The sigma model limit of two-component Ginzburg-Landau theory

Martin Speight
University of Leeds

25 May 2010

J. Geom. Phys. 60 (2010) 599-610

Multicomponent GL theory

- Physical space $M = \mathbb{R}^3$,

$$\psi_a : M \to \mathbb{C}, \qquad a = 1, \dots, k \qquad \text{“condensates”}$$
$$A \in \Omega^1(M) \qquad \text{em gauge potential}$$
$$B = \mathrm{d}A \in \Omega^2(M) \qquad \text{magnetic field}$$
$$\mathrm{d}_A \psi_a = \mathrm{d}\psi_a - iA\psi_a$$
$$E_{GL} = \frac{1}{2}\sum_{a=1}^{k} \|\mathrm{d}_A \psi_a\|^2 + \frac{1}{2}\|B\|^2 + \int_M U(\psi)$$

where $\|\cdot\| = L^2$ norm: $\|B\|^2 = \langle B, B \rangle$, $\langle B, C \rangle = \int_M B \wedge *C$

- Field equations:

$$-*\mathrm{d}_A * \mathrm{d}_A \psi_a + 2\frac{\partial U}{\partial \bar{\psi}_a} = 0$$
$$-*\mathrm{d}*B = C$$

where $C = \frac{i}{2}\sum_a (\bar{\psi}_a \mathrm{d}_A \psi_a - \psi_a \overline{d_A \psi_a})$ =supercurrent

Sigma model limit

- Sigma model limit: assume U strongly confines ψ to $S^{2k-1} \subset \mathbb{C}^k$

$$|\psi_1|^2 + \cdots + |\psi_k|^2 = 1$$

e.g. $U = \lambda(1 - |\psi|^2)^2$, $\lambda \to \infty$

- Rewrite E_{GL} in terms of gauge-invariant fields $C \in \Omega^1(M)$ and $\varphi = \pi \circ \psi : M \to \mathbb{C}P^{k-1}$ where $\pi : S^{2k-1} \to \mathbb{C}P^{k-1}$ is the Hopf fibration

$$\pi : (z_1, z_2, \ldots, z_k) \mapsto [z_1, z_2, \ldots, z_k]$$

Note gauge transformations move ψ along the fibres of π.

- Key observation: $C = A + \psi^* \nu$ where $\nu = -\mathrm{Im}\frac{\mathrm{d}z}{|z|^2}$, one-form on $\mathbb{C}^k \backslash \{0\}$.

$$E_{GL} = \frac{1}{2}(\|\mathrm{d}\psi\|^2 - \|\psi^*\nu\|^2) + \frac{1}{2}\|C\|^2 + \frac{1}{2}\|\mathrm{d}(C - \psi^*\nu)\|^2$$

Sigma model limit

- Give $\mathbb{C}P^{k-1}$ the usual Fubini-Study metric h, with Kähler form ω. Then **by definition** the lift of ω to $\mathbb{C}^k \backslash \{0\}$ is

$$\pi^*\omega = 2i\partial\bar{\partial}\log|z|^2 = -2\mathrm{d}\nu$$

Hence

$$\mathrm{d}(\psi^*\nu) = \psi^*\mathrm{d}\nu = -\frac{1}{2}\psi^*(\pi^*\omega) = -\frac{1}{2}(\pi\circ\psi)^*\omega = -\frac{1}{2}\varphi^*\omega$$

- Furthermore, for any $X \in T(\mathbb{C}^k \backslash \{0\})$,

$$|X|^2 - \nu(X)^2 = \frac{1}{4}\pi^*h(X,X)$$

Hence

$$\|\mathrm{d}\psi\|^2 - \|\psi^*\nu\|^2 = \frac{1}{4}\sum_j \pi^*h(\mathrm{d}\psi E_j, \mathrm{d}\psi E_j) = \frac{1}{4}\sum_j h(\mathrm{d}\varphi E_j, \mathrm{d}\varphi E_j)$$

