EXTREME SKYMIONS AND RESTRICTED HARMONIC MAPS

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1. Harmonic maps

$$\varphi : (M, g) \to (N, h) \qquad E_2(\varphi) = \frac{1}{2} \int_M |d\varphi|^2$$

φ is harmonic if it's a critical pt of E_2

i.e. for all smooth variations φ_t, $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = 0.$$

$$\parallel$$

$$-\langle X, \operatorname{tr} \nabla d\varphi \rangle_{L^2} \quad \text{where}$$

$$X = \partial_t|_{t=0} \varphi_t \in \Gamma(\varphi^{-1} TN)$$

eg: $N = \mathbb{R}$ $\quad \varphi : M \to \mathbb{R}$ harmonic function

$$\Delta \varphi = 0.$$

$M = \mathbb{R} (/ \mathbb{Z})$ $\qquad$ (closed) geodesic in (N, h)

2. Restricted harmonic maps

$\mathrm{SDiff}(M, g)$ = volume preserving diffeos of (M, g)

$$\varphi^* \mathrm{vol}_g = \mathrm{vol}_g.$$

~~Defn~~

of compact support

$$(\{x \in M : \varphi(x) \neq x\})$$

Defn

$\varphi : (M, g) \to (N, h)$ is restricted harmonic

if it's a critical pt of E_2 restricted to its SDiff(M, g) orbit i.e. for all smooth curves $\psi_t \in \mathrm{SDiff}(M, g)$, $\psi_0 = \mathrm{id}_M$,

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi \circ \psi_t) = 0.$$

[id_M is a c.p. of $E_\varphi : \mathrm{SDiff}(M, g) \to \mathbb{R}$
$E_\varphi(\psi) = E_2(\varphi \circ \psi)$.]

Clearly harmonic $\underset{\not\Leftarrow}{\Rightarrow}$ restricted harmonic

eg $M = \mathbb{R}/\mathbb{Z}$ $\mathrm{vol}_g = dx$
$\psi(x) = x + a$ – isometry

$\Rightarrow$ all maps $\mathbb{R}/\mathbb{Z} \to N$ are R.H.
(but only closed geodesics are harmonic)

3. Motivation: Skyrme model

$$\varphi : M = \mathbb{R}^3 \longrightarrow N = SU(2) = S^3$$

$$E_{\mathrm{Skyrme}}(\varphi) = \int_M \frac{1}{2}|d\varphi|^2 + \ldots = E_2(\varphi) + \ldots$$

b.c. $\varphi(\infty) = \mathbb{1}_2$ degree of $\varphi : \underbrace{\mathbb{R}^3 \cup \{\infty\}}_{S^3} \to S^3$
$B \in \mathbb{Z}$

E minimizer in degree B class: atomic nucleus of $\underbrace{\text{baryon number}}_{\#\text{ protons + neutrons}}$ B

$$E_B < BE_1 \qquad \text{binding energy per nucleon} = \frac{BE_1 - E_B}{BE_1} \sim \begin{cases} 15\text{–}20\% & \text{Skyrme model} \\ 1\% & \text{real nuclei} \end{cases}$$

— Skyrmions are much too tightly bound.

Suggestion:
$$E(\varphi) = E_0 + E_6 + \varepsilon E_2 = \int_M \tfrac{1}{2}\left(U(\varphi)^2 + |\varphi^*\omega_h|^2 + \varepsilon |d\varphi|^2\right)$$

$\varepsilon > 0$ small.

$\varepsilon = 0$ model has exactly zero binding energy!

$$0 \le \tfrac{1}{2} \|\varphi^*\omega_h - *U(\varphi)\|_{L^2}^2 = E_0 + E_6 - \int_M \varphi^*(U\omega_h)$$

$\Rightarrow$ $E_0 + E_6 \ge \left(\int_N U\right) B$ with equality

$$\Longleftrightarrow \boxed{\varphi^*\omega_h = *U\circ\varphi.} \quad \text{BOG}$$

If φ satisfies BOG, so does $\varphi\circ\psi$ $\forall\, \psi \in SDiff(M,g)$

Consider minimisers of $E_0 + E_6 + \varepsilon E_2$ in limit $\varepsilon \to 0$.

Q: To which solution of BOG does φ tend?

A(?): Whichever one minimizes E_2

In particular, φ should minimize E_2 over its $SDiff(M,g)$ orbit
$\Rightarrow$ φ should be R.H. !

