/r$
- Unique solution (decaying at infinity)

$$f_*(re^{i\theta}) = \frac{4}{r}(1 - rK_1(r))\cos\theta$$

Self similarity as $\varepsilon \longrightarrow 0$



The metric on $M_{1,1}^0$

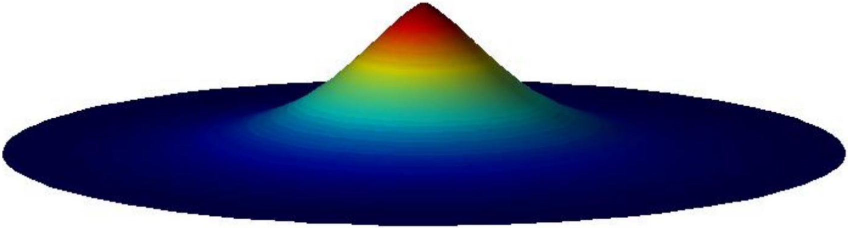
- Predict, for small ε,

$$\widehat{h}(w_1 + i0) \approx \varepsilon f_*(\varepsilon w_1) = \frac{4}{w_1}(1 - \varepsilon w_1 K_1(\varepsilon w_1))$$

whence we extract predictions for $\varepsilon b(\varepsilon)$, $F(\varepsilon)$

$$g^0 = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2)$$

- Conjecture: $F(\varepsilon) \sim -8\pi \log \varepsilon$ as $\varepsilon \to 0$
- $M_{1,1}$ is **incomplete**, with unbounded curvature



Vortex-antivortex pairs on S^2_R

$$g_{S^2} = \Omega(z)dzd\bar{z} = \frac{4R^2 dzd\bar{z}}{(1+|z|^2)^2} = d(2Rz)d(2R\bar{z}) + \cdots$$

- Regularized Taubes equation, $(\pm)$-vortex at $z = \pm\varepsilon$:

$$\nabla_w^2 \widehat{h} - 2\varepsilon^2 \Omega(\varepsilon w) \frac{|w-1|^2 e^{\widehat{h}} - |w+1|^2}{|w-1|^2 e^{\widehat{h}} + |w+1|^2} = 0$$

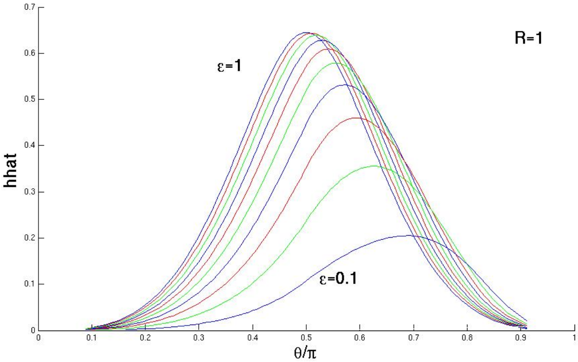
- Squared length of $\partial/\partial\varepsilon$

$$g(\partial/\partial\varepsilon, \partial/\partial\varepsilon) = F(\varepsilon) = 2\pi\left(2\Omega(\varepsilon) + \frac{1}{\varepsilon}\beta'(\varepsilon)\right)$$

where, again, $\beta(\varepsilon) = \varepsilon b_+(\varepsilon, -\varepsilon) = \left.\frac{\partial \widehat{h}}{\partial w_1}\right|_{w=1} - 1$

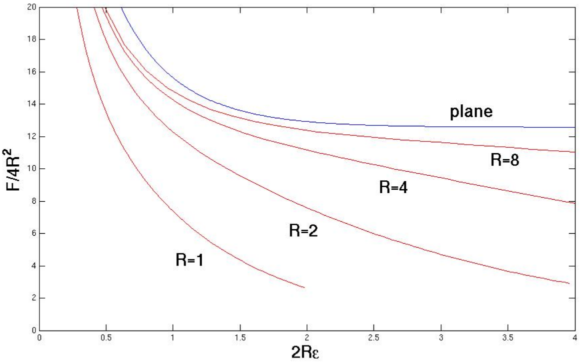
- Again, can solve (TAUBES) numerically for $\varepsilon \in (0,1]$ ($\varepsilon \mapsto 1/\varepsilon$ is an isometry)

Vortex-antivortex pairs on S_R^2

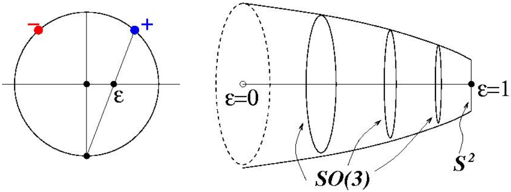


Note that $\beta(1) = -1 = \lim_{\varepsilon \to 0} \beta(\varepsilon)$

The planar limit $R \longrightarrow \infty$



The metric on $M_{1,1}(S^2) = (S^2 \times S^2)\backslash\Delta$



- g is $SO(3)$-invariant, kähler, and invariant under $(z_+, z_-) \mapsto (z_-, z_+)$
- Every such metric takes the form

$$g = -\frac{A'(\varepsilon)}{\varepsilon}(d\varepsilon^2 + \varepsilon^2\sigma_3^2) + A(\varepsilon)\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\sigma_1^2 + \frac{1+\varepsilon^2}{1-\varepsilon^2}\sigma_2^2\right),$$

for some smooth decreasing $A : (0,1] \to \mathbb{R}$ with $A(1) = 0$.
(Here σ_i are the usual left-invariant one-forms on $SO(3)$)