- Finally [Hindmarsh (general k), Babaev-Faddeev-Niemi ($k=2$)]:

$$E(\varphi, C) = \frac{1}{8}\|\mathrm{d}\varphi\|^2 + \frac{1}{2}\|\mathrm{d}C + \frac{1}{2}\varphi^*\omega\|^2 + \frac{1}{2}\|C\|^2$$

Supercurrent Coupled Faddeev-Skyrme Model.
Makes mathematical sense for $\varphi : M \to N$, $C \in \Omega^1(M)$ where $M =$ any Riemannian mfd, $N =$ any Kähler mfd.

- Take $k = 2$ (TCGL) so $N = \mathbb{C}P^1 = S^2$.

$$\begin{aligned} J : T_\varphi S^2 &\to T_\varphi S^2, \qquad X \mapsto \varphi \times X \qquad \text{almost cx structure} \\ \omega(X, Y) &= h(JX, Y) = (\varphi \times X)\cdot Y = \varphi \cdot (X \times Y) \\ |\varphi^*\omega|^2 &= \sum_{j<k} (\varphi \cdot (\partial_j\varphi \times \partial_k\varphi))^2 \\ E(\varphi, 0) &= \frac{1}{8}\int_{\mathbb{R}^3} \left\{ \sum_j |\partial_j\varphi|^2 + \sum_{j<k} (\varphi\cdot(\partial_j\varphi \times \partial_k\varphi))^2 \right\} = \frac{1}{4}E_{FS}(\varphi) \end{aligned}$$

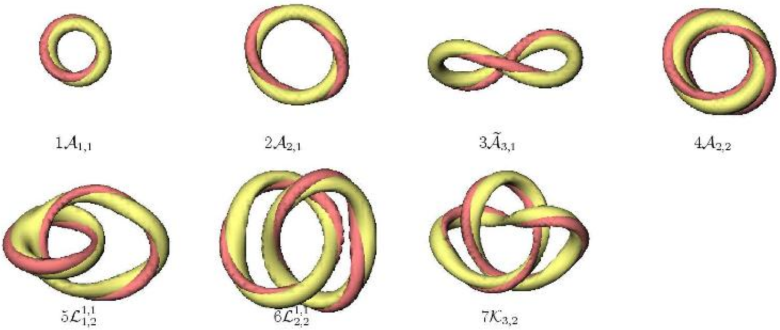
- E_{FS} supports **knot solitons**

Knot solitons

- $\varphi : \mathbb{R}^3 \to S^2$, b.c. $\varphi(\infty) = (0,0,1)$
- Hopf degree $Q = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} A \wedge \mathrm{d}A$ where $\varphi^*\omega = \mathrm{d}A$
- $\varphi^{-1}(\text{reg. value}) =$ oriented link in $\mathbb{R}^3$.
  $Q =$ linking number of different regular preimages.
- Numerics: for some Q, $\varphi^{-1}(0,0,-1)$ is knotted.
- Vakulenko-Kapitanskii bound: $E_{FS}(\varphi) \geq c|Q|^{\frac{3}{4}}$. The power is sharp.

Knot solitons

Studied numerically by Battye and Sutcliffe, Hietarinta and Salo, and many others



[Sutcliffe, Proc. Roy. Soc. Lond. **A463** (2007) 3001]

Babaev-Faddeev-Niemi conjecture

- Since $E(\varphi, 0) = \frac{1}{4}E_{FS}(\varphi)$ and C field is massive (so should be small), the TCGL model (with a confining potential) should support knot solitons too.

