Geometry of dissolving vortices

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What are vortices?

$$\mathcal{L} = \frac{1}{2}\overline{D_\mu \varphi} D^\mu \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{8}(1 - |\varphi|^2)^2$$

- $D_\mu \varphi = (\partial_\mu - iA_\mu)\varphi$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- $B = F_{12}$, $e_i = F_{0i}$
- Finite energy: $\varphi \sim e^{i\chi}$ at large r, winding number $n \in \mathbb{Z}$.
- Finite energy: $D_i \varphi \sim 0$ at large r: $A = A_i dx^i \sim d\chi$

$$\int_{\mathbb{R}^2} B = \int_{\mathbb{R}^2} dA = \oint_{S^1_\infty} A = 2\pi n$$

Flux quantization

What are vortices?

- Vortex: energy minimizer with $n = 1$

$$\varphi = f(r)e^{i\theta}, \qquad A = a(r)d\theta$$

- Multivortices: for any $n \geq 2$, static solutions

$$\varphi = f_n(r)e^{in\theta}, \qquad A = a_n(r)d\theta$$

  Stable if $\lambda < 1$, unstable if $\lambda > 1$. Unique in both cases

- Critical coupling: $\lambda = 1$, space of static solutions **much** more interesting

Bogomol'nyi argument

$$
\begin{aligned}
E &= \frac{1}{2}\int_{\mathbb{R}^2}\left\{|D_1\varphi|^2+|D_2\varphi|^2+B^2+\frac{1}{4}(1-|\varphi|^2)\right\} \\
0 &\leq \frac{1}{2}\int_{\mathbb{R}^2}\left\{|D_1\varphi+iD_2\varphi|^2+[B-\frac{1}{2}(1-|\varphi|^2)]^2\right\} \\
&= E-\frac{1}{2}\int_{\mathbb{R}^2}B \\
&= E-\pi n
\end{aligned}
$$

- Hence $E \geq \pi n$ with equality iff

$$
\begin{aligned}
(D_i+iD_2)\varphi &= 0 \qquad (BOG1) \\
B &= \frac{1}{2}(1-|\varphi|^2) \qquad (BOG2)
\end{aligned}
$$

Moduli space

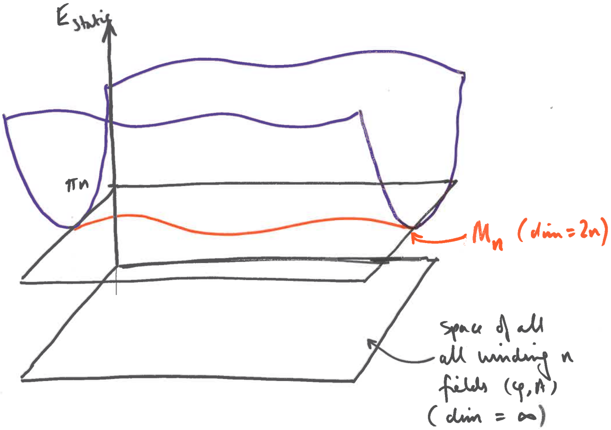
- Taubes: gauge equivalence classes of solns of $(BOG1), (BOG2) \leftrightarrow$ unordered collections of n points in $\mathbb{R}^2 = \mathbb{C}$ (not nec. distinct)
- $\leftrightarrow$ unique monic polynomial whose roots are the marked points

$$P(z) = (z - z_1)(z - z_2)\cdots(z - z_2) = z^n + a_1 z^{n-1} + \cdots + a_n$$

- $\leftrightarrow (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$
- Hence the **moduli space** of n-vortex solutions $M_n \cong \mathbb{C}^n$

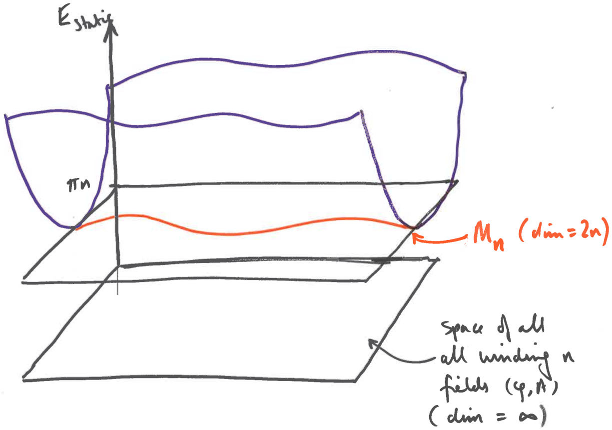
Geodesic approximation

$$L = \frac{1}{2}\int_{\mathbb{R}^2} (|\dot{\varphi}|^2 + |\dot{A}|^2) - E_{static}(\varphi, A)$$



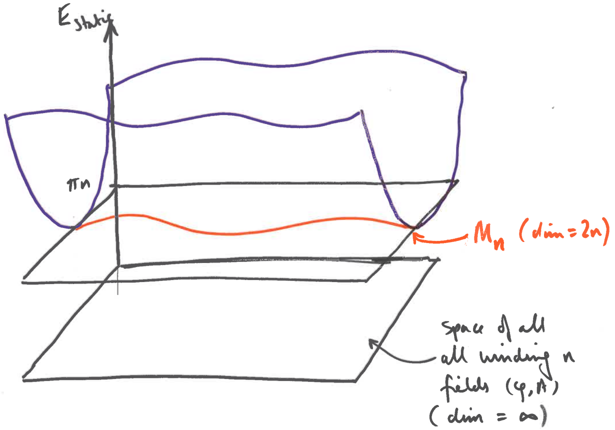
Geodesic approximation

$$L|_{M_n} = \frac{1}{2}\int_{\mathbb{R}^2}(|\sum \frac{\partial \varphi}{\partial z_r}\dot{z}_r|^2 + |\sum \frac{\partial A}{\partial z_r}\dot{z}_r|^2) - \pi n$$



