The ground state of a chiral ferromagnet

Martin Speight
https://cp1lump.github.io
Joint work with Tom Winyard (Dundee), Bernd Schroers (Edinburgh)

Funded by EPSRC

17/3/26

University of Leeds

The standard model

$$\boldsymbol{m} : \mathbb{R}^2 \to S^2$$

The standard model

$$\boldsymbol{m} : \mathbb{R}^2 \to S^2$$

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

The standard model

$$\boldsymbol{m} : \mathbb{R}^2 \to S^2$$

$$\begin{aligned} E &= \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x \\ &= E_{exch} + E_{DMI} + E_{pot} \end{aligned}$$

The standard model

$$\boldsymbol{m} : \mathbb{R}^2 \to S^2$$

$$\begin{aligned} E &= \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x \\ &= E_{exch} + E_{DMI} + E_{pot} \\ &= \int_{\mathbb{R}^2} \mathscr{E} \end{aligned}$$

The standard model

$$\boldsymbol{m} : \mathbb{R}^2 \to S^2$$

$$\begin{aligned} E &= \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x \\ &= E_{exch} + E_{DMI} + E_{pot} \\ &= \int_{\mathbb{R}^2} \mathscr{E} \end{aligned}$$

$$P_{\boldsymbol{m}} \left(-\nabla^2 \boldsymbol{m} - 2 \sum_i \boldsymbol{d}_i \times \partial_i \boldsymbol{m} - \boldsymbol{H} \right) = \boldsymbol{0}$$

Ground state?

- The solution $\boldsymbol{m} : \mathbb{R}^2 \to S^2$ of

$$P_{\boldsymbol{m}} \left(-\nabla^2 \boldsymbol{m} - 2 \sum_i \boldsymbol{d}_i \times \partial_i \boldsymbol{m} - \boldsymbol{H} \right) = \boldsymbol{0}$$

  that has minimal $\langle \mathscr{E} \rangle$ (if that exists...)

Ground state?

- The solution $\boldsymbol{m} : \mathbb{R}^2 \to S^2$ of

$$P_{\boldsymbol{m}} \left(-\nabla^2 \boldsymbol{m} - 2 \sum_i \boldsymbol{d}_i \times \partial_i \boldsymbol{m} - \boldsymbol{H} \right) = \boldsymbol{0}$$

  that has minimal $\langle \mathscr{E} \rangle$ (if that exists...)
- We will consider only
  - Homogeneous states: $\boldsymbol{m} =$ constant
  - Translation invariant states: $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x})$
  - Doubly periodic states: $\boldsymbol{m}(\boldsymbol{x} + n_1 \boldsymbol{v}_1 + n_2 \boldsymbol{v}_2) \equiv \boldsymbol{m}(\boldsymbol{x})$

Reduction of DMI

- By rotating $\boldsymbol{x} \mapsto R\boldsymbol{x}$ our spatial coordinate system, we can assume:

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| \geq |\boldsymbol{d}_2|$$

Reduction of DMI

- By rotating $\boldsymbol{x} \mapsto R\boldsymbol{x}$ our spatial coordinate system, we can assume:

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| \geq |\boldsymbol{d}_2|$$

- By dilating $\boldsymbol{x} \mapsto \lambda\boldsymbol{x}$, we can assume:

$$|\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \in [0, 1]$$

Reduction of DMI

- By rotating $\boldsymbol{x} \mapsto R\boldsymbol{x}$ our spatial coordinate system, we can assume:

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| \geq |\boldsymbol{d}_2|$$

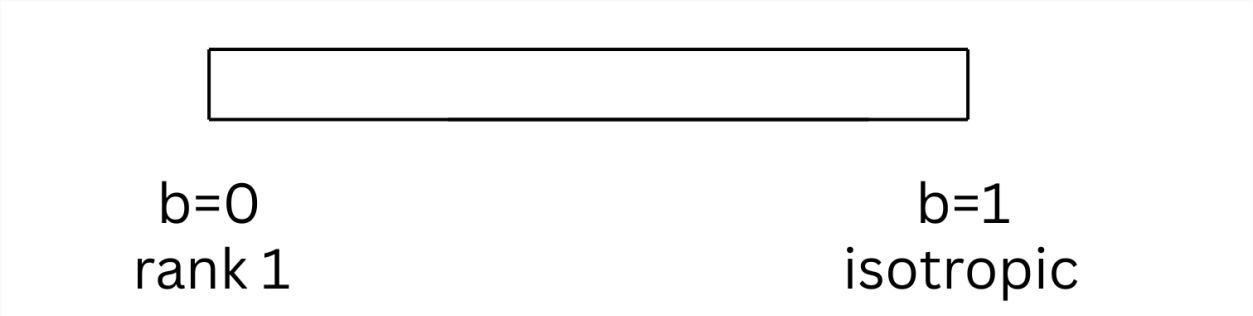
- By dilating $\boldsymbol{x} \mapsto \lambda\boldsymbol{x}$, we can assume:

$$|\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \in [0, 1]$$

- By rotating our target sphere (and $\boldsymbol{H}$), we can assume:

$$\boldsymbol{d}_1 = (1, 0, 0), \qquad \boldsymbol{d}_2 = (0, b, 0)$$

Reduction of DMI



- WARNING: $\boldsymbol{m}, \boldsymbol{H}$ oriented relative to plane spanned by $\boldsymbol{d}_1, \boldsymbol{d}_2$, **not** the sample plane

Skyrmions

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m}) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

- FM state $\boldsymbol{m}(\boldsymbol{x}) = \hat{\boldsymbol{H}}$, always a solution, $E = 0$.

Skyrmions

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m}) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

- FM state $\boldsymbol{m}(\boldsymbol{x}) = \hat{\boldsymbol{H}}$, always a solution, $E = 0$.
- If $\boldsymbol{m}(\boldsymbol{x}) \to \hat{\boldsymbol{H}}$ as $|\boldsymbol{x}| \to \infty$, have degree

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} \boldsymbol{m} \cdot (\partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m}) d^2x \in \mathbb{Z}$$

Skyrmions

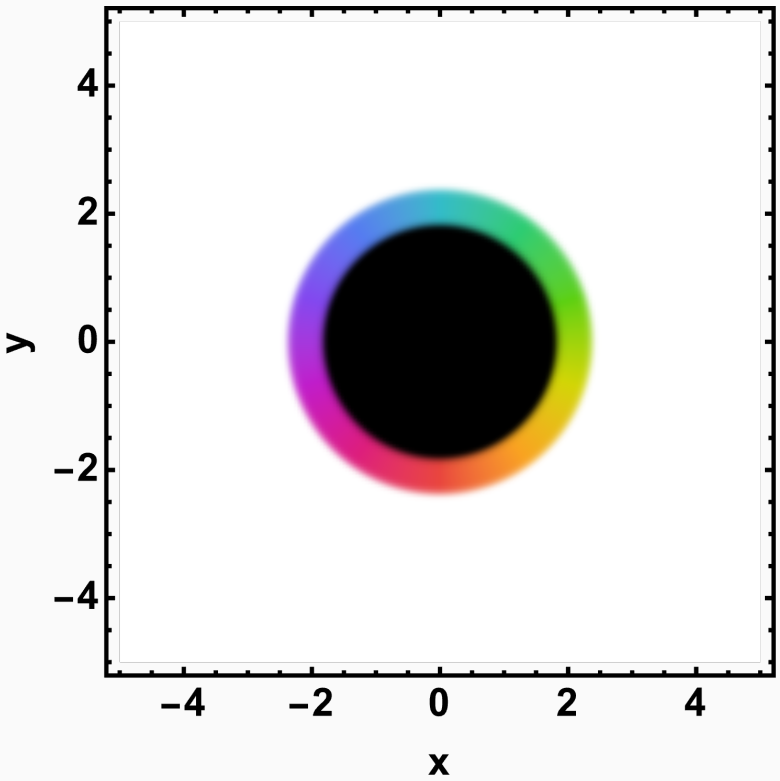
$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m}) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

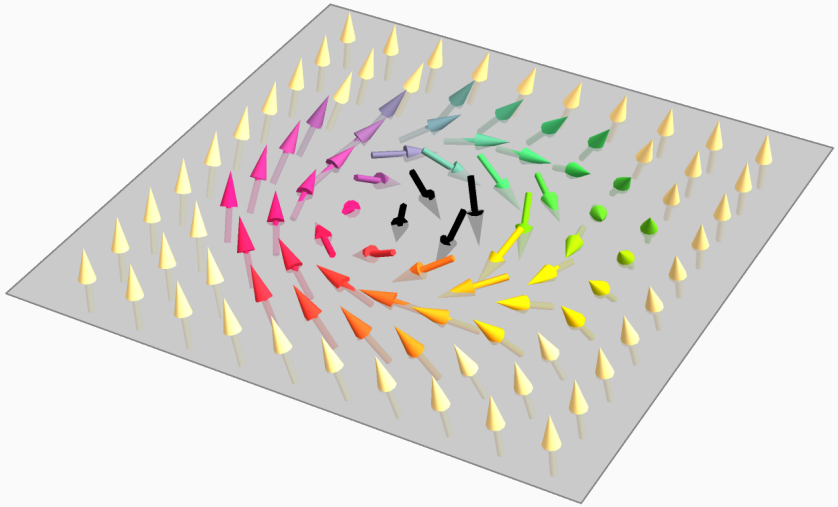
- FM state $\boldsymbol{m}(\boldsymbol{x}) = \hat{\boldsymbol{H}}$, always a solution, $E = 0$.
- If $\boldsymbol{m}(\boldsymbol{x}) \to \hat{\boldsymbol{H}}$ as $|\boldsymbol{x}| \to \infty$, have degree

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} \boldsymbol{m} \cdot (\partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m}) d^2x \in \mathbb{Z}$$

- Can seek minimizers of E of fixed Q. Skyrmions $Q = -1$ (or $Q = +1$)

Skyrmions





The Derrick scaling argument

- Derrick scaling: $\boldsymbol{m}_\lambda(\boldsymbol{x}) = \boldsymbol{m}(\lambda \boldsymbol{x})$

$$E(\lambda) = E(\boldsymbol{m}_\lambda) = E_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda^2} E_{pot}(\boldsymbol{m}).$$

The Derrick scaling argument

- Derrick scaling: $\boldsymbol{m}_\lambda(\boldsymbol{x}) = \boldsymbol{m}(\lambda \boldsymbol{x})$

$$E(\lambda) = E(\boldsymbol{m}_\lambda) = E_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda^2} E_{pot}(\boldsymbol{m}).$$

$\boldsymbol{m}$ critical $\Rightarrow E'(1) = 0 \Rightarrow E_{DMI} = -2E_{pot} < 0$

The Derrick scaling argument

- Derrick scaling: $\boldsymbol{m}_\lambda(\boldsymbol{x}) = \boldsymbol{m}(\lambda \boldsymbol{x})$

$$E(\lambda) = E(\boldsymbol{m}_\lambda) = E_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda^2} E_{pot}(\boldsymbol{m}).$$

  $\boldsymbol{m}$ critical $\Rightarrow E'(1) = 0 \Rightarrow E_{DMI} = -2E_{pot} < 0$

- Possible that $E(\boldsymbol{m}_{skyrmion}) < 0$: skyrmion **lattice** can then form spontaneously

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E_{exch}^{(i)} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E^{(i)}_{exch} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

$$E(\boldsymbol{m}_\lambda) \;=\; \lambda E^{(1)}_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E^{(2)}_{exch}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda} E_{pot}(\boldsymbol{m})$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E^{(i)}_{exch} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

$$E(\boldsymbol{m}_\lambda) = \lambda E^{(1)}_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E^{(2)}_{exch}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda} E_{pot}(\boldsymbol{m})$$

$$\left.\frac{dE(\boldsymbol{m}_\lambda)}{d\lambda}\right|_{\lambda=1} = E^{(1)}_{exch} - E^{(2)}_{exch} - E_{pot} = 0$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E^{(i)}_{exch} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

$$\begin{aligned} E(\boldsymbol{m}_\lambda) &= \lambda E^{(1)}_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E^{(2)}_{exch}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda} E_{pot}(\boldsymbol{m}) \\ \left.\frac{dE(\boldsymbol{m}_\lambda)}{d\lambda}\right|_{\lambda=1} &= E^{(1)}_{exch} - E^{(2)}_{exch} - E_{pot} = 0 \\ E &= E^{(1)}_{exch} + E^{(2)}_{exch} + E_{DMI} + E_{pot} \\ &= 2E^{(2)}_{exch} + 2E_{pot} + E_{DMI} \\ &= 2E^{(2)}_{exch} > 0 \end{aligned}$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

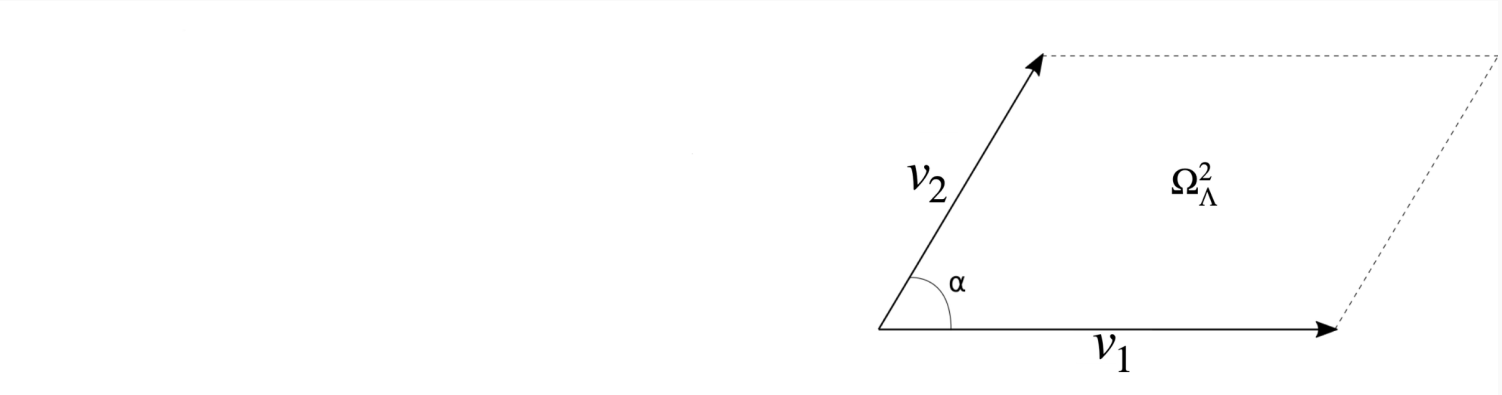
$$E^{(i)}_{exch} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

$$\begin{aligned} E(\boldsymbol{m}_\lambda) &= \lambda E^{(1)}_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E^{(2)}_{exch}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda} E_{pot}(\boldsymbol{m}) \\ \left.\frac{dE(\boldsymbol{m}_\lambda)}{d\lambda}\right|_{\lambda=1} &= E^{(1)}_{exch} - E^{(2)}_{exch} - E_{pot} = 0 \\ E &= E^{(1)}_{exch} + E^{(2)}_{exch} + E_{DMI} + E_{pot} \\ &= 2E^{(2)}_{exch} + 2E_{pot} + E_{DMI} \\ &= 2E^{(2)}_{exch} > 0 \end{aligned}$$

- Ground state is never a skyrmion lattice.