$$E(\varphi, C) = \frac{1}{8}\|\mathrm{d}\varphi\|^2 + \frac{1}{2}\|\mathrm{d}C + \frac{1}{2}\varphi^*\omega\|^2 + \frac{1}{2}\|C\|^2$$
$$= \frac{1}{4}\left\{\frac{1}{2}\|\mathrm{d}\varphi\|^2 + \frac{1}{2}\|\varphi^*\omega\|^2\right\} + \frac{1}{2}\left\{\|\mathrm{d}C\|^2 + \|C\|^2\right\} + \frac{1}{2}\langle \mathrm{d}C, \varphi^*\omega\rangle$$
$$= \frac{1}{4}E_{FS}(\varphi) + E_3(C) + E_4(\varphi, C)$$

- Definitive (?) test: introduce parameter $0 \leq \alpha \leq 1$

$$E = \frac{1}{4}E_{FS}(\varphi) + E_3(C) + \alpha E_4(\varphi, C)$$

Know this has knot solitons when $\alpha = 0$. Do any of them continue to $\alpha = 1$?

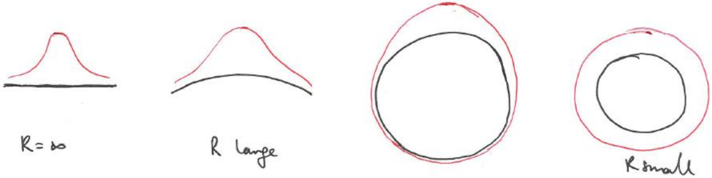
Babaev-Faddeev-Niemi conjecture

$$E = \frac{1}{4}E_{FS}(\varphi) + E_3(C) + \alpha E_4(\varphi, C)$$

- On $\mathbb{R}^3$, must implement numerically (ongoing work with Jäykkä)
- On S^3_R, can answer question ($Q = 1$, $0 < R < 2$) exactly.
- Answer = NO!

Homogenization of solitons on shrinking domains

- Generic phenomenon: topological solitons on compact domains undergo a phase transition as the domain shrinks – they **gain** symmetry



- Happens for Skyrme model, vector meson Skyrme model, Faddeev-Skyrme model on S^3, and abelian Higgs model on any compact Riemann surface
- For $0 < R < 2$ unit hopfion is just Hopf map $\pi : S^3 \to S^2$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $dA = \varphi^*\omega$.

$$\begin{aligned} A &= A_{\text{harmonic}} + dA_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\ E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle dA, dA\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\ 16\pi^2 Q &= \langle A, -*dA\rangle \leq \|A\|\|dA\| \leq \|A\|\sqrt{2E_2(\varphi)} \end{aligned}$$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $dA = \varphi^*\omega$.

$$\begin{aligned} A &= A_{\text{harmonic}} + dA_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\ E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle dA, dA\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\ 16\pi^2 Q &= \langle A, -*dA\rangle \leq \|A\|\|dA\| \leq \|A\|\sqrt{2E_2(\varphi)} \end{aligned}$$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $\mathrm{d}A = \varphi^*\omega$.

$$\begin{aligned}
A &= A_{\text{harmonic}} + \mathrm{d}A_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\
E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle \mathrm{d}A, \mathrm{d}A\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\
16\pi^2 Q &= \langle A, -*\mathrm{d}A\rangle \leq \|A\|\|\mathrm{d}A\| \leq \|A\|\sqrt{2E_2(\varphi)}
\end{aligned}$$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $dA = \varphi^*\omega$.

$$\begin{aligned} A &= A_{\text{harmonic}} + dA_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\ E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle dA, dA\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\ 16\pi^2 Q &= \langle A, -*dA\rangle \leq \|A\|\|dA\| \leq \|A\|\sqrt{2E_2(\varphi)} \end{aligned}$$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $dA = \varphi^*\omega$.

