The L^2 geometry of the moduli space of vortices on the two-sphere in the dissolving limit

Martin Speight (Leeds)
Rene García Lara (Universidad Autonoma de Yucatan)

York, 4/10/22

Rene $\longrightarrow$ 

Vortices on a compact Riemann surface

- Hermitian line bundle (L, h) over (Σ, g_Σ), degree n

$$E(\phi, A) = \frac{1}{2}\|\mathrm{d}_A\phi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|\tau - |\phi|^2\|_{L^2}^2$$

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- Bogomolny bound: $E \geq \tau\pi n$, equality $\Leftrightarrow$

$$\begin{aligned} \overline{\partial}_A\phi &= 0 & (V1) \\ *F_A &= \frac{1}{2}(\tau - |\phi|^2) & (V2) \end{aligned}$$

  Solutions are called **vortices**

The Bogomol'nyi bound

$$\langle F_A, |\phi|^2 \omega_\Sigma \rangle_{L^2} \quad = \quad \langle F_A \phi, \phi \omega_\Sigma \rangle_{L^2} = \langle i \mathrm{d}_A \mathrm{d}_A \phi, \phi \omega_\Sigma \rangle_{L^2}$$

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Hence

$$\begin{aligned}\left\|F_A - \frac{1}{2}(\tau - |\phi|^2)\omega_\Sigma\right\|_{L^2}^2 &= \|F_A\|_{L^2}^2 - \tau\int_\Sigma F_A + \|\partial_A\phi\|_{L^2}^2 - \|\overline{\partial}_A\phi\|_{L^2}^2 \\ &\quad + \frac{1}{4}\|\tau - |\phi|^2\|_{L^2}^2\end{aligned}$$

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$$E(\phi, A) = \frac{1}{2}\left\| F_A - \frac{1}{2}(\tau - |\phi|^2)\omega_\Sigma \right\|^2_{L^2} + \|\overline{\partial}_A\phi\|^2_{L^2} + \frac{\tau}{2}\int_\Sigma F_A.$$

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$$\bar{\partial}_A \phi = 0 \qquad (V1), \qquad *F_A = \frac{1}{2}(\tau - |\phi|^2) \qquad (V2)$$

- Bradlow bound: $\int_\Sigma (V2)$:

$$\begin{aligned} 2\pi n &= \frac{1}{2}\tau|\Sigma| - \frac{1}{2}\|\phi\|^2_{L^2} \\ \|\phi\|^2_{L^2} &= \tau|\Sigma| - 4\pi n =: \varepsilon \geq 0 \end{aligned}$$

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- We'll be interested in limit $\varepsilon \searrow 0$.

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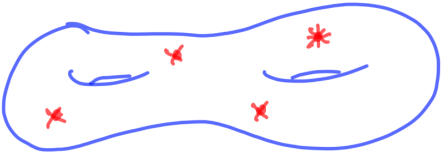
- $\varepsilon = 0$: $\phi = 0$, $*F_A = 2\pi n/|\Sigma|$, constant
- $\varepsilon > 0$: $[(\phi, A)]$ uniquely determined by the **divisor** (ϕ)



$$\leftrightarrow \mathrm{Sym}_n \Sigma = \Sigma^n / S_n$$

Why "vortices"?

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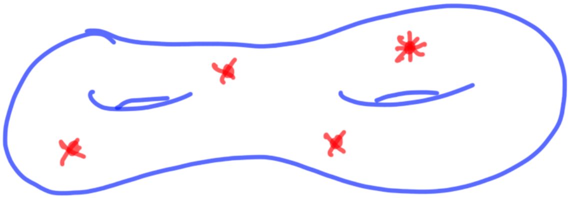

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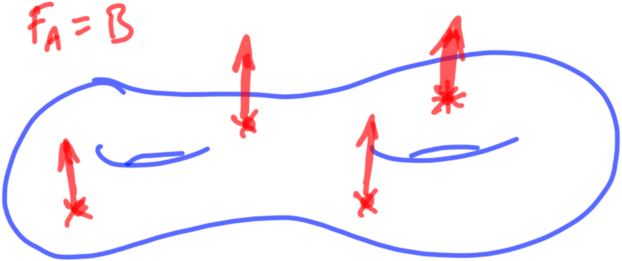

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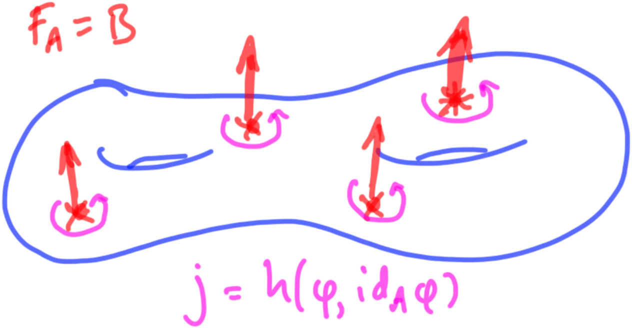