Geodesic approximation

$$L|_{M_n} = \frac{1}{2}\sum_{r,s} \gamma_{rs}\dot{z}_r\dot{\bar{z}}_s - \pi n$$



Geodesic approximation

- **Geodesic** motion in M_n w.r.t. metric γ induced by K.E.
- In maths literature, γ is called the "L^2 metric"
- Obviously hermitian

$$J : T_pM_n \to T_pM_n, \qquad \gamma(JX, JY) \equiv \gamma(X, Y)$$

- Kähler form $\omega(X, Y) = \gamma(JX, Y)$
- M_n is **kähler**: $d\omega = 0$
- **Quantum** geodesic motion: $i\partial_t\Psi = \frac{1}{2}\Delta\Psi$

Vortices on compact surfaces

- Spacetime $\Sigma \times \mathbb{R}$, $\eta = dt^2 - g_\Sigma$
- Why?
  - $\Sigma = T^2 = \mathbb{C}/\Lambda$: vortex lattices
  - More generally: vortex "gas"
  - Maths: equivariant Gromov-Witten theory
- Need a bit more mathematical sophistication: hermitian line bundle L over Σ, φ a section, A a connexion

$$E(\varphi, A) = \frac{1}{2}\|d_A\varphi\|^2 + \frac{1}{2}\|F_A\|^2 + \frac{1}{8}\|1 - |\varphi|^2\|^2$$

- Still have flux quantization:

$$\int_\Sigma F_A = 2\pi n$$

$n = \deg(L)$

Vortices on compact surfaces

- Still have Bogomol'nyi argument: $E \geq \pi n$ with equality iff

$$\begin{aligned} \bar{\partial}_A \varphi &= 0 && (BOG1) \\ F_A &= \frac{1}{2}(1 - |\varphi|^2) * 1 && (BOG2) \end{aligned}$$

- Bradlow bound: integrate $(BOG2)$ over Σ

$$2\pi n = \frac{1}{2} Area(\Sigma) - \frac{1}{2}\|\varphi\|^2 \leq \frac{1}{2} Area(\Sigma)$$

  - No vortex solutions if $Area(\Sigma) < 4\pi n$.
  - If $Area(\Sigma) = 4\pi n$ all solutions have $\varphi \equiv 0$, $*F_A$ constant
  - If $Area(\Sigma) > 4\pi n$, vortex solutions $\leftrightarrow$ effective divisors on Σ of degree n
    $M_n = \Sigma^n / S_n$

Dissolved vortices

- Note: $\varphi = 0$, $*F_A = 2\pi n / Area(\Sigma)$ is **always** a solution of the Euler-Lagrange equations

$$E = \frac{2\pi^2 n^2}{Area(\Sigma)} + \frac{1}{8} Area(\Sigma)$$

  Solution not unique (up to gauge) if $H^1(\Sigma) \neq 0$: $M_n^{dis} = T^{2g}$ $(g = genus(\Sigma))$

- $Area(\Sigma) \searrow 4\pi n$: "dissolving" limit
- $|\varphi|$ becomes small, F_A becomes uniform
- $g \gg n$ studied by Manton and Romao
- $g = 0$ studied by Baptista and Manton

Vortices on a sphere

$$M_n \cong \mathbb{C}P^n$$

- Use stereographic coord z on S^2
- $[(z_1, z_2, \ldots, z_n)] \leftrightarrow P(z) = a_0 + a_1 z + \cdots + a_n z^n$
- $a_n = a_{n-1} = \cdots = 0 \Rightarrow$ root(s) at $z = \infty$
- $(a_0, a_1, \ldots, a_n) \sim (\lambda a_0, \lambda a_1, \ldots, \lambda a_n)$
- Metric γ_{L^2} not known exactly, but...
- Manton exactly computed the **volume** of (M_n, γ_{L^2})!

$$Vol(M_n(S^2)) = \frac{\pi^n (Area(S^2) - 4\pi n)^n}{n!}$$

  - valid on **any** sphere
  - shrinks to 0 as $Area(S^2) \searrow 4\pi n$

The conjecture

- Define R s.t. $Area(S^2) = 4\pi R^2$
- Rescale γ_{L^2} to normalize volume: $\gamma'_{L^2} = \gamma_{L^2}/(R^2 - n)$
- **Conjecture** (Baptista, Manton): As $R^2 \searrow n$, γ'_{L^2} converges uniformly to "the" Fubini-Study metric on $\mathbb{C}P^n$
- Originally made for round metric on S^2 - but argument obviously generalizes to any metric
- Huge symmetry gain (at most $SO(3) \to U(n)$)
- So what? E.g. quantum energy spectrum should have unexpected large quasi-degeneracies

What is the FS metric?