$$\begin{aligned} A &= A_{\text{harmonic}} + dA_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\ E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle dA, dA\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\ 16\pi^2 Q &= \langle A, -*dA\rangle \leq \|A\|\|dA\| \leq \|A\|\sqrt{2E_2(\varphi)} \end{aligned}$$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2 \sqrt{\lambda_1}\, Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $dA = \varphi^*\omega$. Hodge decomposition

$$\begin{aligned} A &= A_{\text{harmonic}} + dA_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\ E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle dA, dA\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\ 16\pi^2 Q &= \langle A, -*dA\rangle \leq \|A\|\|dA\| \leq \|A\|\sqrt{2E_2(\varphi)} \end{aligned}$$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $dA = \varphi^*\omega$. $\delta = \pm * d * : \Omega^k \to \Omega^{k-1}$

$$\begin{aligned}
A &= A_{\mathrm{harmonic}} + dA_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\
E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle dA, dA\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\
16\pi^2 Q &= \langle A, -*dA\rangle \leq \|A\|\|dA\| \leq \|A\|\sqrt{2E_2(\varphi)}
\end{aligned}$$

- $\pi : S^3_R \to S^2$ is a critical point of $E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2$ and $E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2$ separately
- Unstable for E_1 (index=4)
- Stable for E_2: in fact minimizes E_2 in its htpy class!
- **Thm (JMS-Svensson)**: Let M be a compact, oriented 3-mfd and $\varphi : M \to S^2$ be algebraically inessential (i.e. $\varphi^*\omega$ exact). Then

$$E_2(\varphi) \geq 8\pi^2\sqrt{\lambda_1}\,Q(\varphi)$$

where $\lambda_1 > 0$ is the lowest eigenvalue of the Laplacian on coexact one-forms on M.

Proof: Choose A s.t. $dA = \varphi^*\omega$. Hodge decomposition

$$\begin{aligned} A &= A_{\text{harmonic}} + dA_0 + \delta A_2 = \delta A_2 \quad \text{w.l.o.g.} \\ E_2(\varphi) &= \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\langle dA, dA\rangle = \frac{1}{2}\langle A, \Delta A\rangle \geq \frac{1}{2}\lambda_1\|A\|^2 \\ 16\pi^2 Q &= \langle A, -*dA\rangle \leq \|A\|\|dA\| \leq \|A\|\sqrt{2E_2(\varphi)} \end{aligned}$$

Stability of the Hopf map

- $\pi : S^3_R \to S^2$ attains this bound
- Examine Hessian of E_2 restricted to negative modes of E_1
- **Cor (JMS-Svensson, Isobe)**: $\pi : S^3_R \to S^2$ is a stable critical point of E_{FS} iff $0 < R \leq 2$.
- $\varphi = \pi$, $C = 0$ stable critical point of

$$E(\varphi, C) = \frac{1}{4} E_{FS}(\varphi) + E_3(C) + \alpha E_4(\varphi, C)$$

 when $\alpha = 0$. How does this continue to $\alpha > 0$?
- Programme: develop variational calculus for $E(\varphi, C)$ in general setting.
  First variation formula = field equations (critical pts)
  Second variation formula = Hessian (stability of critical points)

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1}TN) \quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2 \qquad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1}TN)$ $\quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2$ $\quad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1}TN)$ $\quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2$ $\qquad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1} TN)$ $\quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2$ $\qquad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$

$$\tau(\varphi) = \operatorname{tr}\nabla \mathrm{d}\varphi = \Delta\varphi - (\varphi \cdot \Delta\varphi)\varphi \quad (N = S^2)$$

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1} TN)$ $\quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2$ $\quad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$
- $E_2(t) = \frac{1}{2}\|\varphi_t^*\omega\|^2$ $\quad \dot{E}_2(0) = -\langle X, J\mathrm{d}\varphi\sharp\delta(\varphi^*\omega)\rangle$

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1}TN)$ $\quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2 \qquad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$

- $E_2(t) = \frac{1}{2}\|\varphi_t^*\omega\|^2 \quad \dot{E}_2(0) = -\langle X, J\mathrm{d}\varphi\sharp\delta(\varphi^*\omega)\rangle$

$$\flat : TM \to T^*M, \quad \flat X = g(X,\cdot), \qquad \sharp = \flat^{-1} : T^*M \to TM$$

