Geometry of vortices on the sphere in the dissolving limit

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20/7/22

Rene $\longrightarrow$ 

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- Hermitian line bundle L over $\Sigma = (S^2, g_\Sigma)$, degree n

$$E(\phi, A) = \frac{1}{2}\|\mathrm{d}_A\phi\|^2_{L^2} + \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{8}\|\tau - |\phi|^2\|^2_{L^2}$$

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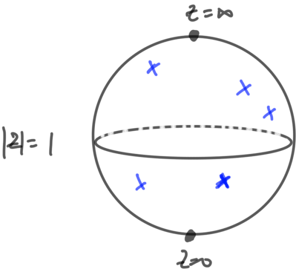
- $\varepsilon > 0$: $[(\phi, A)]$ uniquely determined by **divisor** (ϕ)



$$\leftrightarrow p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

$$\leftrightarrow [a_0, a_1, \ldots, a_n]$$

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- Baptista-Manton conjecture: $\lim_{\varepsilon \to 0} g_\varepsilon =$ Fubini-Study metric

The conjecture

- Equip L with hol struc $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

$$H^0(L) = \{\phi \in \Gamma(L) : \overline{\partial}_{\widehat{A}}\phi = 0\} \equiv \mathbb{C}^{n+1}$$

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$$\pi : S \to (\mathbb{P}(H^0(L)), g_{FS}), \qquad \widehat{\phi} \mapsto \{c\widehat{\phi}\}$$

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- Baptista-Manton conjecture: $\lim_{\varepsilon \to 0} g_\varepsilon = g_0$
- Surprising? Massive gain in symmetry

The theorem

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More precisely:

There exists $C > 0$ such that, for all $v \in TM_n$ and all $\varepsilon \in (0, 1)$

$$|g_\varepsilon(v, v) - g_0(v, v)| \leq C\varepsilon g_0(v, v)$$

The proof

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  $$(\phi, A) = (\sqrt{\varepsilon}\widehat{\phi}e^{u/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u)$$

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- Energy estimate, elliptic estimate, Sobolev $\Rightarrow$

$$\|u\|_{C^0} \leq C\varepsilon.$$

  Vortices are uniformly well approximated by pseudovortices (for small ε)

The proof (cont)

- Take a curve of vortex solutions

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$$\Delta \dot{u} + \varepsilon |\widehat{\phi}|^2 e^u \dot{u} = -2\varepsilon e^u h(\widehat{\phi}, \dot{\widehat{\phi}})$$

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- Lax-Milgram Lemma $\Rightarrow$ estimate $\|\dot{u}\|_{H^1} \leq C\varepsilon \|\dot{\widehat{\phi}}\|_{L^2}$
- Just enough to get bound on $|g_\varepsilon - g_0|$.

A corollary

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Then

$$\lambda_k(g) = \inf\{\Lambda_g(V) : \dim V = k + 1\}$$

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- **Corollary** (JMS,RGL): There exists $C > 0$ such that, for all $k \in \mathbb{Z}^+$
$$\left| \frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} - 1 \right| \leq C\varepsilon$$

- Spectrum of M_n converges uniformly to spectrum of FS

Open questions

- Convergence of geodesics? Need $g_\varepsilon \to g_0$ in C^1
- Convergence of curvature? Need $g_\varepsilon \to g_0$ in C^2
- n-dependence of the bounds?
- Leading correction to g_0?
- Higher genus? Much more subtle (Manton, Romao)