- Unique kähler-einstein metric on $\mathbb{C}P^n$
- In inhomogeneous coords $[1, w_1, \dots, w_n]$

$$\gamma_{FS} = \frac{\sum_i dw_i d\overline{w}_i}{1 + |w|^2} - \frac{(\sum_i \overline{w}_i dw_i)(\sum_j w_j d\overline{w}_j)}{(1 + |w|^2)^2}.$$

- Hopf fibration $\mathbb{C}^{n+1} \supset S^{2n+1} \to \mathbb{C}P^n$:
  $\pi : (a_0, a_1, \dots, a_n) \mapsto [a_0, a_1, \dots, a_n]$
- γ_{FS} is the unique riemannian metric on $\mathbb{C}P^n$ such that $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a **riemannian submersion**:
  - $T_p S^{2n+1} = \ker d\pi_p \oplus \mathcal{H}_p$
  - $d\pi_p : \mathcal{H}_p \to T_{\pi(p)}\mathbb{C}P^n$ is an isometry

Intuition

- In dissolving limit $\varphi \to 0$ and $A \to$ constant curvature connexion
- On $L \to S^2$, const curv connexion is unique (up to gauge)
- Choose and fix such a connexion A, assume φ is "small" and solves $(BOG1)$:

$$\overline{\partial}_A \varphi = 0$$

  φ holomorphic w.r.t. holo structure defined by A
- In a holomorphic trivialization of L over $S^2 \backslash pt$

$$\varphi = a_0 + a_1 z + \cdots + a_n z^n$$

  Remaining gauge freedom: $\varphi \mapsto e^{ic}\varphi$

Intuition

- No such φ can solve $(BOG2)$

$$F_A = \frac{1}{2}(1 - |\varphi|^2) * 1$$

Demand only that it solves $(BOG2)$ "on average"

$$2\pi n = \frac{1}{2} Area(S^2) - \frac{1}{2}\|\varphi\|^2$$

i.e. $\varphi \in S^{2n+1}_{\rho} \subset H^0(L) = \mathbb{C}^{n+1}$, $\rho^2 = 4\pi(R^2 - n)$

- Curve of solutions: A constant, $\varphi(t)$ moving orthogonal to gauge orbit

$$T = \frac{1}{2}\|\dot{\varphi}\|^2$$

Hence induces FS metric on $\mathbb{C}P^n = S^{2n+1}/\sim$

Testing the conjecture

- Underlying idea: $\varphi \longrightarrow$ holomorphic section of fixed L^2 norm
- On round sphere, can write these down explicitly
- Solve Bogomol'nyi equations numerically on round sphere, investigate limit $R^2 \searrow n$

Numerics

- $g_\Sigma = \Omega dz d\bar{z}$, $\Omega = \frac{4R^2}{(1+|z|^2)^2}$
- Define $h = \log|\varphi|^2$
- Can use $(BOG1)$ to eliminate A from $(BOG2)$

$$\nabla^2 h + \Omega(1 - e^h) = 4\pi \sum_r \delta(z - z_r)$$

- Consider case $n = 2$, $z_1 = \varepsilon \in (0,1)$, $z_2 = -\varepsilon$
- Regularize: $h(z) = f_+(z) + \log|z - \varepsilon|^2 + \log|z + \varepsilon|^2$

$$\nabla^2 f_+ + \Omega(1 - |z^2 - \varepsilon^2|^2 e^{f_+}) = 0 \qquad (*)$$

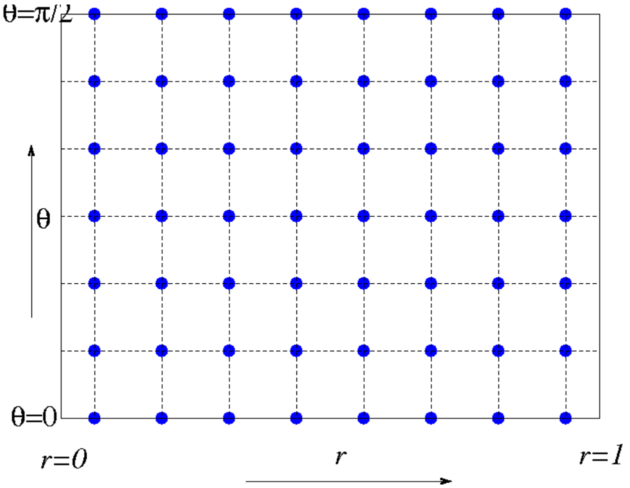
- Repeat in opposite coord patch $w = 1/z$

$$\nabla^2 f_- + \Omega(1 - |w^2 - \varepsilon^{-2}|^2 e^{f_-}) = 0 \qquad (**)$$

- Solve $(*)$ on disk $|z| \leq 1$, $(**)$ on disk $|w| \leq 1$, impose matching condition on equator $|z| = |w| = 1$.

Numerics

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \Omega(r)(1 - |r^2 e^{2i\theta} - \varepsilon^{\pm 2}|^2 e^f) = 0$$



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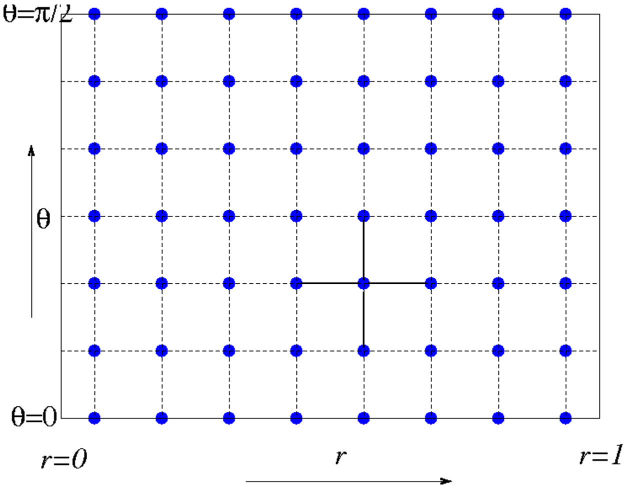