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1}TN)$ $\quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2 \qquad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$
- $E_2(t) = \frac{1}{2}\|\varphi_t^*\omega\|^2 \qquad \dot{E}_2(0) = -\langle X, J\mathrm{d}\varphi\sharp\delta(\varphi^*\omega)\rangle$
- $E_3(t) = \frac{1}{2}\|\mathrm{d}C_t\|^2 + \frac{1}{2}\|C_t\|^2 \qquad \dot{E}_3(0) = \langle Y, \delta\mathrm{d}C + C\rangle$

First variation

Smooth variations: $\varphi_t : M \to N$ $\qquad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1}TN) \qquad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|d\varphi_t\|^2 \qquad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$
- $E_2(t) = \frac{1}{2}\|\varphi_t^*\omega\|^2 \qquad \dot{E}_2(0) = -\langle X, Jd\varphi\sharp\delta(\varphi^*\omega)\rangle$
- $E_3(t) = \frac{1}{2}\|dC_t\|^2 + \frac{1}{2}\|C_t\|^2 \qquad \dot{E}_3(0) = \langle Y, \delta dC + C\rangle$
- $E_4(t) = \frac{1}{2}\langle dC_t, \varphi_t^*\omega\rangle \qquad \dot{E}_4(0) = \frac{1}{2}\langle Y, \delta(\varphi^*\omega)\rangle - \frac{1}{2}\langle X, Jd\varphi\sharp\delta dC\rangle$

First variation

Smooth variations: $\varphi_t : M \to N$ $\quad C_t \in \Omega^1(M)$

$X = \partial_t \varphi_t|_{t=0} \in \Gamma(\varphi^{-1}TN) \quad Y = \partial_t C_t|_{t=0} \in \Omega^1(M)$

- $E_1(t) = \frac{1}{2}\|\mathrm{d}\varphi_t\|^2 \qquad \dot{E}_1(0) = -\langle X, \tau(\varphi)\rangle,$
- $E_2(t) = \frac{1}{2}\|\varphi_t^*\omega\|^2 \qquad \dot{E}_2(0) = -\langle X, J\mathrm{d}\varphi\sharp\delta(\varphi^*\omega)\rangle$
- $E_3(t) = \frac{1}{2}\|\mathrm{d}C_t\|^2 + \frac{1}{2}\|C_t\|^2 \qquad \dot{E}_3(0) = \langle Y, \delta\mathrm{d}C + C\rangle$
- $E_4(t) = \frac{1}{2}\langle \mathrm{d}C_t, \varphi_t^*\omega\rangle \qquad \dot{E}_4(0) = \frac{1}{2}\langle Y, \delta(\varphi^*\omega)\rangle - \frac{1}{2}\langle X, J\mathrm{d}\varphi\sharp\delta\mathrm{d}C\rangle$

$$\dot{E}(0) = -\frac{1}{4}\langle X, \tau(\varphi) + J\mathrm{d}\varphi\sharp\delta(\varphi^*\omega + 2\alpha\mathrm{d}C)\rangle + \langle Y, \delta\mathrm{d}C + C + \frac{\alpha}{2}\delta(\varphi^*\omega)\rangle = 0$$

for all X, Y

$$\begin{aligned} \delta(\mathrm{d}C + \frac{\alpha}{2}\varphi^*\omega) + C &= 0, \\ \tau(\varphi) - \frac{2}{\alpha} J\mathrm{d}\varphi\sharp[C + (1-\alpha^2)\delta\mathrm{d}C] &= 0. \end{aligned}$$

- **Fact:** $\delta C = 0$, so $\operatorname{div}\sharp C = 0$ on M^3
- **Fact:** For fixed $\varphi : M \to N$, there can be at most one C s.t. (φ, C) is critical.
  [Assume (φ, C') also a solution. Then $C'' = C - C'$ solves $\delta\mathrm{d}C'' + C'' = 0$
  $$\Rightarrow \quad 0 = \langle C'', \delta\mathrm{d}C'' + C''\rangle = \|\mathrm{d}C''\|^2 + \|C''\|^2$$
  so $C'' = 0$.]
- **Defn:** Solution (φ, C) is an embedding of φ if φ is a critical point of E_{FS}
- **Note:** If an embedding of φ exists, it's unique.