4. 1st Variation formula

Key observation: $E_2(\varphi, g)$ is geometrically natural:

$$E(\varphi \circ \psi, g) \equiv E(\varphi, (\psi^{-1})^* g)$$

$\forall \varphi, g$, ψ diffeo $M \circlearrowleft$

$\Rightarrow$ Variation $\varphi \circ \psi_t$, g fixed $\longleftrightarrow$ φ fixed, $(\psi_t^{-1})^* g$.

Stress tensor $\quad g_t, \; g_0 = 0, \; \partial_t|_{t=0} g_t = \varepsilon$

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi, g_t) = \frac{1}{2} \left\langle S_E(\varphi), \varepsilon \right\rangle_{L^2}$$

E_2: $\quad S_{E_2} = \frac{1}{2} |d\varphi|^2 g - \varphi^* h$

Theorem $\varphi : (M,g) \to (N,h)$ is restricted harmonic $\Leftrightarrow$ $\operatorname{div} \varphi^* h$ is exact.

$$\left[\quad (\operatorname{div} \tau)(X_1, \dots, X_{p-1}) = \sum_i (\nabla_{e_i} \tau)(e_i, X_1, \dots, X_p) \right]$$

Proof $(\Rightarrow)$: Let ψ_t = flow of $X \in \mathring{\Gamma}(TM)$ divergenceless

$$\frac{d}{dt}\Big|_{t=0} E_2(\underbrace{\varphi \circ \psi_t}_{h}, g) = \frac{d}{dt}\Big|_{t=0} E_2(\varphi, \psi_{-t}^* g)$$

$$= -\frac{1}{2} \langle S(\varphi), \mathcal{L}_X g \rangle_{L^2}$$

$$= \int_M (\operatorname{div} S)(X)$$

$$= \langle \operatorname{div} S, \flat X \rangle_{L^2}$$

$\operatorname{div} X = 0 \Leftrightarrow \delta \flat X = 0$

φ RH $\Rightarrow$ $\operatorname{div} S \perp_{L^2}$ coclosed 1-forms of compact support

$\Rightarrow$ $\operatorname{div} S \perp_{L^2} \delta\omega \quad \forall \omega \in \Omega^2(M)$ compact support

$\Rightarrow$ $d(\operatorname{div} S) = 0$

Poincaré pairing $H^1(M) \times H^{m-1}_c(M) \to \mathbb{R}$

$$([\alpha], [\beta]) \longmapsto \int_M \alpha \wedge \beta$$

is nondegenerate.

φ RH $\Rightarrow$ $([\operatorname{div} S], [\beta]) \longmapsto 0 \quad \forall [\beta]$

$\Rightarrow$ $[\operatorname{div} S] = 0$ i.e. $\operatorname{div} S = -\operatorname{div} \varphi^* h$ is exact. $\square$

Examples (i)

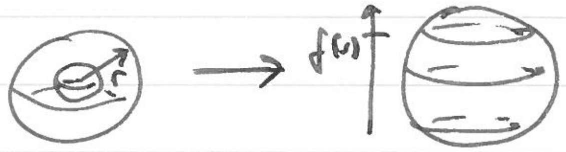
$$\begin{tikzcd} (M\times N,\ g+f^2h) \arrow[r, "\text{RH, but not H.}"] \arrow[d, "H \Rightarrow RH"] & (M,g) \\ (N,h) & \end{tikzcd}$$

(ii) Weakly conformal maps $\varphi : M \to N$

$$\varphi^* h = f^* g$$

$$\Rightarrow \operatorname{div} \varphi^* h = df .$$

eg. inverse stereographic projection $\mathbb{R}^3 \to S^3$

(iii) $\mathbb{R}^3$  S^3

Any suspension of $id : S^2 \to S^2$

(iv) $\varphi : M \to \mathbb{R}$ $\quad H^1(M) = 0$

$$d(\Delta\varphi) \wedge d\varphi = 0 \qquad \text{— cone}$$

eg all eigenfunctions of Δ

eg $F(\varphi)$, φ a soln, $F : \mathbb{R} \to \mathbb{R}$ arbitrary.

5. Second variation

Two parameter variation $\varphi_{s,t} = \varphi \circ \psi_{s,t}$ "surface" $\subset SDiff(M,g)$

$$\psi_{0,0} = id_M , \quad \text{~~tangent space~~}$$

$$X = \partial_s \psi_{s,t}\big|_{0,0} \in T_0(TM)$$

$$Y = \partial_t \psi_{s,t}\big|_{0,0} \in T_0(TM)$$

$$\underset{\uparrow}{\mathrm{Hess}}(X,Y) := \left.\frac{\partial^2}{\partial s\,\partial t}\right|_{s=t=0} E_2(\varphi \circ \psi_{s,t})$$

Symmetric bilinear form $T_0(\mathcal{D}M) \times T_0(\mathcal{D}M) \to \mathbb{R}$.