Skyrmion lattices

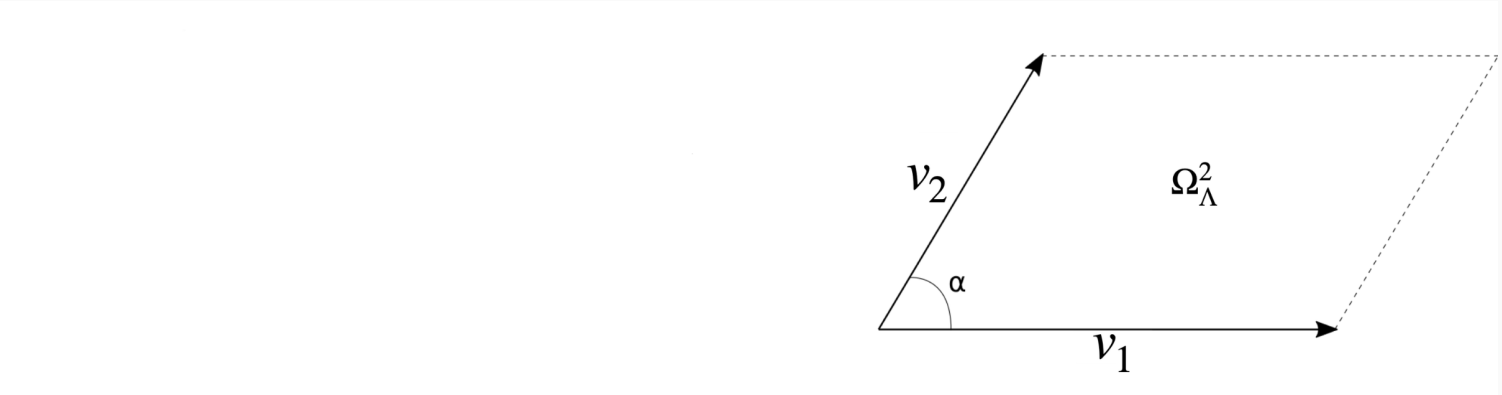


- Assume $\boldsymbol{m} : \mathbb{R}^2 \to S^2$ is periodic w.r.t. lattice Λ generated by $\boldsymbol{v}_1, \boldsymbol{v}_2$

$$\boldsymbol{m}(\boldsymbol{x} + n_1\boldsymbol{v}_1 + n_2\boldsymbol{v}_2) = \boldsymbol{m}(\boldsymbol{x})$$

for all $(n_1, n_2) \in \mathbb{Z}^2$.

Skyrmion lattices

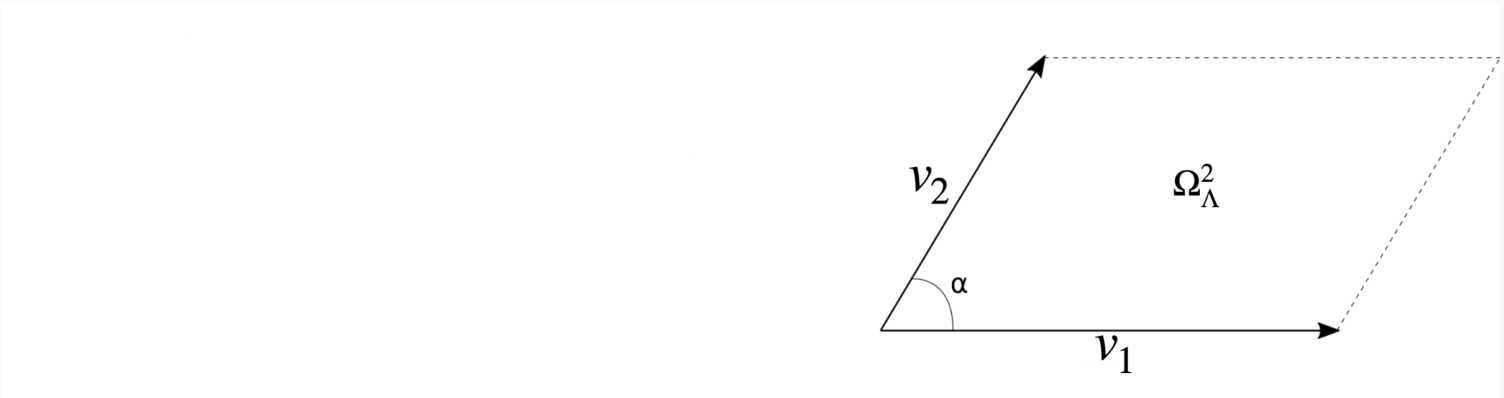


- Assume $\boldsymbol{m} : \mathbb{R}^2 \to S^2$ is periodic w.r.t. lattice Λ generated by $\boldsymbol{v}_1, \boldsymbol{v}_2$

$$\boldsymbol{m}(\boldsymbol{x} + n_1\boldsymbol{v}_1 + n_2\boldsymbol{v}_2) = \boldsymbol{m}(\boldsymbol{x})$$

  for all $(n_1, n_2) \in \mathbb{Z}^2$.
- It lives on torus $\Omega_\Lambda = \mathbb{R}^2/\Lambda$

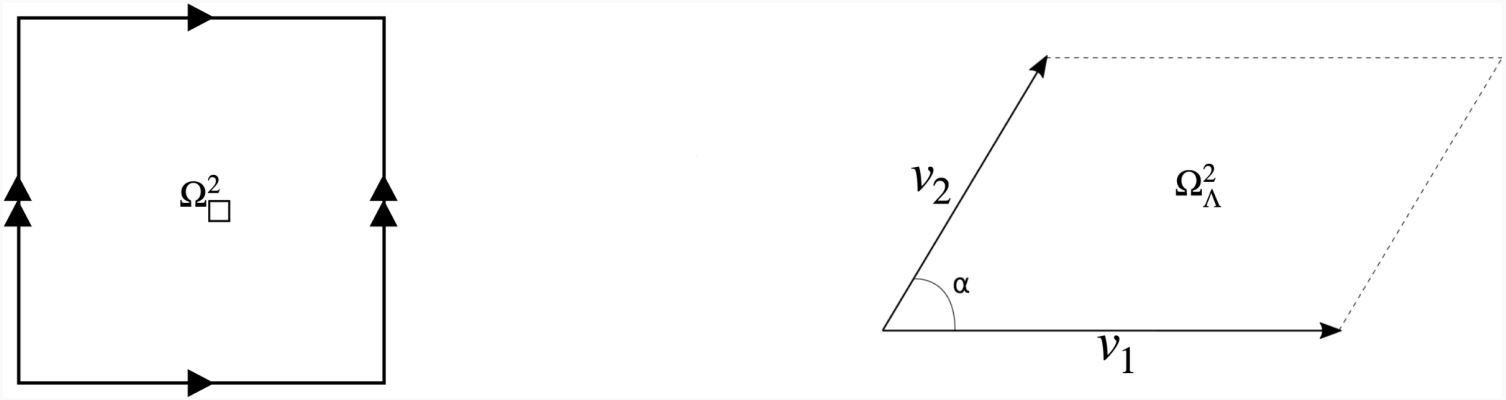
Skyrmion lattices



- Energy per unit area is

$$\langle \mathscr{E} \rangle = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \left\{ \frac{1}{2} |d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i}) + V(\boldsymbol{m}) \right\} d^2x$$

Skyrmion lattices



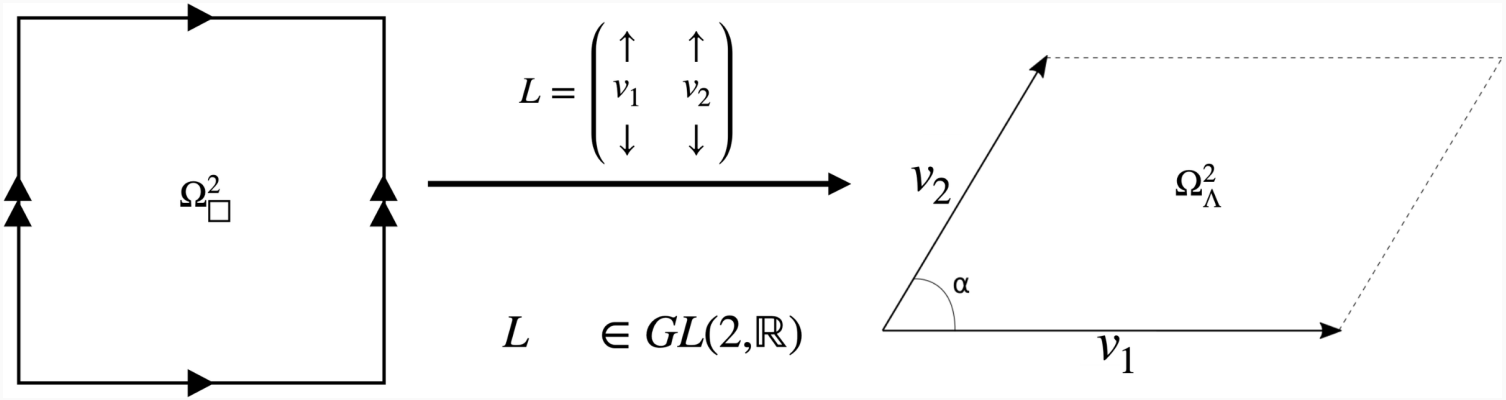
- Energy per unit area is

$$\langle \mathscr{E} \rangle = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \left\{ \frac{1}{2} |d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i}) + V(\boldsymbol{m}) \right\} d^2x$$

- Define new coords X_1, X_2,

$$\boldsymbol{x} = X_1 \boldsymbol{v}_1 + X_2 \boldsymbol{v}_2 = L\boldsymbol{X}$$

Skyrmion lattices



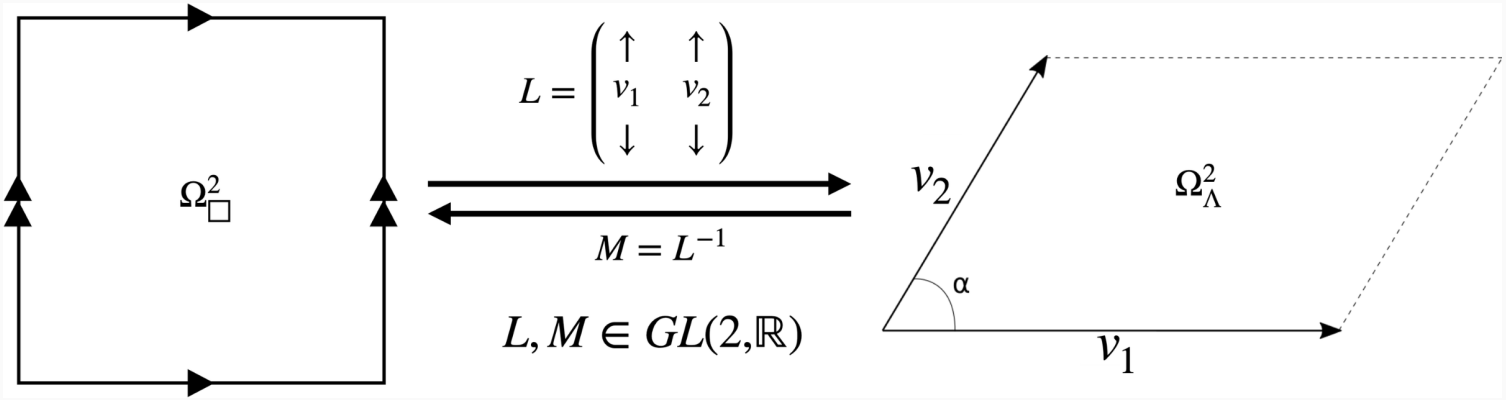
- Energy per unit area is

$$\langle \mathscr{E} \rangle = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \left\{ \frac{1}{2} |d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i}) + V(\boldsymbol{m}) \right\} d^2x$$

- Define new coords X_1, X_2,

$$\boldsymbol{x} = X_1 \boldsymbol{v}_1 + X_2 \boldsymbol{v}_2 = L\boldsymbol{X}$$