Embedded Hopf map $S^3_R \longrightarrow S^2$

$$
\begin{aligned}
\delta(dC + \frac{\alpha}{2}\varphi^*\omega) + C &= 0 \qquad (1) \\
\tau(\varphi) - \frac{2}{\alpha} J d\varphi \sharp [C + (1-\alpha^2)\delta dC] &= 0 \qquad (2)
\end{aligned}
$$

- $G = SU(2)$, $K = \{\mathrm{diag}(\lambda, \bar{\lambda}) : \lambda \in U(1)\}$, $\varphi : x \mapsto xK$
  Left-invariant vector fields θ_a, one-forms σ_a
  At $x = e$, $\theta_a = \frac{i}{2}\tau_a$.Radius $R \Rightarrow |\theta_a| = \frac{R}{2}$

$$
\begin{aligned}
\text{Try } C &= \mu\sigma_3 \quad \text{[then (2) holds automatically]} \\
\varphi^*\omega &= -\sigma_1 \wedge \sigma_2 \\
\delta(\varphi^*\omega) &= -*d*\sigma_1 \wedge \sigma_2 = -\frac{4}{R^2}\sigma_3 \\
(1) \Leftrightarrow \mu &= \frac{2\alpha}{4+R^2}
\end{aligned}
$$

Embedded Hopf map $S^3_R \longrightarrow S^2$

- So un-coupled charge 1 hopfion continues for all $0 \leq \alpha \leq 1$ as

$$(\varphi = \text{hopf map}, C = \frac{2\alpha}{4+R^2}\sigma_3)$$

  Note this is the *unique* embedding of the Hopf map
- Great. But is it stable at $\alpha = 1$?
- Need second variation formula. . .

Second variation

$$\varphi_{s,t} : M \longrightarrow N \qquad C_{s,t} \in \Omega^1(M)$$

$$X = \partial_s \varphi_{s,t}|_{s=t=0}, \qquad Y = \partial_s C_{s,t}|_{s=t=0},$$

$$\hat{X} = \partial_t \varphi_{s,t}|_{s=t=0} \in \varphi^{-1}(TN) \qquad \hat{Y} = \partial_t C_{s,t}|_{s=t=0} \in \Omega^1(M)$$

$$\begin{aligned} \mathrm{Hess}((\hat{X}, \hat{Y}), (X, Y)) &= \frac{\partial}{\partial s \partial t} E(\varphi_{s,t}, C_{s,t})\Big|_{s=t=0} \\ &= \left\langle \begin{pmatrix} \hat{X} \\ \hat{Y} \end{pmatrix}, \mathscr{H} \begin{pmatrix} X \\ Y \end{pmatrix} \right\rangle \end{aligned}$$

- Symmetric bilinear form on $\Gamma(\mathscr{E})$, $\mathscr{E} = \varphi^{-1} TN \oplus T^* M$
- $\mathscr{H}$ self-adjoint, 2nd order linear diff-op $\Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ "Jacobi operator"
- Jacobi operators for E_1 (Smith, Urakawa...) and E_2 (JMS-Svensson) well understood

$$\begin{aligned} \mathscr{J} X &= \bar{\Delta}_\varphi X - \mathscr{R}_\varphi X \\ \mathscr{L} X &= -J\left(\nabla^\varphi_{\sharp \delta \varphi^* \omega} X + d\varphi(\sharp \delta d \varphi^* \iota_X \omega)\right) \qquad \iota_X \omega(\cdot) = \omega(X, \cdot) \end{aligned}$$