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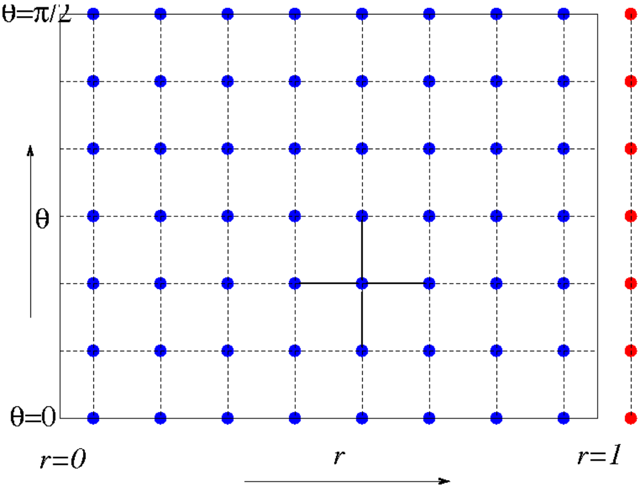


Numerics

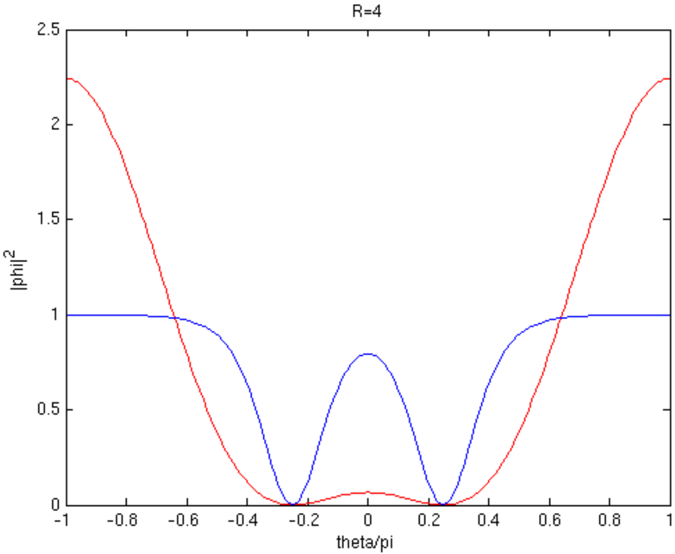
$$G : \mathbb{R}^{2n_r n_\theta} \to \mathbb{R}^{2n_r n_\theta}, \qquad G(f_+, f_-) = 0$$

- Newton-Raphson method, $n_r = n_\theta = 50$
- Integral constraint on numerical solutions:

$$\frac{1}{2} \int_{S^2} (1 - e^h) = 2\pi n$$

Holds almost to machine precision (!) (error $\sim 10^{-15}$)

Convergence of φ



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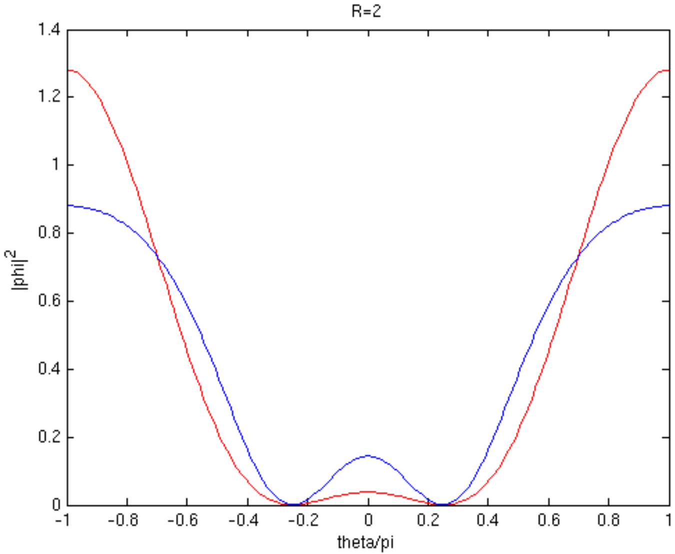

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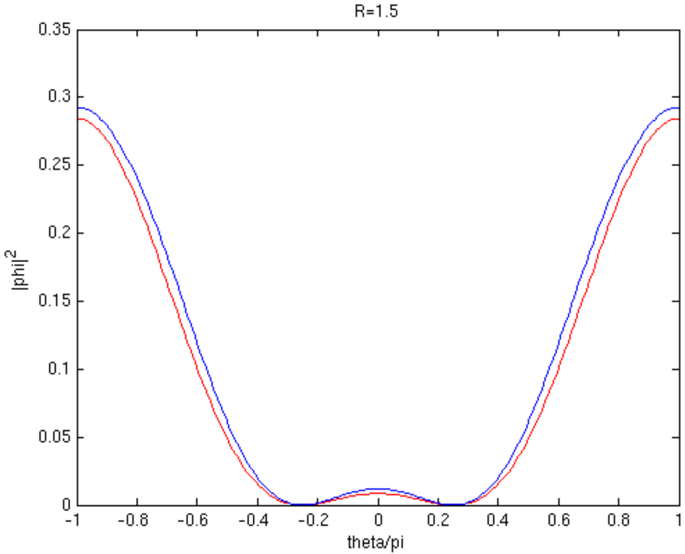