Skyrmion lattices



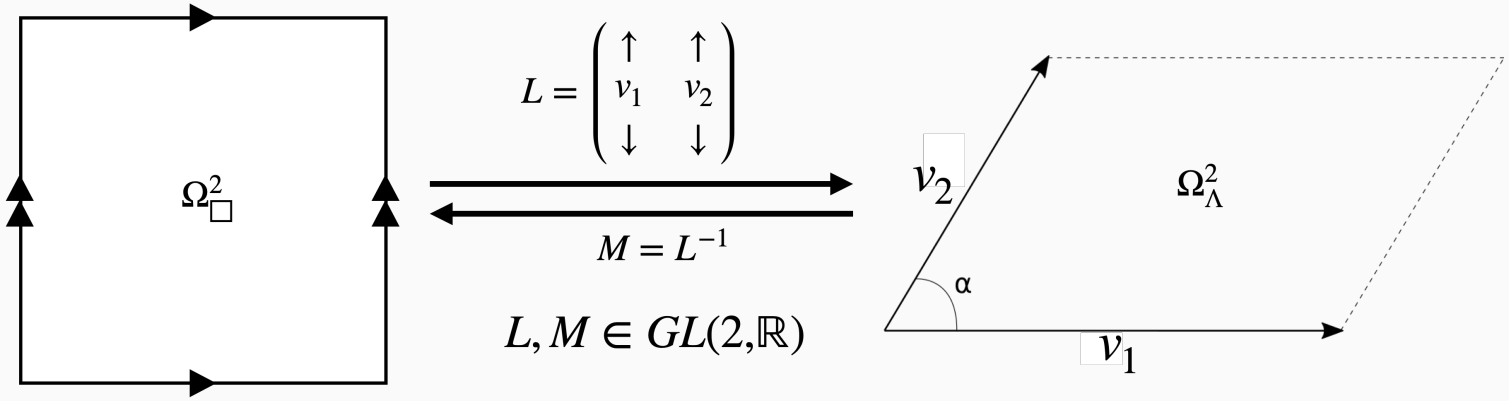
- Energy per unit area is

$$\langle \mathscr{E} \rangle = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \left\{ \frac{1}{2} |d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i}) + V(\boldsymbol{m}) \right\} d^2x$$

- Define new coords X_1, X_2,

$$\boldsymbol{x} = X_1 \boldsymbol{v}_1 + X_2 \boldsymbol{v}_2 = L\boldsymbol{X}$$

Skyrmion lattices



$$\begin{aligned}
\langle \mathscr{E} \rangle &= \frac{1}{2}\,\mathrm{tr}(M^T G(\boldsymbol{m}) M) + \mathrm{tr}(\mathscr{D}(\boldsymbol{m}) M) + C(\boldsymbol{m}) \\
G_{ij}(\boldsymbol{m}) &= \int_\square \frac{\partial \boldsymbol{m}}{\partial X_i} \cdot \frac{\partial \boldsymbol{m}}{\partial X_j} dX_1 dX_2 \\
\mathscr{D}_{ij}(\boldsymbol{m}) &= \boldsymbol{d}_i \cdot \int_\square \boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial X_j} dX_1 dX_2 \\
C(\boldsymbol{m}) &= \int_\square V(\boldsymbol{m}) dX_1 dX_2
\end{aligned}$$

Skyrmion lattices

- Switch viewpoint: $\boldsymbol{m}$ is a field on the fixed torus $\Omega_\square = \mathbb{R}^2/\mathbb{Z}^2$, but it minimizes a functional that depends parametrically on $M \in GL(2,\mathbb{R})$ (the lattice geometry)

Skyrmion lattices

- Switch viewpoint: $\boldsymbol{m}$ is a field on the fixed torus $\Omega_\square = \mathbb{R}^2/\mathbb{Z}^2$, but it minimizes a functional that depends parametrically on $M \in GL(2,\mathbb{R})$ (the lattice geometry)
- Optimal lattice minimizes $\langle \mathscr{E} \rangle$ with respect to **both** $\boldsymbol{m}$ and M!

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_{\square} \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2} \operatorname{tr}(M^T G M) + \operatorname{tr}(\mathscr{D} M) + C$$

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_{\square} \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2} \operatorname{tr}(M^T G M) + \operatorname{tr}(\mathscr{D} M) + C$$

attains global min at

$$M_* = -G^{-1} \mathscr{D}^T$$

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_{\square} \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2} \operatorname{tr}(M^T G M) + \operatorname{tr}(\mathscr{D} M) + C$$

attains global min at

$$M_* = -G^{-1} \mathscr{D}^T$$

- G symmetric pos def (if $Q \neq 0$), certainly invertible

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_{\square} \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2} \operatorname{tr}(M^T G M) + \operatorname{tr}(\mathscr{D} M) + C$$

attains global min at

$$M_* = -G^{-1} \mathscr{D}^T$$

- G symmetric pos def (if $Q \neq 0$), certainly invertible
- $\det M_* = \det \mathscr{D} / \det G = 0$ if $\mathscr{D}$ not invertible!
  - $b \neq 0$: $\det \mathscr{D} \neq 0$ for a.e. $\boldsymbol{m}$
  - $b = 0$: $\det \mathscr{D} = 0$ for a.e. $\boldsymbol{m}$ [no stable lattices for rank 1]
- $\det M_* < 0$ if $\det \mathscr{D} < 0$

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_\square \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2} \operatorname{tr}(M^T G M) + \operatorname{tr}(\mathscr{D} M) + C$$

attains global min at

$$M_* = -G^{-1} \mathscr{D}^T$$

- G symmetric pos def (if $Q \neq 0$), certainly invertible
- $\det M_* = \det \mathscr{D} / \det G = 0$ if $\mathscr{D}$ not invertible!
  - $b \neq 0$: $\det \mathscr{D} \neq 0$ for a.e. $\boldsymbol{m}$
  - $b = 0$: $\det \mathscr{D} = 0$ for a.e. $\boldsymbol{m}$ [no stable lattices for rank 1]
- $\det M_* < 0$ if $\det \mathscr{D} < 0$
  - diffeo flips orientation, $Q \mapsto -Q$
  - Feature not bug!

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2, \mathbb{R})$

$$\langle \mathscr{E} \rangle : C^\infty(\Omega_\square, S^2) \to \mathbb{R}.$$

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2, \mathbb{R})$

$$\langle \mathscr{E} \rangle : C^\infty(\Omega_\square, S^2) \to \mathbb{R}.$$

- Minimize by **arrested Newton flow**

$$\ddot{\boldsymbol{m}} = -\operatorname{grad}_{L^2} \langle \mathscr{E} \rangle$$

$\boldsymbol{m}(0)$ some initial guess, $\dot{\boldsymbol{m}}(0) = \mathbf{0}$

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2, \mathbb{R})$

$$\langle \mathscr{E} \rangle : C^\infty(\Omega_\square, S^2) \to \mathbb{R}.$$

- Minimize by **arrested Newton flow**

$$\ddot{\boldsymbol{m}} = -\operatorname{grad}_{L^2} \langle \mathscr{E} \rangle$$

  $\boldsymbol{m}(0)$ some initial guess, $\dot{\boldsymbol{m}}(0) = \mathbf{0}$
- Arrest (set $\dot{\boldsymbol{m}}(t) = \mathbf{0}$) if

$$\langle \dot{\boldsymbol{m}}(t), \operatorname{grad}_{L^2} \langle \mathscr{E} \rangle \rangle_{L^2} > 0$$

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2, \mathbb{R})$

$$\langle \mathscr{E} \rangle : C^\infty(\Omega_\square, S^2) \to \mathbb{R}.$$

- Minimize by **arrested Newton flow**

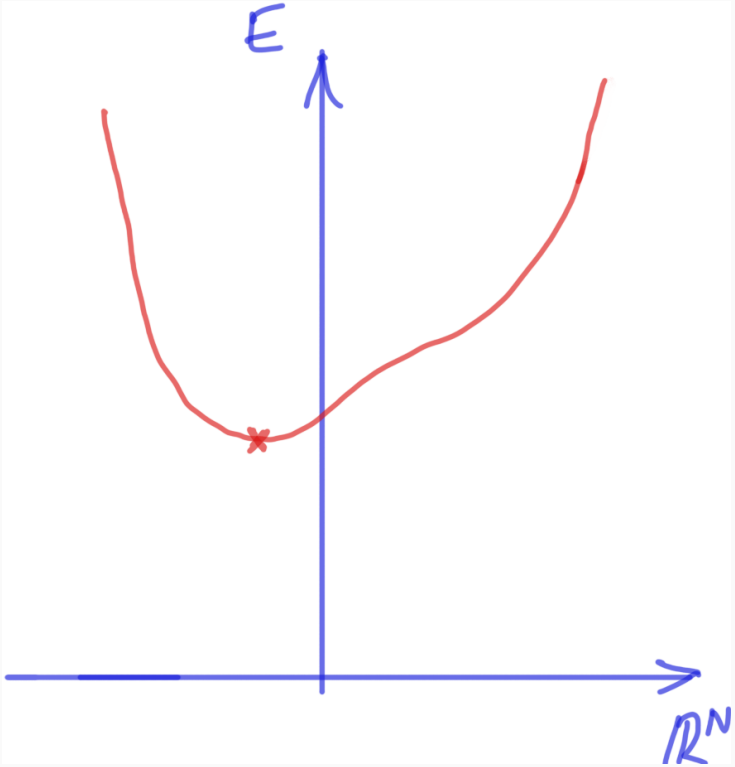
$$\ddot{\boldsymbol{m}} = -\operatorname{grad}_{L^2} \langle \mathscr{E} \rangle$$

  $\boldsymbol{m}(0)$ some initial guess, $\dot{\boldsymbol{m}}(0) = \mathbf{0}$
- Arrest (set $\dot{\boldsymbol{m}}(t) = \mathbf{0}$) if

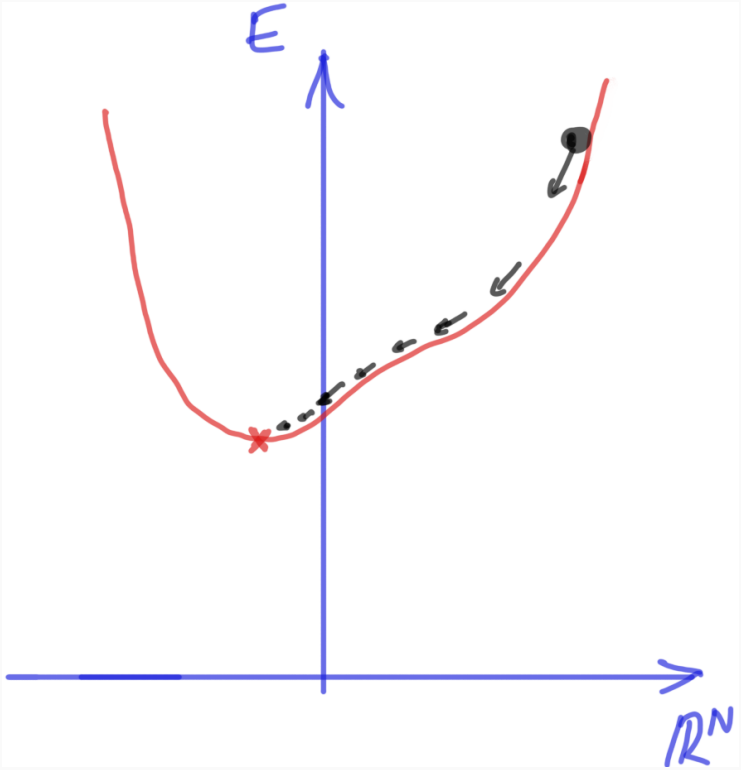
$$\langle \dot{\boldsymbol{m}}(t), \operatorname{grad}_{L^2} \langle \mathscr{E} \rangle \rangle_{L^2} > 0$$

- Continue until $\| \operatorname{grad}_{L^2} \langle \mathscr{E} \rangle \|_{C^0} < tol$

Skyrmion lattices: optimizing the field

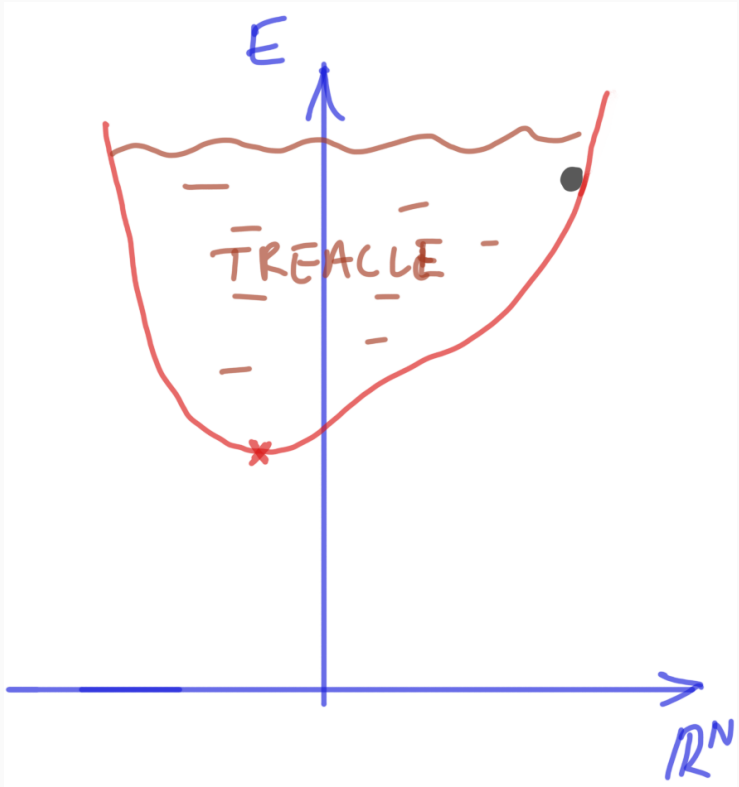


Skyrmion lattices: optimizing the field



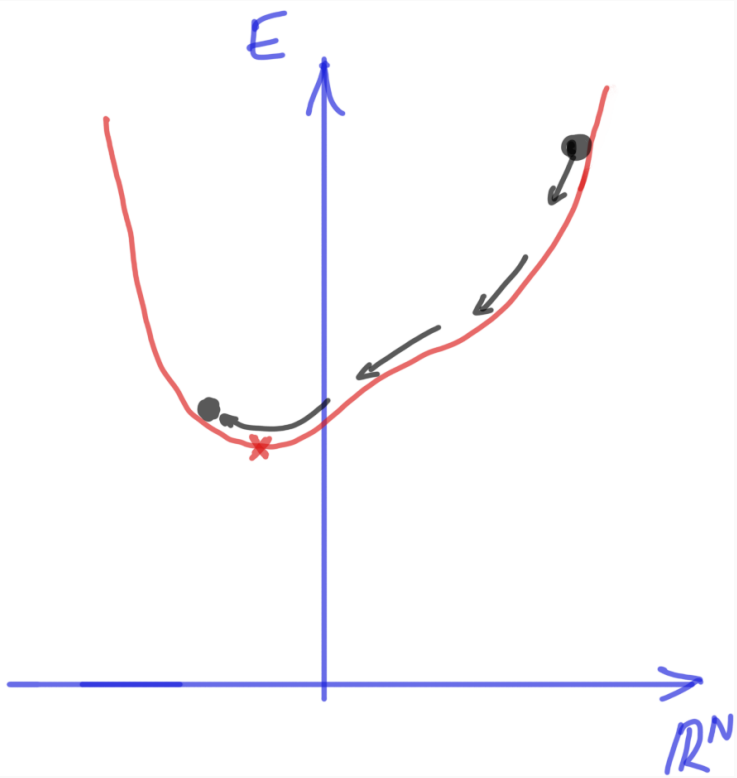
Gradient flow: $\dot{\boldsymbol{m}} = -\operatorname{grad} E$