Second variation

$$\mathscr{H}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\mathscr{J} + \frac{1}{4}\mathscr{L} + \alpha\mathscr{C} & \frac{1}{2}\alpha\mathscr{A} \\ \frac{1}{2}\alpha\mathscr{B} & \delta d + 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

where

$$\begin{aligned} &\mathscr{A} : \Omega^1(M) \to \Gamma(\varphi^{-1}TN) && \mathscr{A} : Y \mapsto -J d\varphi \sharp \delta d Y \\ &\mathscr{B} : \Gamma(\varphi^{-1}TN) \to \Omega^1(M) && \mathscr{B} : X \mapsto \delta d(\varphi^* \iota_X \omega) \\ &\mathscr{C} : \Gamma(\varphi^{-1}TN) \to \Gamma(\varphi^{-1}TN) && \mathscr{C} : X \mapsto -\frac{1}{2} J \nabla^{\varphi}_{\sharp \delta d C} X. \end{aligned}$$

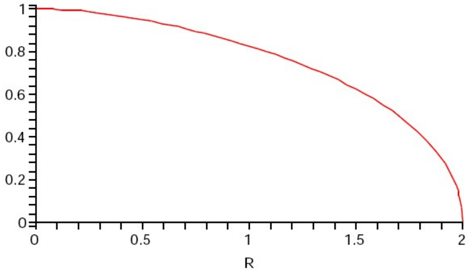
Hessian of the embedded Hopf map

$$G = SU(2), \quad K = S(U(1)\times U(1)), \qquad \varphi(x) = xK, \quad C = \frac{2\alpha\sigma_3}{4+R^2}$$

- Left translation $\Rightarrow \mathscr{E} \equiv G \times (\mathfrak{k}^\perp \oplus \mathfrak{g}^*)$
  $X = f_1 \mathrm{d}\varphi\theta_1 + f_2 \mathrm{d}\varphi\theta_2$, $Y = f_3\sigma_1 + f_4\sigma_2 + f_5\sigma_3$, $f_1, \ldots, f_5 \in C^\infty(G)$
- $\mathscr{J}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{4}{R^2}\begin{pmatrix} -(\theta_1^2+\theta_2^2+\theta_3^2) & -2\theta_3 \\ 2\theta_3 & -(\theta_1^2+\theta_2^2+\theta_3^2) \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$
- Similar expressions for $\mathscr{L}, \mathscr{A}, \mathscr{B}, \mathscr{C}, \delta\mathrm{d}$ as matrices of diffops
- Peter-Weyl theorem: matrix elements of unitary irreps of G form basis for $L^2(G)$. Expand each $f_a : G \to \mathbb{R}$ in this basis.
- $\mathscr{H}$ preserves G-invariant subspaces of $\Gamma(\mathscr{E})$, so $\mathscr{H}$ decomposes into blocks $\mathscr{H}^{(s)}$, indexed by "spin" $s \in \frac{1}{2}\mathbb{N}$ of irrep.

Hessian of the embedded Hopf map

- Fundamental irrep $s = \frac{1}{2}$: replace each θ_a by $\frac{i}{2}\tau_a$. Get 10×10 matrix representing $\mathscr{H}^{(\frac{1}{2})}$
- Maple finds one eigenvalue which becomes negative for $\alpha > \alpha_0(R)$. Total multiplicity 4



$$\alpha_0(R) = \frac{1}{2}\sqrt{\frac{144 + 16R^2 - 9R^4 - R^6}{36 + 19R^2}}$$

- No other negative eigenvalues $0 \leq s \leq 10$. Index probably 4.