The L^2 metric on M_n

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- Kähler. $[\omega_{L^2}]$ known explicitly (Baptista)

$$|M_n| = \frac{\pi^n}{n!}\sum_{i=0}^{r}\left(\begin{array}{c} r \\ i \end{array}\right)\left(\begin{array}{c} n \\ i \end{array}\right) i!(4\pi)^i \varepsilon^{n-i}$$

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- Limit $\varepsilon \to \infty$ studied by Mundet i Riera & Romao and (independently) Nagy (2017): vortices become pointlike, g converges to product metric on Σ^n/S_n
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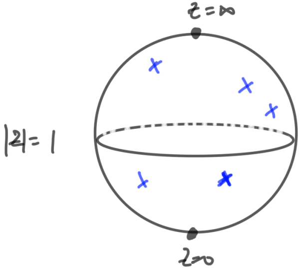
- $M_n \equiv \mathbb{C}P^n$

Vortices on S^2



$$\leftrightarrow p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$
$$\leftrightarrow [a_0, a_1, \ldots, a_n]$$

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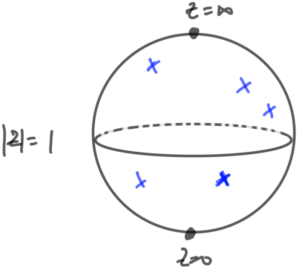

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- Baptista-Manton conjecture (2003): $\lim_{\varepsilon \to 0} g_\varepsilon =$ Fubini-Study metric

The conjecture

- Choose and fix const curv connexion $\widehat{A}$ on L
- Equip L with hol structure $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

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$$\pi : S \to (\mathbb{P}(H^0(L)), g_{FS}), \qquad \widehat{\phi} \mapsto \{c\widehat{\phi}\}$$

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- "The" Fubini-Study metric: $g_0 := f^* g_{FS}$
- Baptista-Manton conjecture: $\lim_{\varepsilon \to 0} g_\varepsilon = g_0$
- Surprising? Massive gain in symmetry

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More precisely:

There exists $C > 0$ such that, for all $v \in TM_n$ and all $\varepsilon \in (0,1)$

$$|g_\varepsilon(v,v) - g_0(v,v)| \leq C\varepsilon g_0(v,v)$$

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- How do we turn this intuition into a proof?

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$$(\phi, A) = (\sqrt{\varepsilon}\widehat{\phi}e^{u/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u)$$

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- Energy estimate, elliptic estimate, Sobolev $\Rightarrow$

$$\|u\|_{C^0} \leq C\varepsilon.$$

  Vortices are uniformly well approximated by pseudovortices (for small ε)

Unpacking that a little...

- Sobolev: $\|u\|_{C^0} \leq C\|u\|_{H^2}$

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- Decompose $u = u_0 + \overline{u}$ where $\overline{u} = |\Sigma|^{-1} \int_\Sigma u$

$$\Delta u_0 - \frac{\varepsilon}{|\Sigma|} + \varepsilon|\widehat{\phi}|^2 e^{\overline{u}} e^{u_0} = 0.$$

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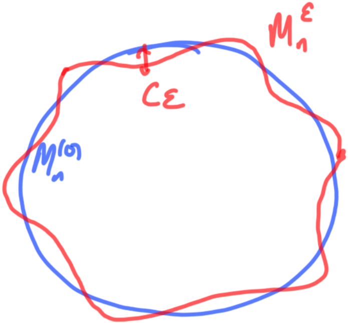

- Now estimate the **metric**...

Convergence of spec Δ

- Spectrum of Δ on (M, g)

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \cdots$$

- **Corollary** (JMS,RGL): There exists $C > 0$ such that, for all $k \in \mathbb{Z}^+$ and all $\varepsilon \in (0, 1/C)$,

$$\frac{(1 - C\varepsilon)^n}{(1 + C\varepsilon)^{n+1}} \leq \frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} \leq \frac{(1 + C\varepsilon)^n}{(1 - C\varepsilon)^{n+1}}.$$

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- Spectrum of M_n converges uniformly to spectrum of FS!
- Surprising this follows from only C^0 convergence!

$$\Delta = -g^{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} + \Gamma_{ij}^k \frac{\partial}{\partial x_k}\right)$$

Convergence of spec Δ

- Urakawa-Bando (1983): for any finite dimensional subspace $V \subset C^\infty(M)$

$$\Lambda_g(V) := \sup \left\{ \frac{\|\mathrm{d}f\|_{L^2}^2}{\|f\|_{L^2}^2} : f \in V \setminus \{0\} \right\}.$$

  Then

$$\lambda_k(g) = \inf\{\Lambda_g(V) : \dim V = k+1\}$$

- Corollary easily follows

Open questions

- Convergence of geodesics? Need $g_\varepsilon \to g_0$ in C^1
- Convergence of curvature? Need $g_\varepsilon \to g_0$ in C^2
- n-dependence of the bounds?
- Leading correction to g_0?
- Higher genus? Much more subtle (Manton, Romao)