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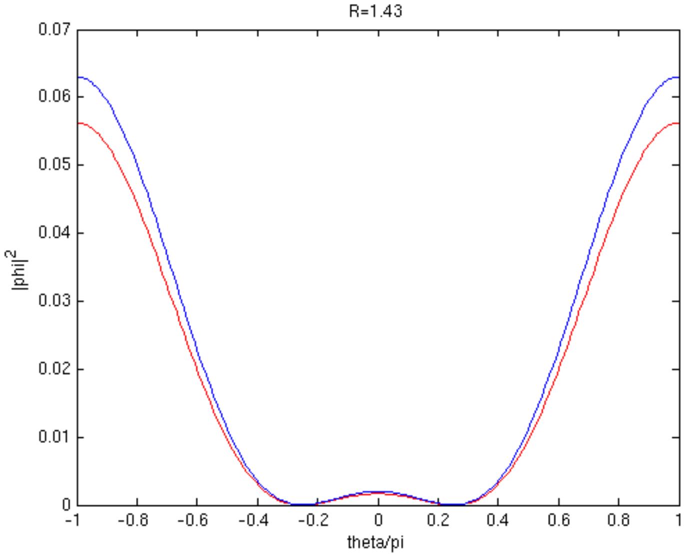


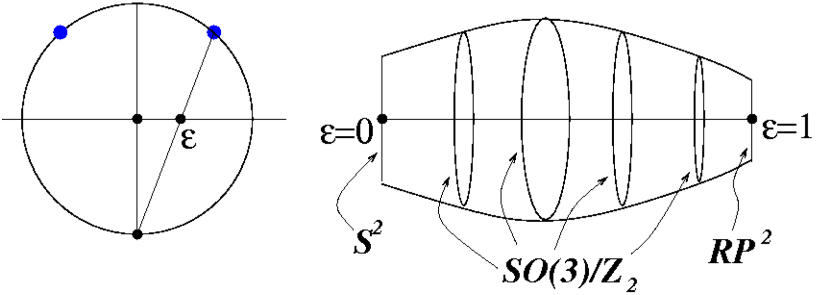
Convergence of γ_{L^2}

$$h = \log|\varphi|^2 = \log|z - z_r|^2 + a_r + \frac{1}{2}b_r(\bar{z} - \bar{z}_r) + \frac{1}{2}\bar{b}_r(z - z_r) + \cdots$$

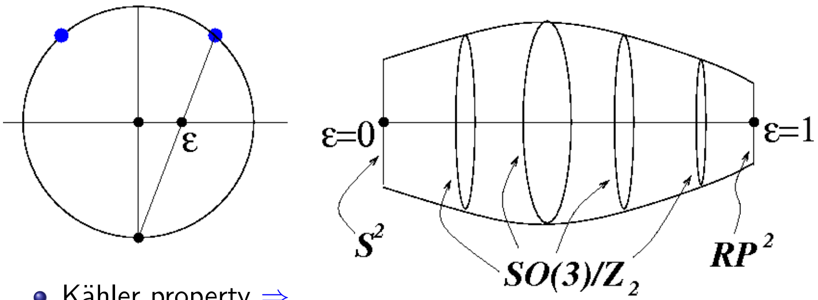
- Defines $(0,1)$ form $b = \sum_r b_r d\bar{z}_r$ on $M_n \backslash \Delta$, holomorphic
- Strachan-Samols localization formula:

$$\omega_{L^2} = \pi \sum_r \Omega(z_r) \frac{i}{2} dz_r \wedge d\bar{z}_r + i\pi db$$

The two-vortex moduli space



$$\gamma = A_0(\varepsilon)d\varepsilon^2 + A_1(\varepsilon)\sigma_1^2 + A_2(\varepsilon)\sigma_2^2 + A_3(\varepsilon)\sigma_3^2$$

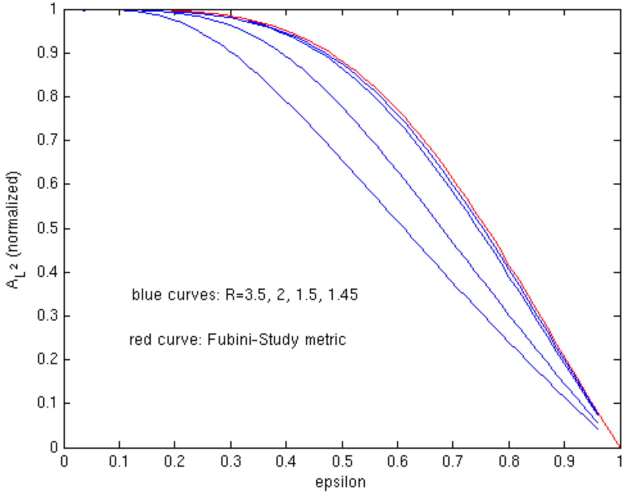


- Kähler property $\Rightarrow$

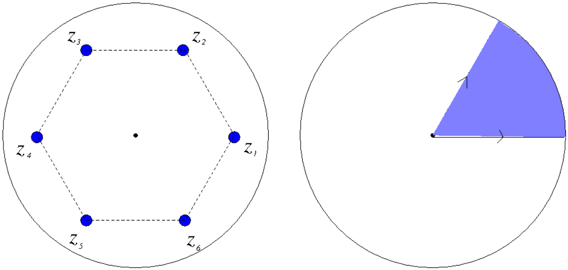
$$\gamma = -\frac{A'(\varepsilon)}{\varepsilon}(d\varepsilon^2 + \varepsilon^2\sigma_3^2) + A(\varepsilon)\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\sigma_1^2 + \frac{1+\varepsilon^2}{1-\varepsilon^2}\sigma_2^2\right)$$

  where $A : (0,1) \to (0,\infty)$ is smooth and decreasing
- Applies to **any** $SO(3)$ invariant kähler metric on M_2, hence both γ_{L^2} and γ_{FS}
- $\gamma_{L^2} \to \gamma_{FS}$ iff $A_{L^2} \to A_{FS}$