Skyrmion lattices: optimizing the field



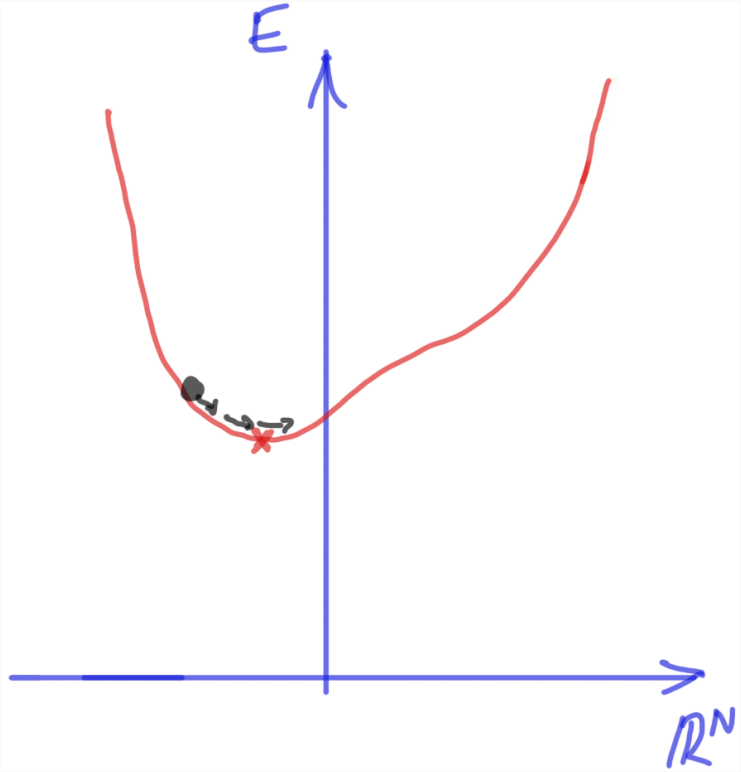
Gradient flow: $\dot{\boldsymbol{m}} = -\operatorname{grad} E$

Skyrmion lattices: optimizing the field



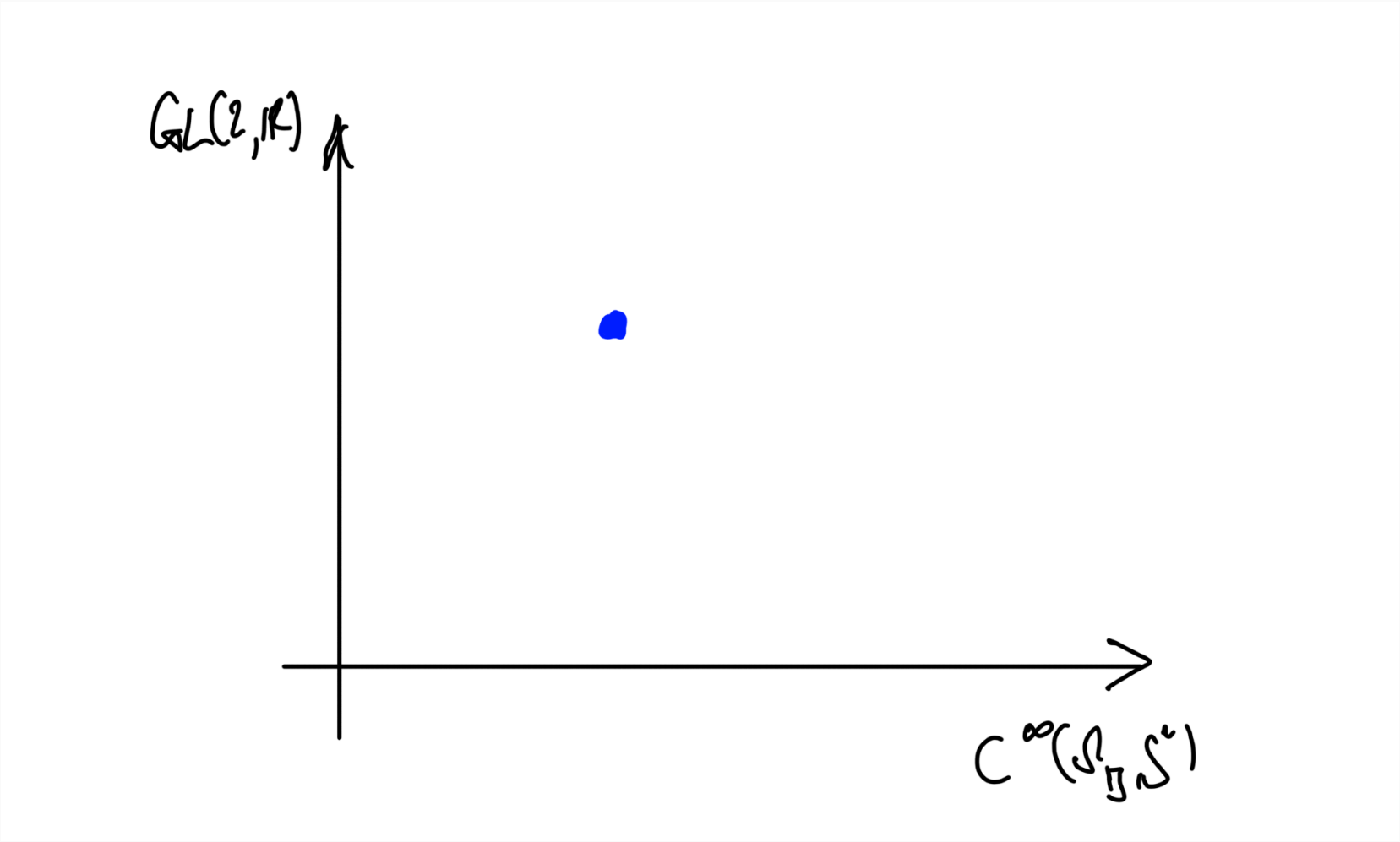
Arrested Newton flow: $\ddot{\boldsymbol{m}} = -\operatorname{grad} E$