φ a <u>stable</u> RHM if Hess ≥ 0.

<u>Theorem</u> $\mathrm{Hess}(X,Y) = \frac{1}{2}\langle \mathcal{L}_X \varphi^* h, \mathcal{L}_Y g\rangle_{L^2}$

<u>Cor</u> If φ is weakly conformal, it is <u>stable</u> RH

Proof: $\varphi^* h = f^2 g \Rightarrow$

$$\mathrm{Hess}(X,X) = \frac{1}{2}\langle \mathcal{L}_X(f^2 g), \mathcal{L}_X g\rangle_{L^2}$$

$$= \underbrace{\langle X[f]g, \mathcal{L}_X g\rangle_{L^2}}_{0} + \|f\mathcal{L}_X g\|^2_{L^2} \geq 0. \quad \square$$

$\left(\mathrm{div}\, X = \frac{1}{2}\langle X, \mathcal{L}_X g\rangle\right)$

Note: <u>unstable</u> HM can be stable RHM

eg $\mathrm{id} : S^{m\geq 3} \to S^{m\geq 3}$ harmonic, unstable

but conformal, hence stable RHM.

6. Open questions

(i) Existence questions eg RHM $S^3 \to S^2$
hopf degree $\neq k^2$?

eg $\varphi : M \to N$ s.t. $d(\operatorname{div} \varphi^* h) = 0$ but not exact?

(local but not global RHM)?

eg unstable RHM?

(ii) Stability / rigidity

$$\operatorname{Hess}(X, Y) = -\left\langle \# \operatorname{div}(\mathcal{L}_X \varphi^* h), Y \right\rangle_{L^2}$$

is $J_\varphi X = -\# \operatorname{div}(\mathcal{L}_X \varphi^* h)$ like "Jacobi operator"?

i.e. $J_\varphi : \Gamma_0(TM) \to \Gamma_0(TM) \subset \Gamma(TM)$?

A solution to a variational problem is "rigid" if it's locally unique up to isometries.

Usually $\Leftrightarrow$ ker J_φ exhausted by symmetries.

Not for restricted variational problems like this — deformations could be transverse to SDiff $\circ\, \varphi$

Rigidity: $\partial_t \big|_{t=0} d(\operatorname{div} \varphi_t^* h) = 0$

Given RHM φ, what is space of deformations which remain RH to 1st order

("tangent space" to space of RHMs at φ)?

$$\partial_t\Big|_{t=0} d(\mathrm{div}\,\varphi_t^* h) = 0 \quad \text{linear PDE for } Y = \partial_t|_{t=0}\varphi_t \in \Gamma(\varphi^{-1}TN)$$

Certainly, if $\mathrm{Hess}(X,X)=0$ then $Y = d\varphi X$ is a deformation.

Usually Y a deformation $\Leftrightarrow$ $\mathrm{Hess}(Y,Y)=0$ but not here — Y may be transverse to $SDiff\cdot\varphi$.

i.e. Rigidity / Stability "decouple".

Jacobi operator $J_\varphi : T_0(M) \to T_0(M)$ self-adjoint st.

$$\mathrm{Hess}(X,Y) = \langle J_\varphi X, Y\rangle_{L^2} \quad ?$$

$$\shortparallel$$

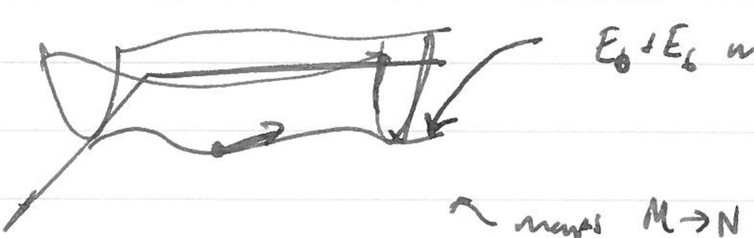
$$\langle -\#\,\mathrm{div}(\mathcal{L}_X \varphi^* h), Y\rangle_{L^2}$$

(iii) <u>Dynamics</u> how do we introduce time evolution?

$$\varphi : M \to N \quad \rightsquigarrow \quad \varphi : \underset{dt^2-g}{\mathbb{R}\times M} \to N$$

$$E_0 + E_6 \qquad\qquad S_0 + S_6$$

"Geodesic" approximation: geodesic flow in $SDiff(M,g)$



c.f. perfect incompressible fluid flow (Arnold)