Convergence of γ_{L^2} on M_2



Vortex polygons



- Vortex polygons on a surface of revolution ($\Omega = \Omega(|z|)$): $z_1 = \varepsilon e^{i\psi}$, $z_r = \lambda^{r-1} z_1$
- Totally geodesic submanifold $M_n^0 \cong S^2$ in M_n
- Induced metric

$$\gamma_{L^2}| \;=\; F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2)$$

- Can compute F from localization formula

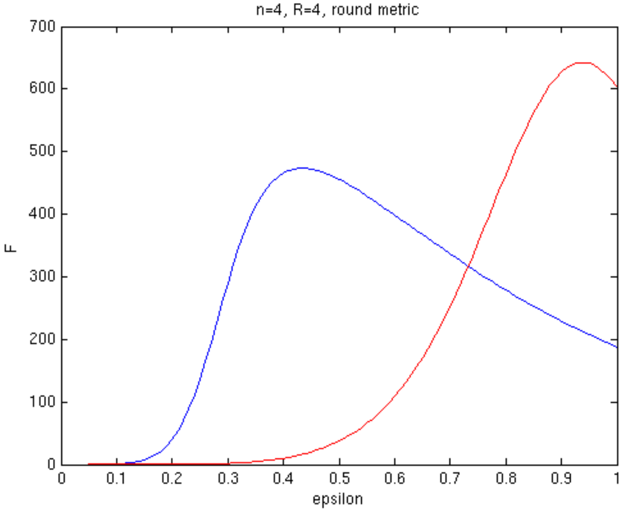
Vortex polygons

- Compare with metric induced by Fubini-Study
- $P(z) = z^n - \varepsilon^n \leftrightarrow [1, 0, \dots, \varepsilon^n] \in \mathbb{C}P^n$

$$F_{FS}(\varepsilon) = \left| \frac{\partial}{\partial \varepsilon} \right|^2 = 4\pi(R^2 - n) \frac{n^2 \varepsilon^{2n-2}}{(1 + \varepsilon^{2n})^2}$$

- Convergence for $n = 2$ ($g_\Sigma =$ round) follows from previous work
- Even n technically simpler: $n = 4$

Convergence of $\gamma_{L^2}|$ on M_4^0



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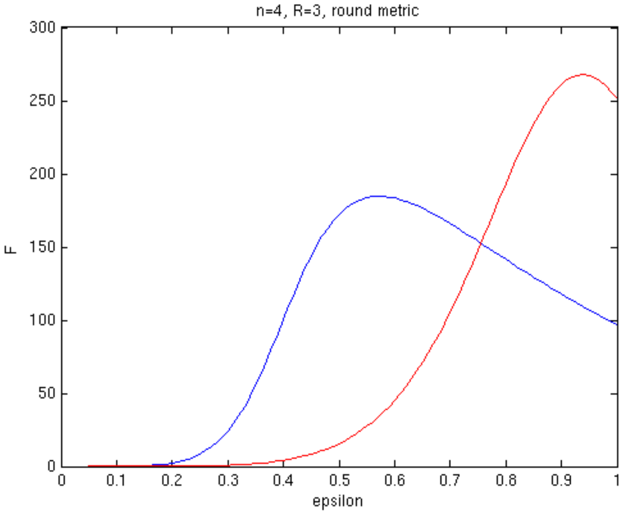

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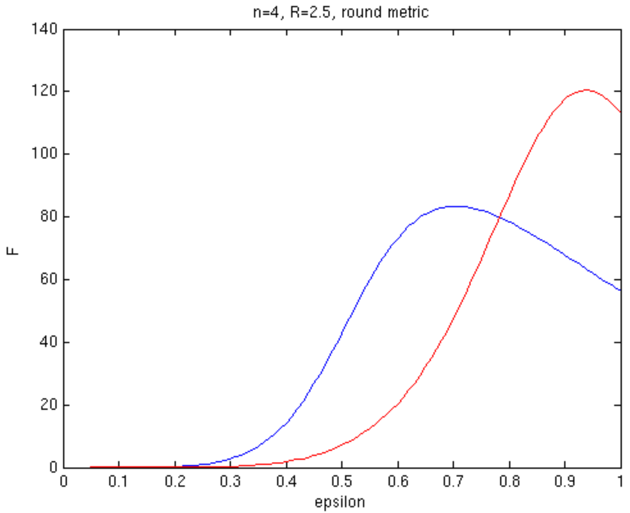

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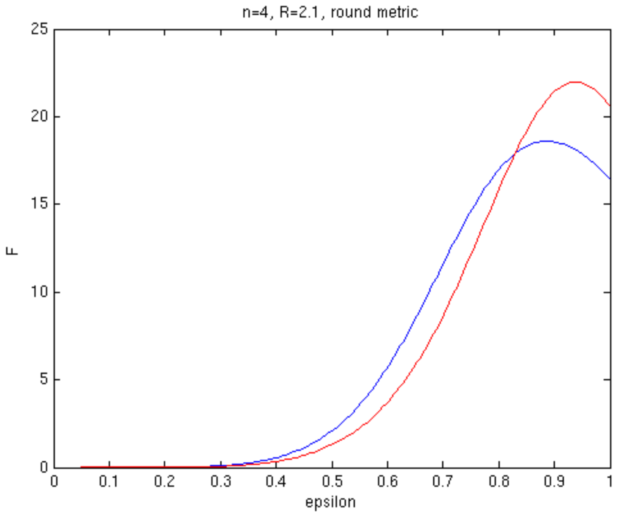


Non-round spheres

- Recall informal "derivation" of conjecture works on **any** topological sphere
- Test this numerically? Deform $g_{S^2} = \Omega(dr^2 + r^2 d\theta^2)$
- Want to keep $z \mapsto 1/z$ isometry, $SO(2)$ symmetry
- Ω a rational function of r