Skyrmion lattices: optimizing the field

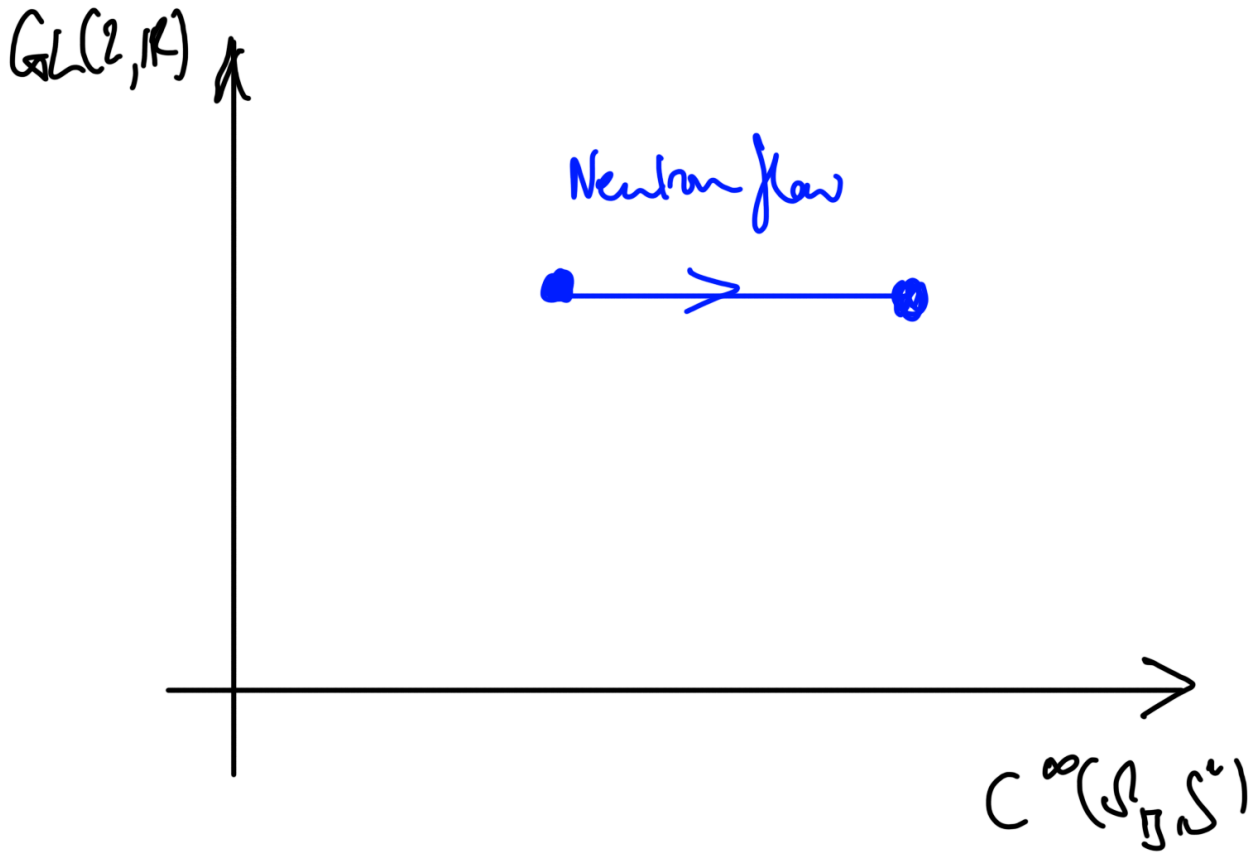


Arrested Newton flow: $\ddot{\boldsymbol{m}} = -\operatorname{grad} E$

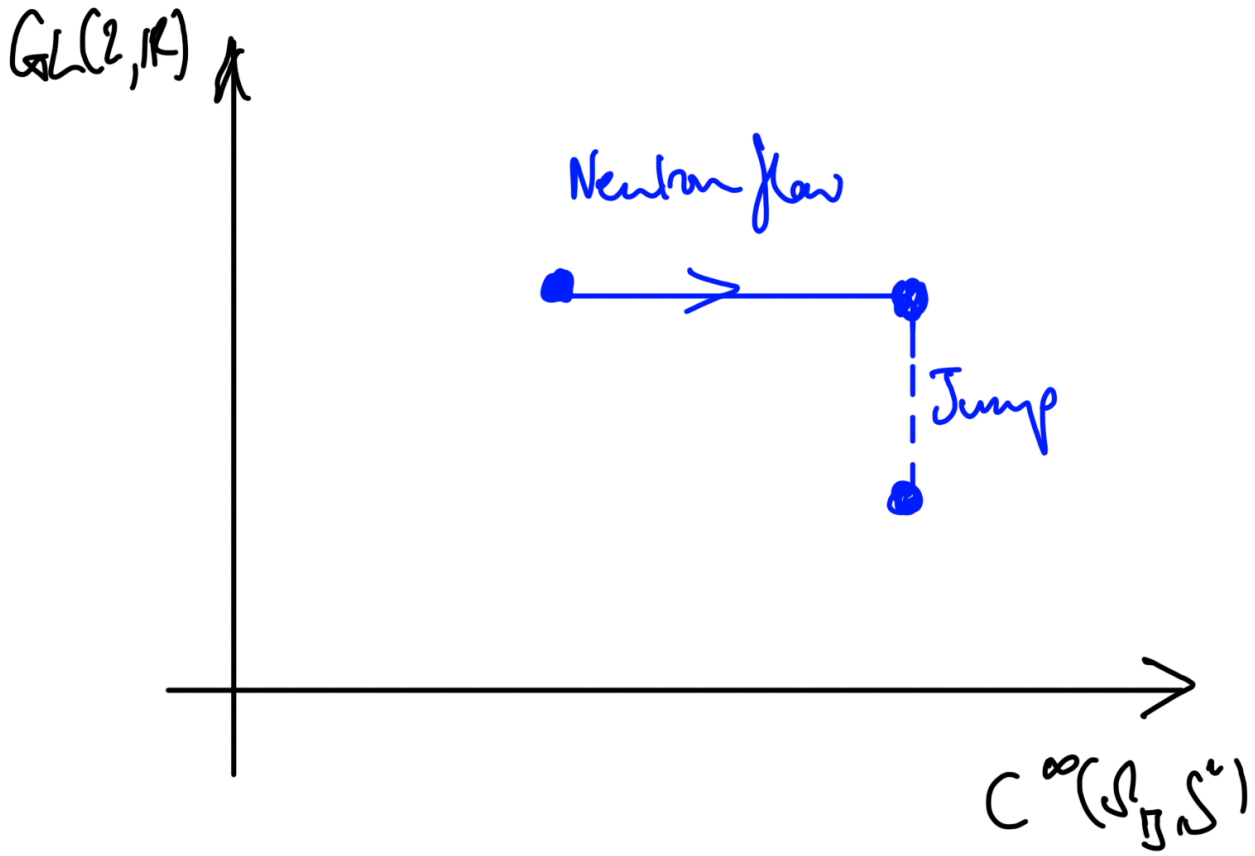
Skyrmion lattices: alternate



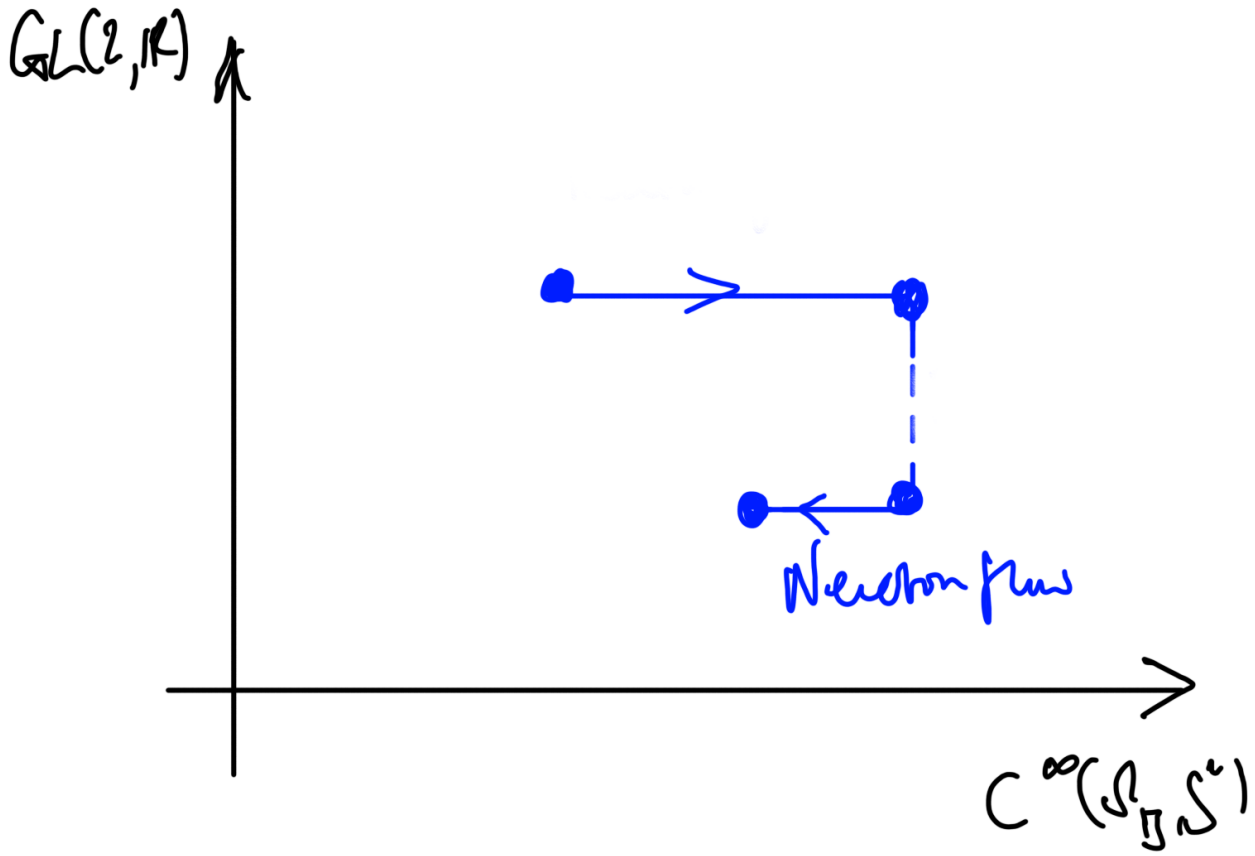
Skyrmion lattices: alternate



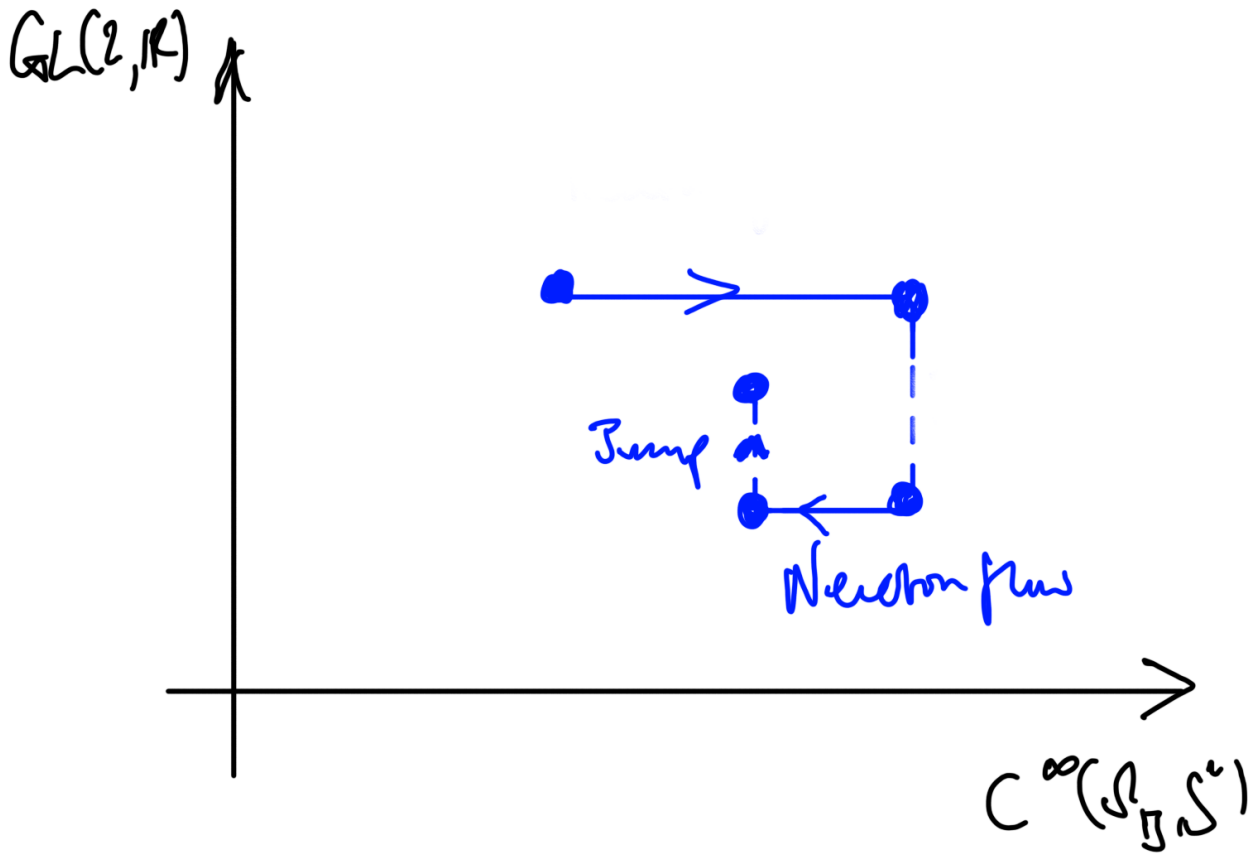
Skyrmion lattices: alternate



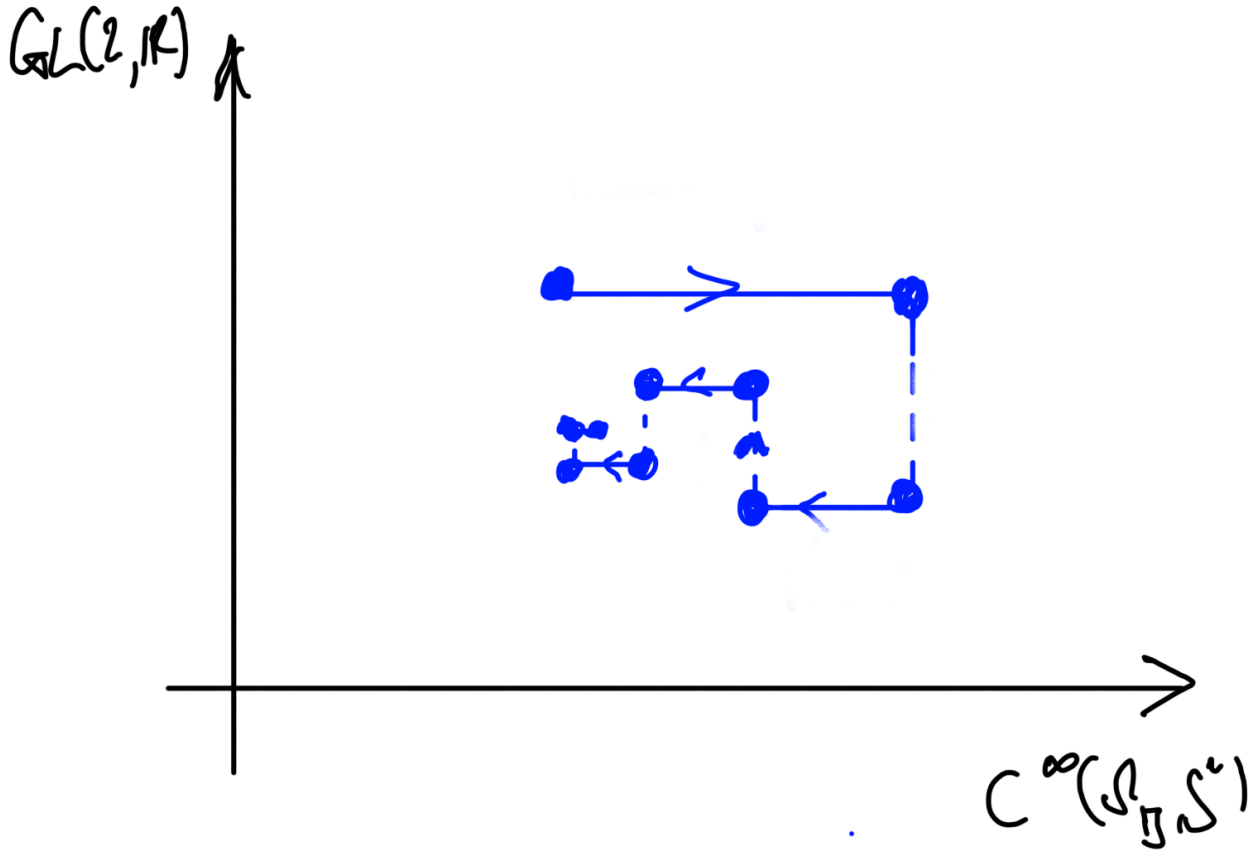
Skyrmion lattices: alternate



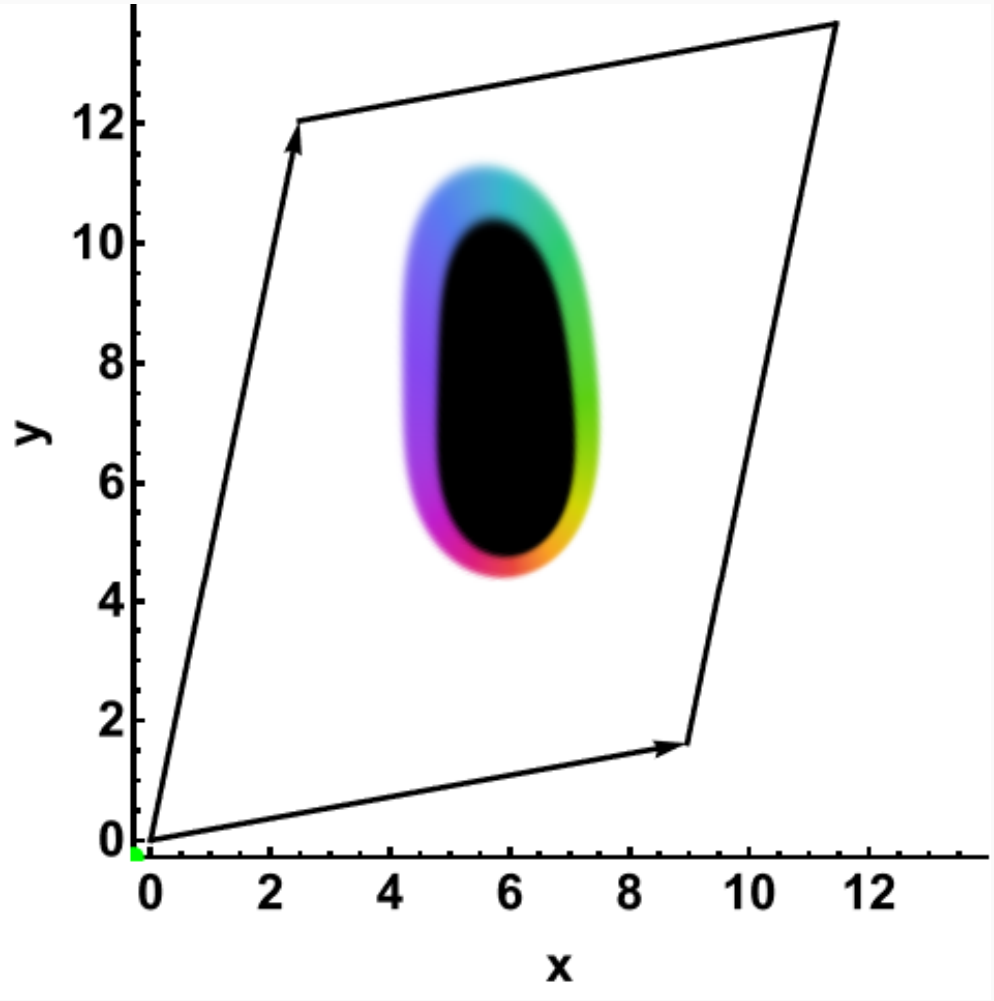
Skyrmion lattices: alternate



Skyrmion lattices: alternate



A Skyrmion lattice

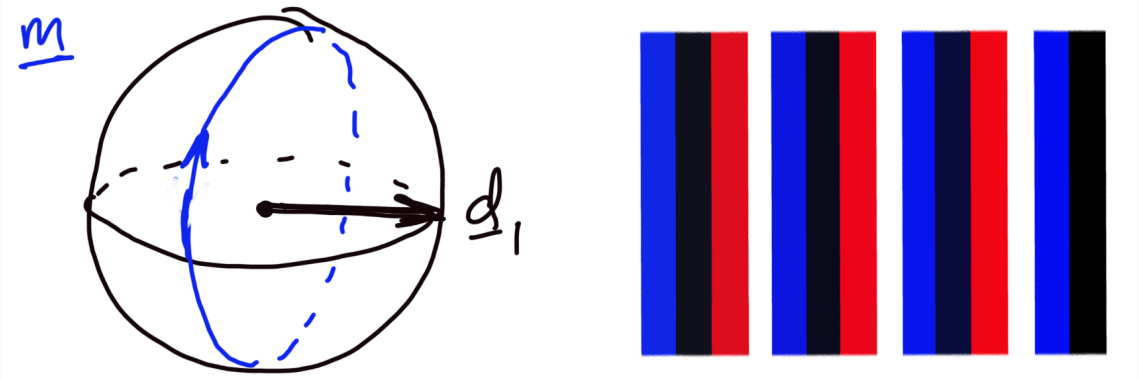


$b = 0.8$, $\boldsymbol{H} = (0.26, 0.15, 0.51)$, $\langle \mathscr{E} \rangle < 0$

Spiral phase

- Ground state for $\boldsymbol{H} = \boldsymbol{0}$?