- Has finite total volume iff A is bounded

$$Vol(M_{1,1}) = -\int_{(0,1]\times SO(3)} A'A d\varepsilon \wedge \sigma_{123} = \frac{1}{4}(4\pi)^2 \lim_{\varepsilon\to 0} A(\varepsilon)^2$$

The metric on $M_{1,1}(S^2) = (S^2 \times S^2) \backslash \Delta$

- For the vortex metric

$$A(\varepsilon) = 2\pi \left(\frac{4R^2}{1+\varepsilon^2} - \beta(\varepsilon) - 2R^2 - 1 \right)$$

- Recall numerics suggest $\lim_{\varepsilon \to 0} \beta(\varepsilon) = -1$, whence

$$Vol(M_{1,1}) = (2\pi)^2 (4\pi R^2)^2$$

- Coincides with the volume of configuration space of a pair of distinct point particles of mass 2π moving on S^2_R
- **Conjecture:** The L^2 metric on $M_{1,1}(S^2_R)$ is incomplete, with unbounded curvature, and has finite total volume $(2\pi)^2(4\pi R^2)^2$

The volume of $M_{n,n}(S^2)$

- $M_{n,n}(S^2) = \{\text{disjoint pairs of } n\text{-divisors on } S^2\} = (\mathbb{P}^n \times \mathbb{P}^n) \backslash \Delta$
- Consider gauged **linear** sigma model:
  - fibre $\mathbb{C}^2$
  - gauge group $\widetilde{U}(1) \times U(1) : (\varphi_1, \varphi_2) \mapsto (e^{i(\widetilde{\theta}+\theta)}\varphi_1, e^{i\widetilde{\theta}}\varphi_2)$

$$
\begin{aligned}
E_{\widetilde{e}} &= \frac{1}{2}\int_\Sigma \left\{ \frac{|\widetilde{F}|^2}{\widetilde{e}^2} + |F|^2 + |\mathrm{d}_{\widetilde{A}}\varphi|^2 + |\mathrm{d}_A\varphi|^2 \right. \\
&\quad \left. + \frac{\widetilde{e}^2}{4}(4 - |\varphi_1|^2 - |\varphi_2|^2)^2 + \frac{1}{4}(2 - |\varphi_1|^2)^2 \right\}
\end{aligned}
$$

- For any $\widetilde{e} > 0$, has compact moduli space of (n, n)-vortices

$$
M_{n,n}^{lin} = \mathbb{P}^n \times \mathbb{P}^n
$$

- Baptista found a formula for $[\omega_{L^2}]$ of $M_{n_1,n_2}(\Sigma)$
- Can compute $Vol(M_{n,n}^{lin}(S^2))$ by evaluating $[\omega_{L^2}]$ on $\mathbb{P}^1 \times \{p\}$, $\{p\} \times \mathbb{P}^1$

The volume of $M_{n,n}(S^2)$

$$
\begin{aligned}
E_{\tilde{e}} &= \frac{1}{2}\int_\Sigma \left\{ \frac{|\widetilde{F}|^2}{\tilde{e}^2} + |F|^2 + |\mathrm{d}_{\widetilde{A}}\varphi|^2 + |\mathrm{d}_A\varphi|^2 \right. \\
&\quad \left. + \frac{\tilde{e}^2}{4}(4 - |\varphi_1|^2 - |\varphi_2|^2)^2 + \frac{1}{4}(2 - |\varphi_1|^2)^2 \right\}
\end{aligned}
$$

- Take formal limit $\tilde{e} \to 0$:
  - $|\varphi_1|^2 + |\varphi_2|^2 = 4$ pointwise
  - $\widetilde{A}$ frozen out, fibre $\mathbb{C}^2$ collapses to $S^3/\widetilde{U}(1) = \mathbb{P}^1$
  - E-L eqn for $\widetilde{A}$ is algebraic: eliminate $\widetilde{A}$ from E_∞

$$
E_\infty = \frac{1}{2}\int_\Sigma |F|^2 + 4\frac{|\mathrm{d}u - iAu|^2}{(1+|u|^2)^2} + \left(\frac{1-|u|^2}{1+|u|^2}\right)^2
$$

where $u = \varphi_1/\varphi_2$

  - Exactly our $\mathbb{P}^1$ sigma model!

The volume of $M_{n,n}(S^2)$

- Leads us to conjecture that

$$Vol(M_{n,n}(S^2)) = \lim_{\tilde{e}\to\infty} Vol(M^{lin}_{n,n}(S^2) = \frac{(2\pi Vol(S^2))^{2n}}{(n!)^2}$$

- More elaborate choice of linear model gives more general conjecture:

$$Vol(M_{n,m}(S^2)) = \frac{(2\pi)^{n+m}}{n!m!}(Vol(S^2) - \pi(n-m))^n(Vol(S^2) + \pi(n-m))^m$$

- Similar limit ($\mathbb{C}^k$ fibre, $U(1)$ gauge $\rightarrow$ *ungauged* $\mathbb{P}^{k-1}$ model) studied rigorously by Chih-Chung Liu.