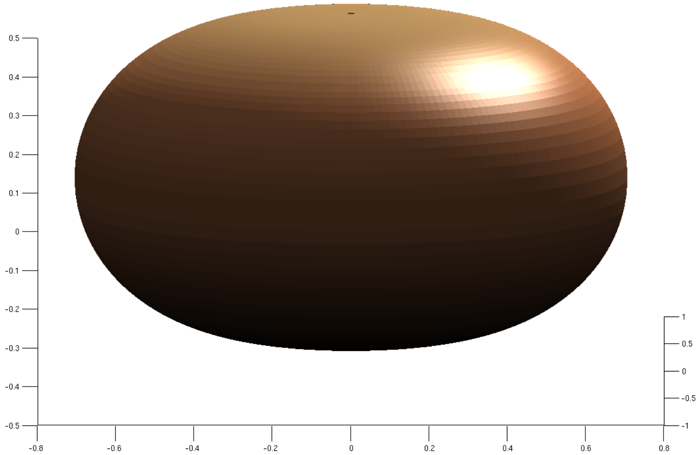
$$\Omega = \frac{p(r^2)}{q(r^2)}$$

$\deg(q) = \deg(p) + 2$, p, q palindromic

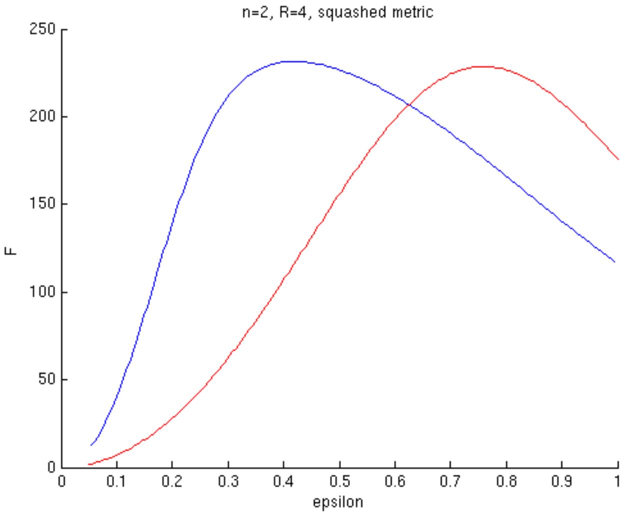
- Round metric: $p = 1$, $q = 1 + 2x + x^2$
- Squashed metric: $p = 1$, $q = 1 + x^2$

$$\Omega = \frac{(8/\pi)R^2}{1 + r^4}$$

Non-round spheres



Convergence of $\gamma_{L^2}|$ on squashed spheres: M_2^0



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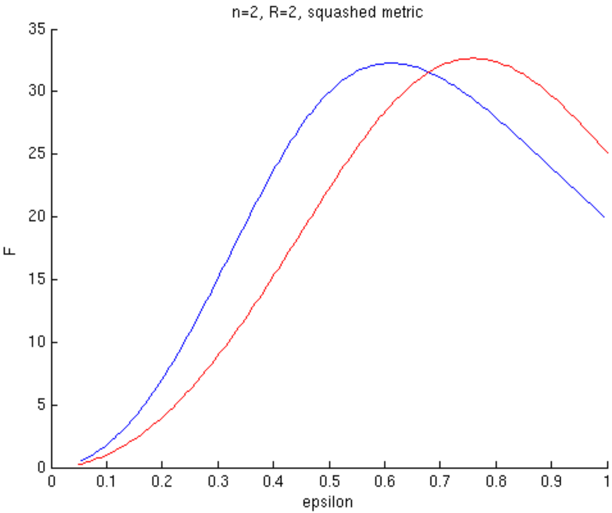

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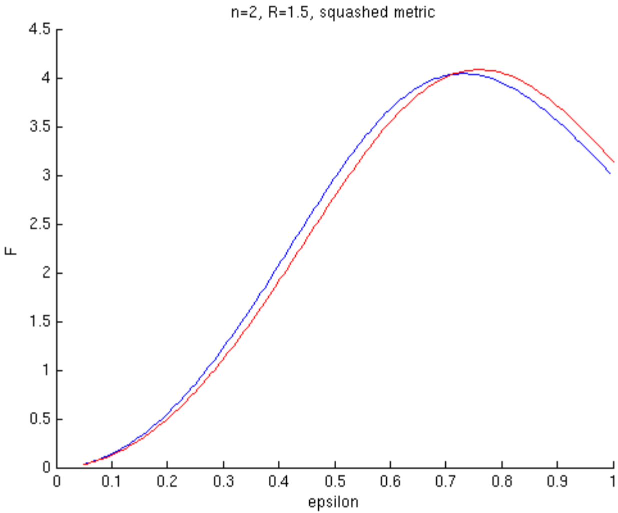