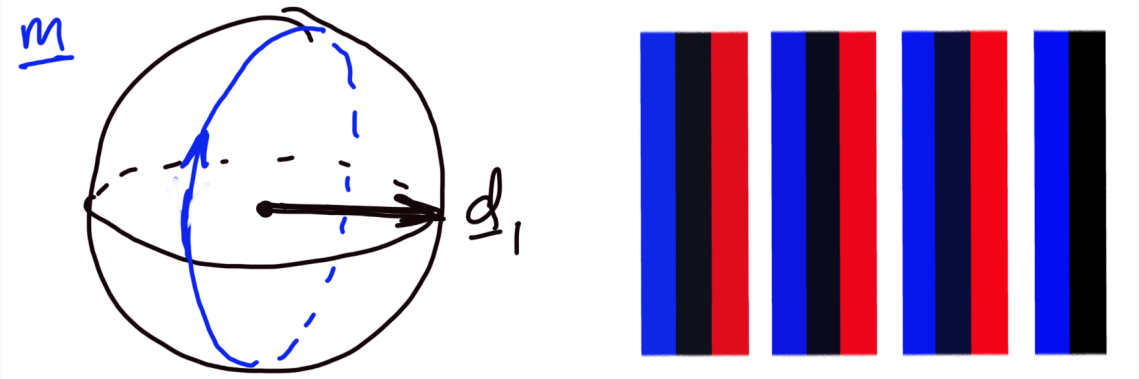
$$\boldsymbol{m}(x_1, x_2) = (0, \cos x_1, -\sin x_1), \qquad \mathscr{E} = -\frac{1}{2}$$



Spiral phase

- Ground state for $\boldsymbol{H} = \boldsymbol{0}$?

$$\boldsymbol{m}(x_1, x_2) = (0, \cos x_1, -\sin x_1), \qquad \mathscr{E} = -\frac{1}{2}$$



- Spiral phase: $\boldsymbol{m}(\boldsymbol{n}\cdot\boldsymbol{x})$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(t+T) \equiv \boldsymbol{m}(t)$

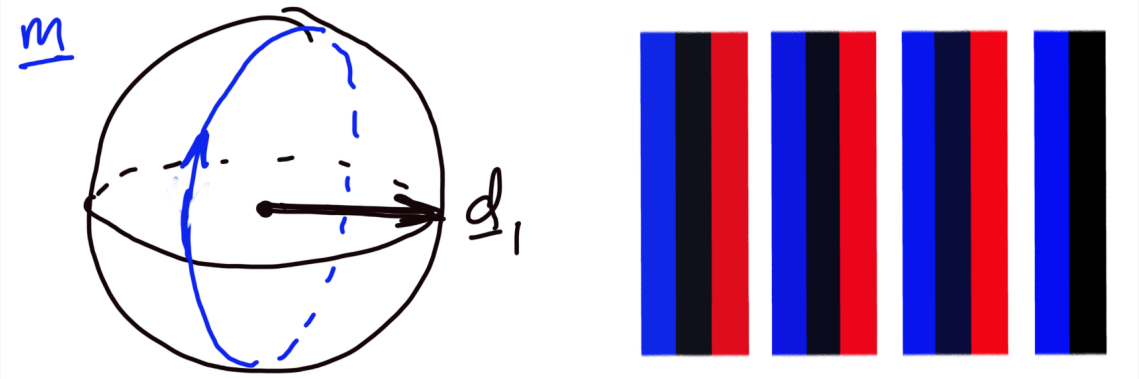
$$\langle \mathscr{E} \rangle = \frac{1}{T}\int_0^T \left\{ \frac{1}{2}|\dot{\boldsymbol{m}}|^2 + \boldsymbol{d}(\boldsymbol{n}) \cdot (\boldsymbol{m} \times \dot{\boldsymbol{m}}) + |\boldsymbol{H}| - \boldsymbol{H}\cdot\boldsymbol{m} \right\} < 0$$

where $\boldsymbol{d}(\boldsymbol{n}) = n_1\boldsymbol{d}_1 + n_2\boldsymbol{d}_2$

Spiral phase

- Ground state for $\boldsymbol{H} = \boldsymbol{0}$?

$$\boldsymbol{m}(x_1, x_2) = (0, \cos x_1, -\sin x_1), \qquad \mathscr{E} = -\frac{1}{2}$$



- Spiral phase: $\boldsymbol{m}(\boldsymbol{n}\cdot\boldsymbol{x})$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(t+T) \equiv \boldsymbol{m}(t)$

$$\langle \mathscr{E} \rangle = \frac{1}{T}\int_0^T \left\{ \frac{1}{2}|\dot{\boldsymbol{m}}|^2 + \boldsymbol{d}(\boldsymbol{n}) \cdot (\boldsymbol{m} \times \dot{\boldsymbol{m}}) + |\boldsymbol{H}| - \boldsymbol{H}\cdot\boldsymbol{m} \right\} < 0$$

  where $\boldsymbol{d}(\boldsymbol{n}) = n_1\boldsymbol{d}_1 + n_2\boldsymbol{d}_2$

- Must exist for $|\boldsymbol{H}|$ small

Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$

Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$
- Nonempty: $B_{1/2}(0) \subseteq \mathscr{H}_b$

Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$
- Nonempty: $B_{1/2}(0) \subseteq \mathscr{H}_b$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow s\boldsymbol{H} \in \mathscr{H}_b$ for all $s \in [0,1]$

Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$
- Nonempty: $B_{1/2}(0) \subseteq \mathscr{H}_b$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow s\boldsymbol{H} \in \mathscr{H}_b$ for all $s \in [0,1]$
- $\mathbb{Z}_2^3$ symmetric: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow (\pm H_1, \pm H_2, \pm H_3) \in \mathscr{H}_b$

Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$
- Nonempty: $B_{1/2}(0) \subseteq \mathscr{H}_b$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow s\boldsymbol{H} \in \mathscr{H}_b$ for all $s \in [0,1]$
- $\mathbb{Z}_2^3$ symmetric: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow (\pm H_1, \pm H_2, \pm H_3) \in \mathscr{H}_b$
- Open

Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$
- Nonempty: $B_{1/2}(0) \subseteq \mathscr{H}_b$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow s\boldsymbol{H} \in \mathscr{H}_b$ for all $s \in [0,1]$
- $\mathbb{Z}_2^3$ symmetric: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow (\pm H_1, \pm H_2, \pm H_3) \in \mathscr{H}_b$
- Open
- Bounded: $\mathscr{H}_b \subseteq B_1(0)$

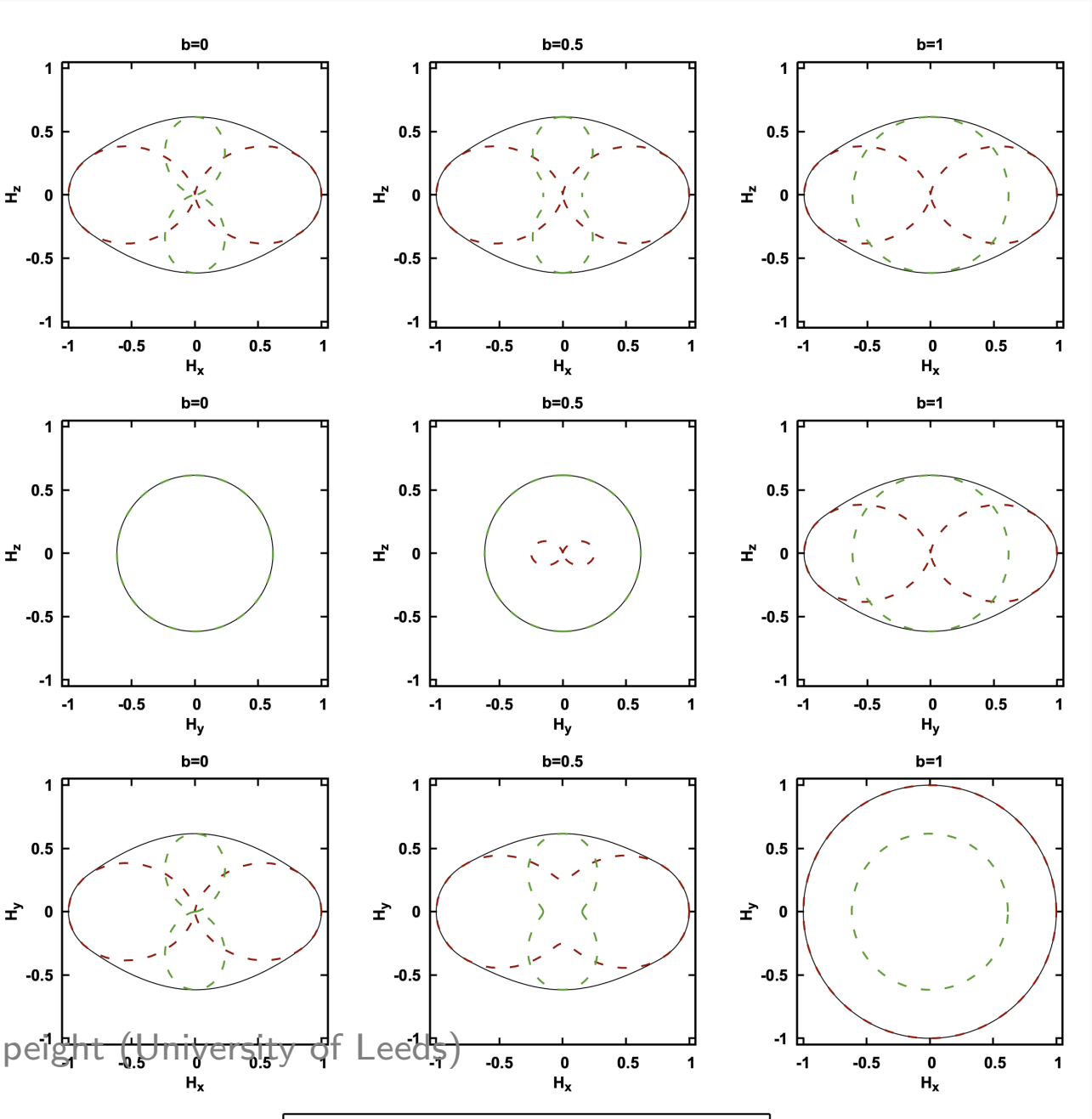
Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$
- Nonempty: $B_{1/2}(0) \subseteq \mathscr{H}_b$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow s\boldsymbol{H} \in \mathscr{H}_b$ for all $s \in [0,1]$
- $\mathbb{Z}_2^3$ symmetric: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow (\pm H_1, \pm H_2, \pm H_3) \in \mathscr{H}_b$
- Open
- Bounded: $\mathscr{H}_b \subseteq B_1(0)$
- Nested: $b \leq b' \Rightarrow \mathscr{H}_b \subseteq \mathscr{H}_{b'}$