Concluding remarks

- On $\mathbb{R}^3$, supercurrent coupling destroys VK bound:
  Can show $\inf\{E(\varphi, C) : Q(\varphi) = \text{fixed}\} = 0$ at $\alpha = 1$ (bound)
- $\Rightarrow$ if TCGL has knot solitons they are at best local minima
- Embedded hopf "soliton" unstable on S^3_R for all R
  Destabilized by supercurrent coupling before we reach E_{GL} (at $\alpha = 1$)
- Analysis relied on deep geometric understanding of variational problems for

$$E_1(\varphi) = \frac{1}{2}\|d\varphi\|^2 \qquad \text{harmonic map problem}$$

$$E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2 \qquad \text{“symplectic harmonic map problem”}$$

Second remains little explored, but has some very beautiful properties

FS versus SCFS

$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)

FS versus SCFS

$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)
- SCFS model: $\inf\{E(\varphi, C) \,:\, Q(\varphi) = n\} = 0$ for all $n \in \mathbb{Z}$

FS versus SCFS

$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)
- SCFS model: $\inf\{E(\varphi, C) : Q(\varphi) = n\} = 0$ for all $n \in \mathbb{Z}$
  Proof: $\exists$ φ with $\varphi = (0, 0, 1)$ outside $\overline{B}$.

FS versus SCFS

$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)
- SCFS model: $\inf\{E(\varphi, C) \,:\, Q(\varphi) = n\} = 0$ for all $n \in \mathbb{Z}$
  Proof: $\exists$ φ with $\varphi = (0, 0, 1)$ outside $\overline{B}$.
  $\exists$ C s.t. $\varphi^*\omega = -2\mathrm{d}C$ ($H^2(M) = 0$).

FS versus SCFS

$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)
- SCFS model: $\inf\{E(\varphi, C) \,:\, Q(\varphi) = n\} = 0$ for all $n \in \mathbb{Z}$
  Proof: $\exists\ \varphi$ with $\varphi = (0, 0, 1)$ outside $\overline{B}$.
  $\exists\ C$ s.t. $\varphi^*\omega = -2\mathrm{d}C$ ($H^2(M) = 0$).
  Can assume $C = 0$ outside $\overline{B}$ ($H^1(M \backslash \overline{B}) = 0$).

FS versus SCFS

$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)
- SCFS model: $\inf\{E(\varphi, C) \,:\, Q(\varphi) = n\} = 0$ for all $n \in \mathbb{Z}$
  Proof: $\exists$ φ with $\varphi = (0, 0, 1)$ outside $\overline{B}$.
  $\exists$ C s.t. $\varphi^*\omega = -2\mathrm{d}C$ ($H^2(M) = 0$).
  Can assume $C = 0$ outside $\overline{B}$ ($H^1(M\backslash\overline{B}) = 0$).
  Let $D_\lambda : M \to M$, $D_\lambda(x) = \lambda x$, $\lambda > 0$.

FS versus SCFS

$M = \mathbb{R}^3$, $N = S^2$

- FS model: $E_{FS}(\varphi) \geq cQ(\varphi)^{\frac{3}{4}}$ (VK bound)
- SCFS model: $\inf\{E(\varphi, C) \,:\, Q(\varphi) = n\} = 0$ for all $n \in \mathbb{Z}$
  Proof: $\exists$ φ with $\varphi = (0, 0, 1)$ outside $\overline{B}$.
  $\exists$ C s.t. $\varphi^*\omega = -2\mathrm{d}C$ ($H^2(M) = 0$).
  Can assume $C = 0$ outside $\overline{B}$ ($H^1(M \backslash \overline{B}) = 0$).
  Let $D_\lambda : M \to M$, $D_\lambda(x) = \lambda x$, $\lambda > 0$.

$$E(\varphi \circ D_\lambda, D_\lambda^* C) = \frac{1}{2\lambda}\|\mathrm{d}\varphi\|^2 + 0 + \frac{1}{2\lambda}\|C\|^2$$

(back)