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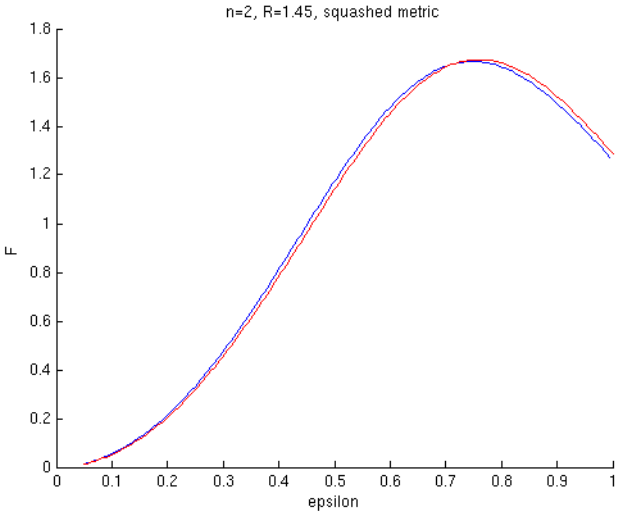
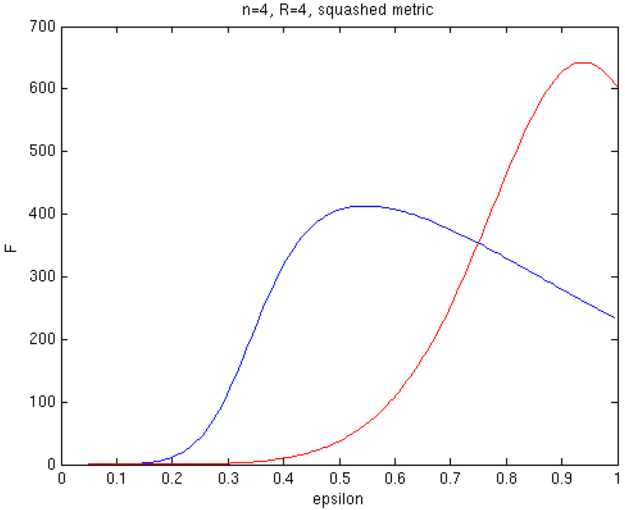
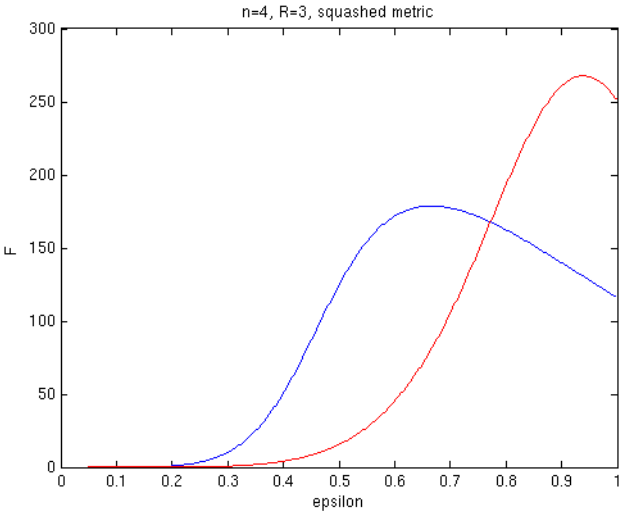


Convergence of $\gamma_{L^2}|$ on squashed spheres: M_4^0



Convergence of $\gamma_{L^2}|$ on squashed spheres: M_4^0



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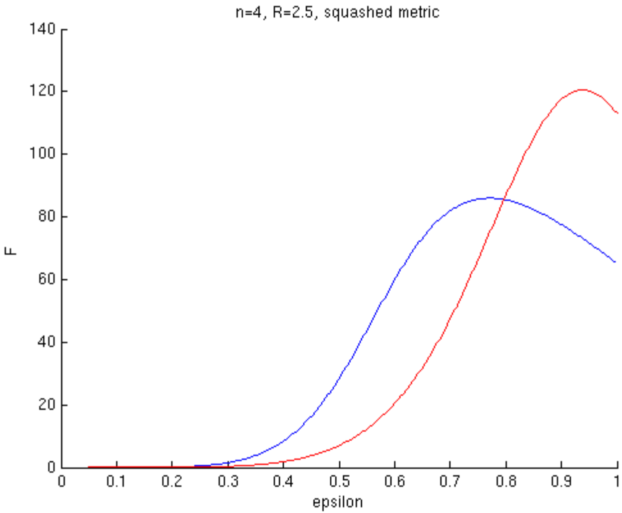
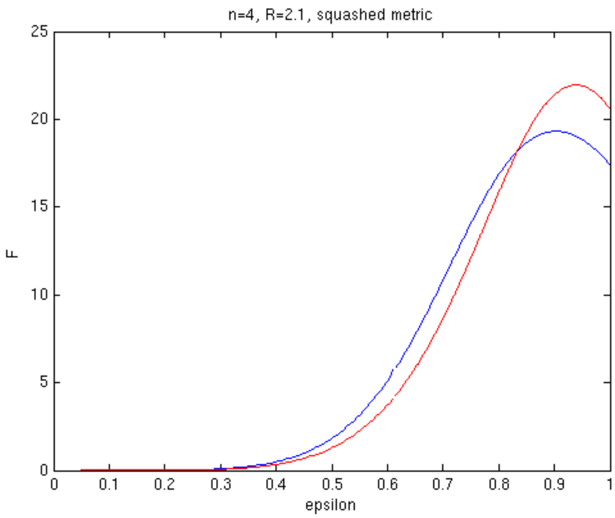


Convergence of $\gamma_{L^2}|$ on squashed spheres: M_4^0



Summary

- Baptista-Manton conjecture: for $M_n(S^2) \equiv \mathbb{C}P^n$, $\gamma_{L^2} \to \gamma_{FS}$ as $Area(S^2) \searrow 4\pi n$
- Very strong numerical evidence for $n = 2$, S^2 round
  - γ_{L^2} cohomogeneity 1, specified by a single $A : (0, 1) \to \mathbb{R}$
  - $A_{L^2} \to A_{FS}$
- Good numerical evidence for $n = 2$, S^2 squashed
  - Good convergence at least on totally geodesic sphere of "centred" vortex pairs
- OK numerical evidence for $n = 4$, S^2 round and squashed
- Interesting open question: what happens under Chern-Simons deformation?