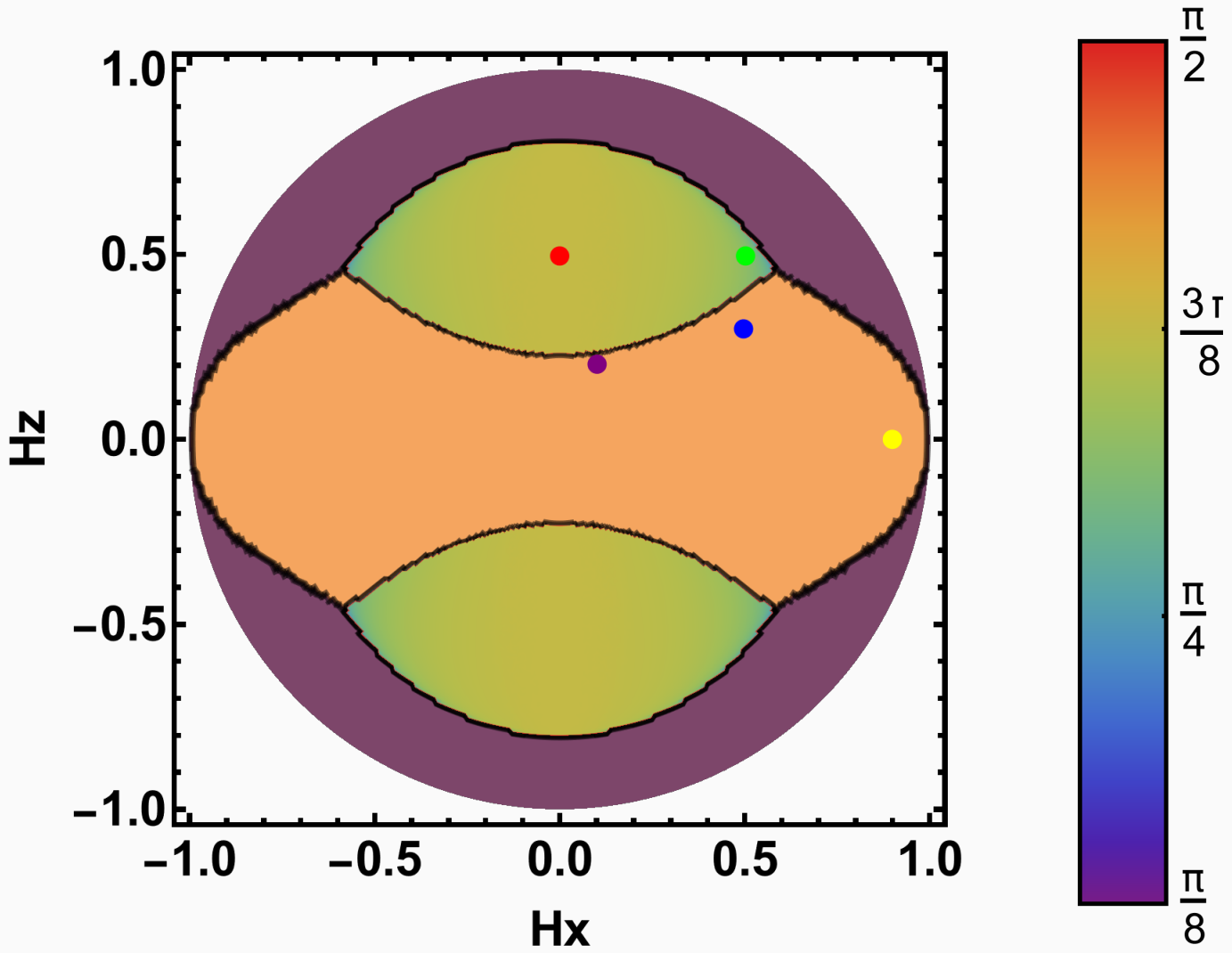
Spiral phase

- Domain $\mathscr{H}_b = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H},b}\}$
- Nonempty: $B_{1/2}(0) \subseteq \mathscr{H}_b$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow s\boldsymbol{H} \in \mathscr{H}_b$ for all $s \in [0,1]$
- $\mathbb{Z}_2^3$ symmetric: $\boldsymbol{H} \in \mathscr{H}_b \Rightarrow (\pm H_1, \pm H_2, \pm H_3) \in \mathscr{H}_b$
- Open
- Bounded: $\mathscr{H}_b \subseteq B_1(0)$
- Nested: $b \leq b' \Rightarrow \mathscr{H}_b \subseteq \mathscr{H}_{b'}$
- $\mathscr{H}_0$ and $\mathscr{H}_1$ are volumes of revolution

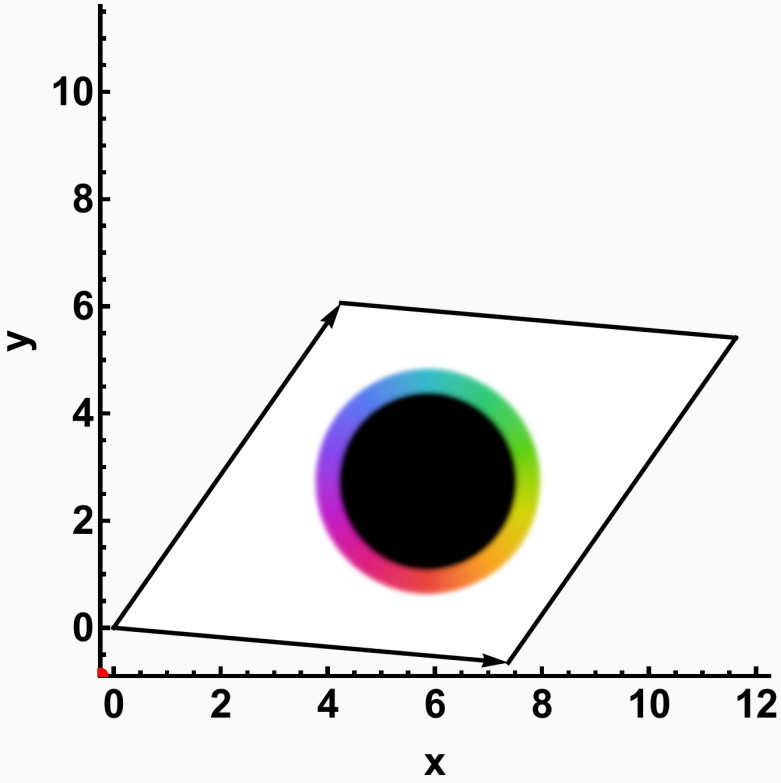
Spiral domain

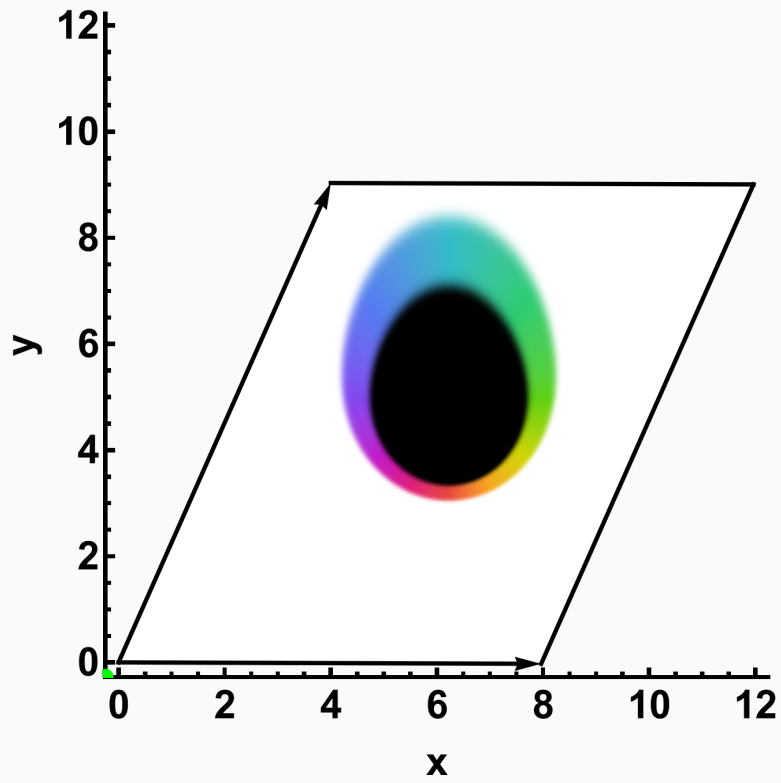


The ground state: $b = 1$

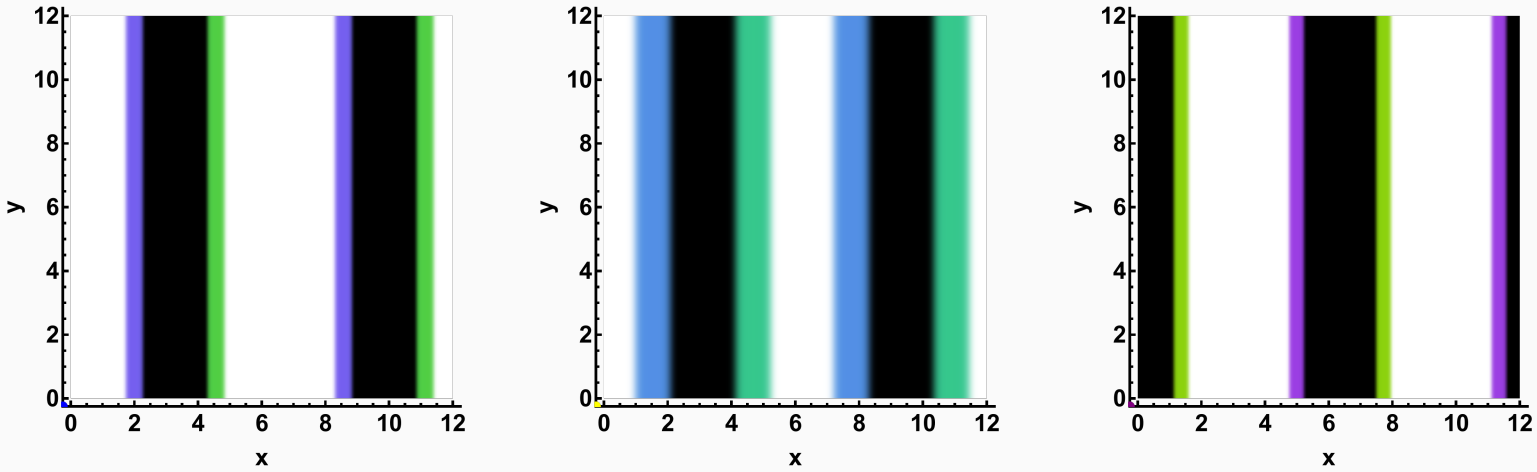


Skyrmion lattices: $b = 1$



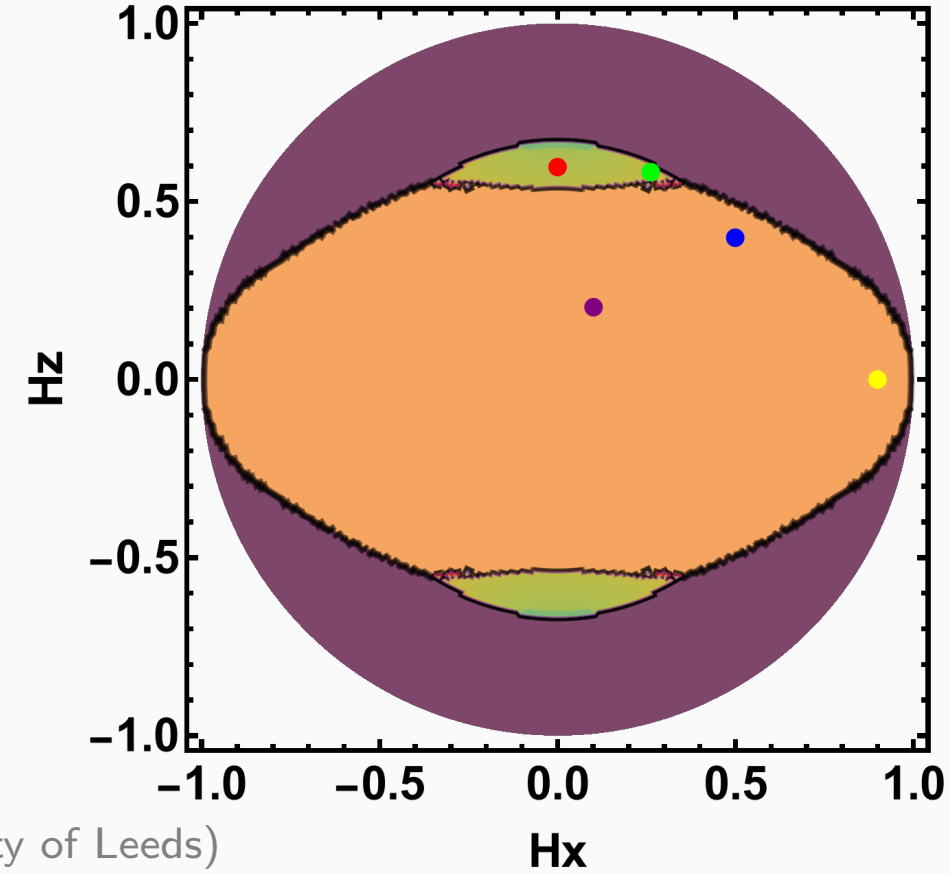


Spiral phases: $b = 1$



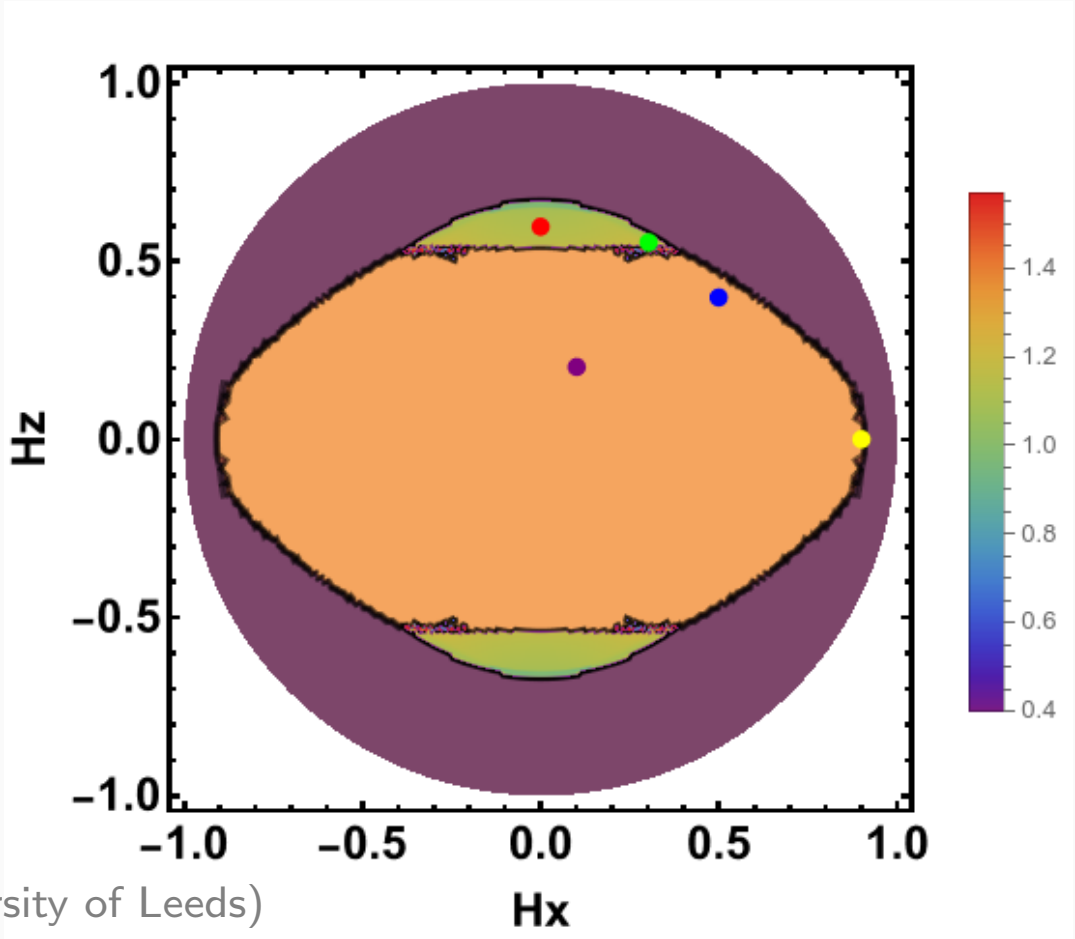
Ground state: $b = 0.8$

$$\varphi = 0$$



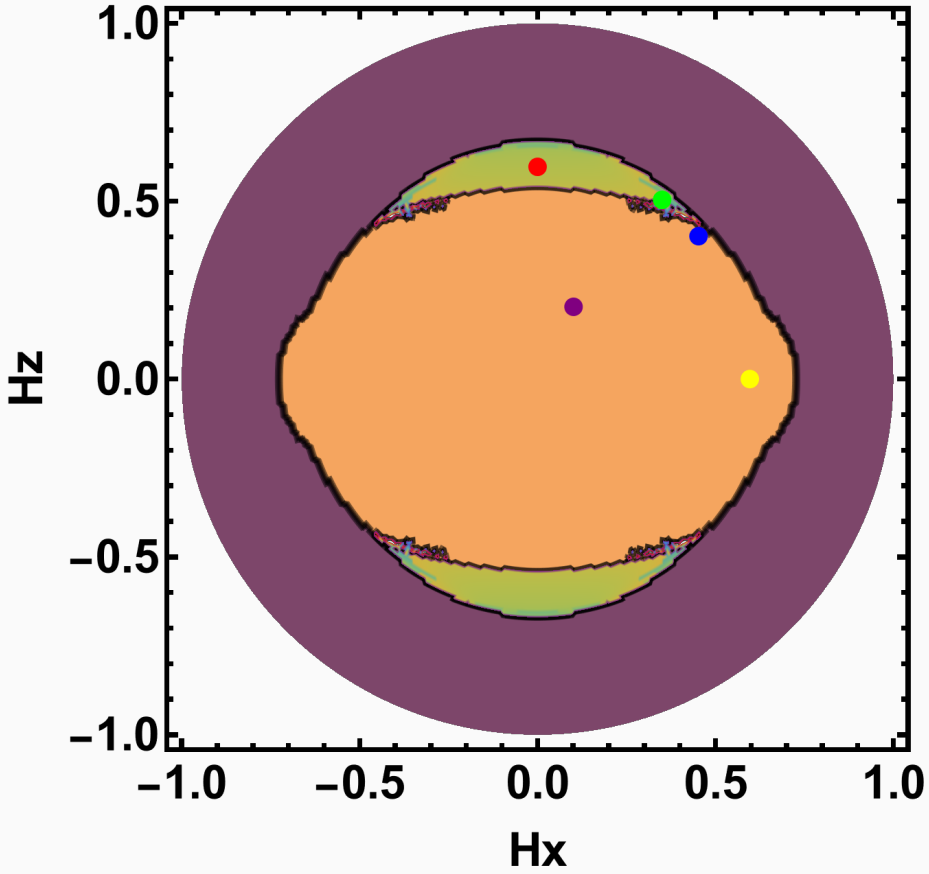
Ground state: $b = 0.8$

$$\varphi = \pi/6$$



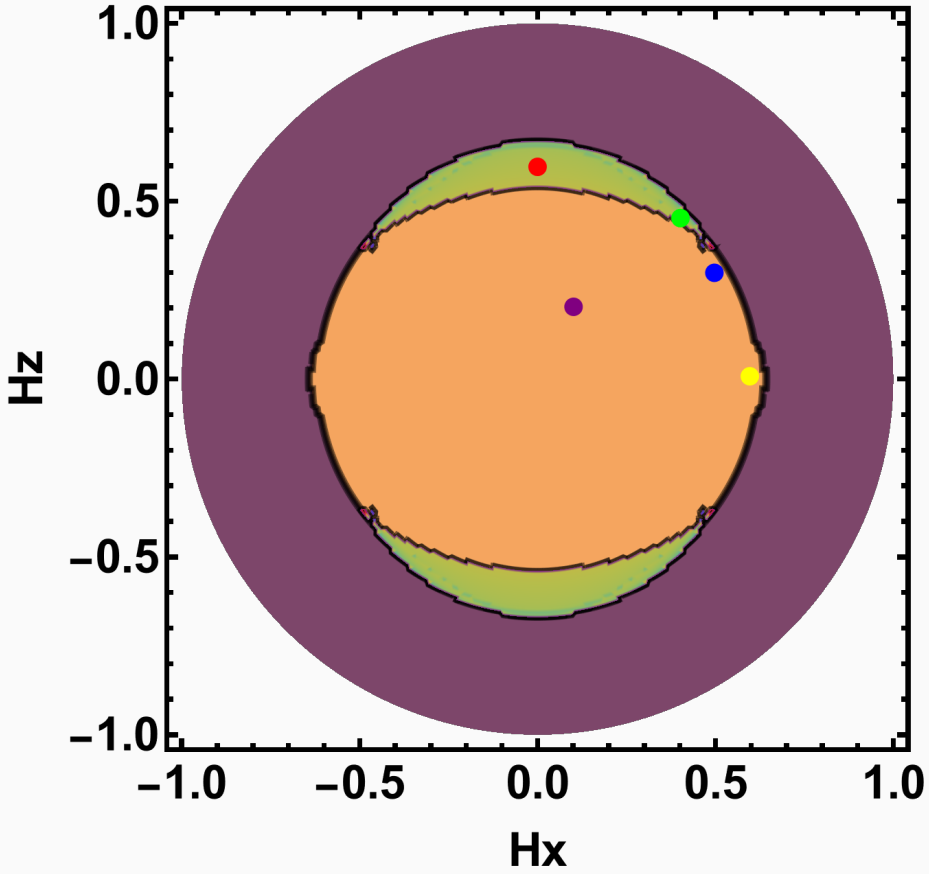
Ground state: $b = 0.8$

$$\varphi = \pi/3$$

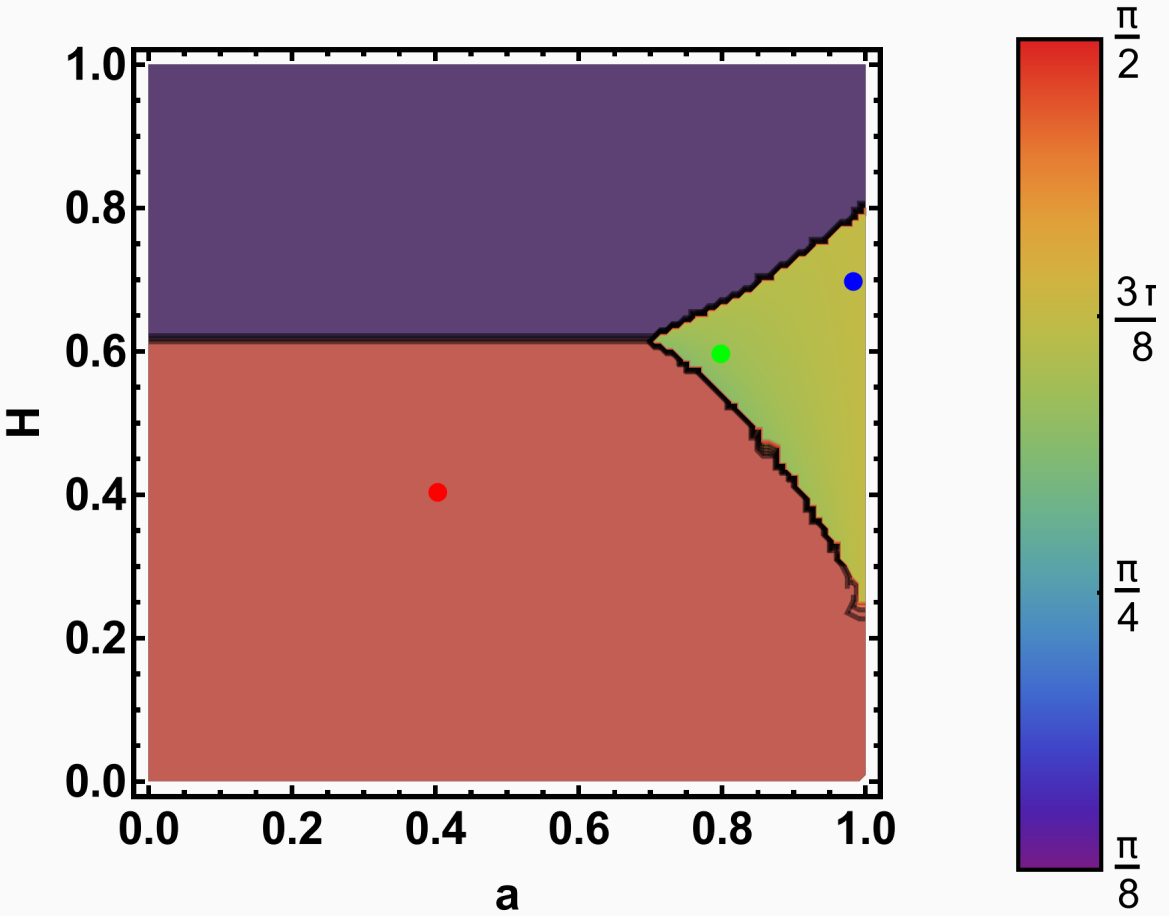


Ground state: $b = 0.8$

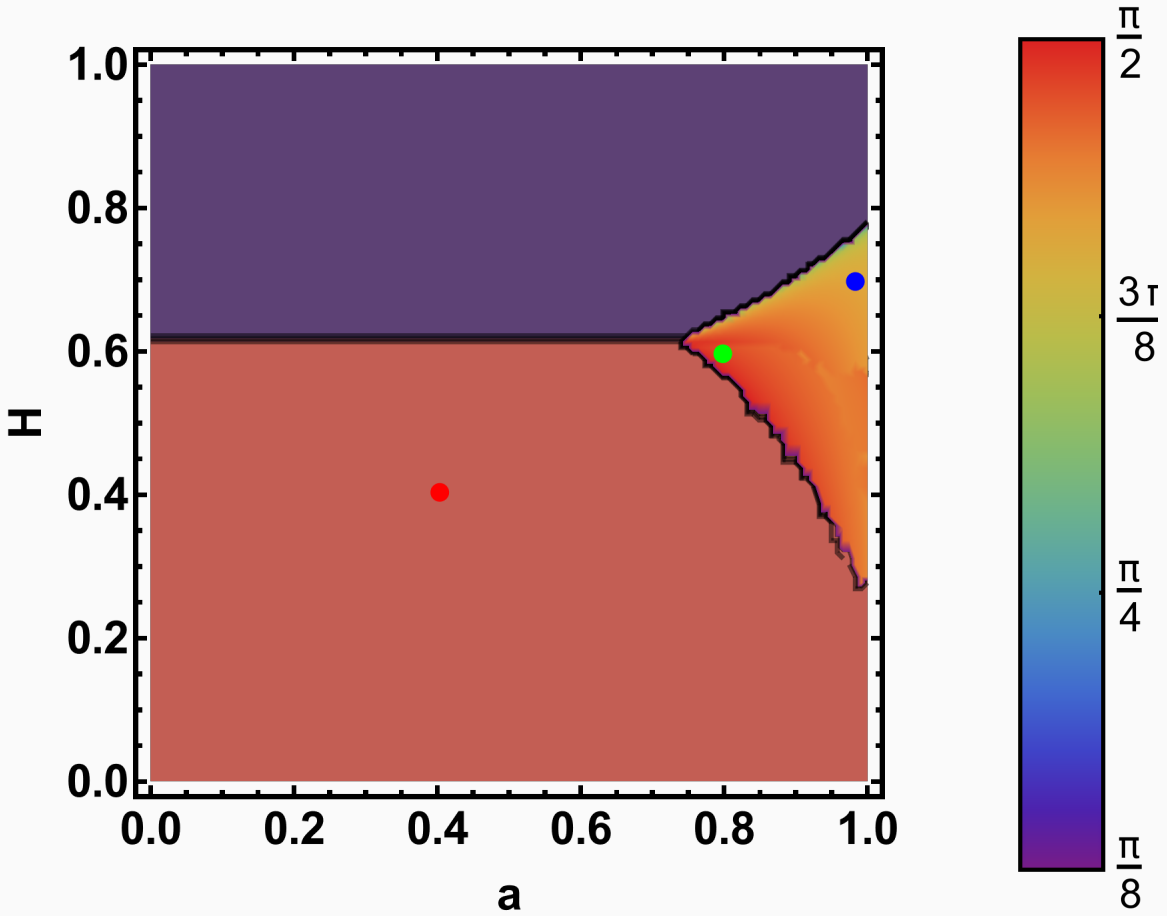
$$\varphi = \pi/2$$



The ground state: $\hat{H} = (0, 0, 1)$



The ground state: $\hat{H} = (0, 1/2, \sqrt{3}/2)$



Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

- At low $|\boldsymbol{H}|$, ground state is a spiral phase

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

- At low $|\boldsymbol{H}|$, ground state is a spiral phase
- Spiral phase domain depends nontrivially on b

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

- At low $|\boldsymbol{H}|$, ground state is a spiral phase
- Spiral phase domain depends nontrivially on b
- Minimal model with $b < b_* \approx 0.75$ never (i.e. for no $\boldsymbol{H}$) has skyrmion lattice as ground state

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

- At low $|\boldsymbol{H}|$, ground state is a spiral phase
- Spiral phase domain depends nontrivially on b
- Minimal model with $b < b_* \approx 0.75$ never (i.e. for no $\boldsymbol{H}$) has skyrmion lattice as ground state
- Minimal model with $\boldsymbol{H}$ to close to the $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ plane never (i.e. for no b) has skyrmion lattice as ground state

Conclusions

- Skyrmion lattices (when they exist) always have $Q = \pm 1$ per unit cell

Conclusions

- Skyrmion lattices (when they exist) always have $Q = \pm 1$ per unit cell
- Cell geometry gets distorted if $\boldsymbol{H}$ not perpendicular to $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ plane, but nothing really weird happens.

Open questions

- Could skyrmion lattices have technological applications? Is the ability to manipulate their geometry by changing $\boldsymbol{H}$ useful?
- Can we **prove** existence of skyrmion lattices?