Dipole interactions in chiral ferromagnets

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https://cp1lump.github.io
Joint work with Paul Leask (KTH Stockholm)
arXiv:2504.17772
16/5/25

University of Leeds

The standard model

- $\boldsymbol{m} : \mathbb{R}^3 \to S^2$

$$E(\boldsymbol{m}) = \int_{\mathbb{R}^3} \underbrace{\frac{1}{2}|\mathrm{d}\boldsymbol{m}|^2}_{\text{exchange}} + \underbrace{\sum_{i=1}^{3} \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m})}_{\text{DMI}} + \underbrace{V(\boldsymbol{m})}_{\text{potential}}$$

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V(\boldsymbol{m}) &= -\boldsymbol{B}^{ext} \cdot \boldsymbol{m} + K(1 - m_3^2)
\end{aligned}
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- $\boldsymbol{B}^{ext} = (0, 0, B)$, assume $\partial_3 \boldsymbol{m} = \boldsymbol{0}$

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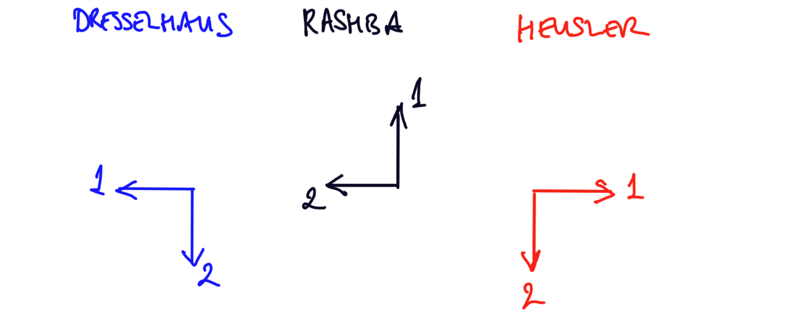
\end{aligned}
$$

- $\boldsymbol{B}^{ext} = (0, 0, B)$, assume $\partial_3 \boldsymbol{m} = \boldsymbol{0}$
- **BULK SKYRMION LINES** not thin film

3 classes of DMI

$$\sum_{i=1}^{2} \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m})$$



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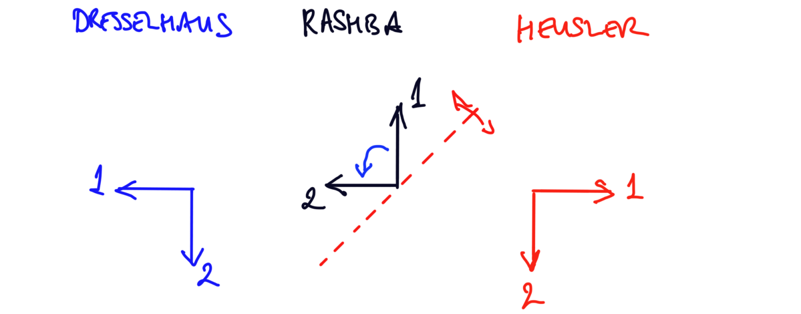

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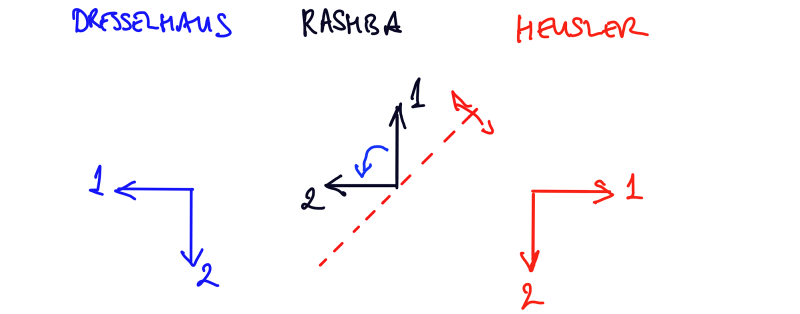
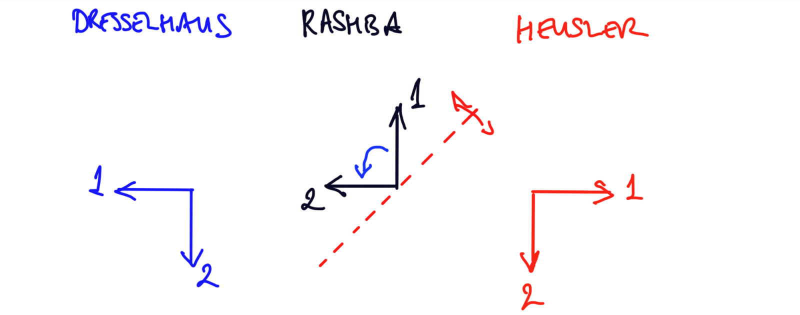


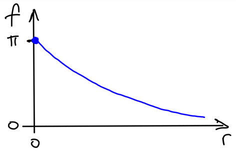
$$E_D(\boldsymbol{m}) = E_R(m_2, -m_1, m_3) \qquad E_H(\boldsymbol{m}) = E_R(-m_2, -m_1, m_3)$$

3 classes of DMI



$$E_D(\boldsymbol{m}) = E_R(m_2, -m_1, m_3) \qquad E_H(\boldsymbol{m}) = E_R(-m_2, -m_1, m_3)$$

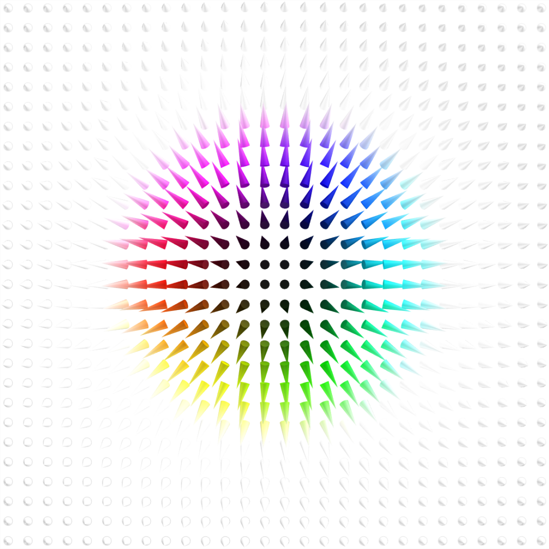
- Degenerate
- $\boldsymbol{m}_R(\boldsymbol{x}) = (\sin f(r)\cos\theta, \sin f(r)\sin\theta, \cos f(r))$



3 (equivalent) classes of DMI

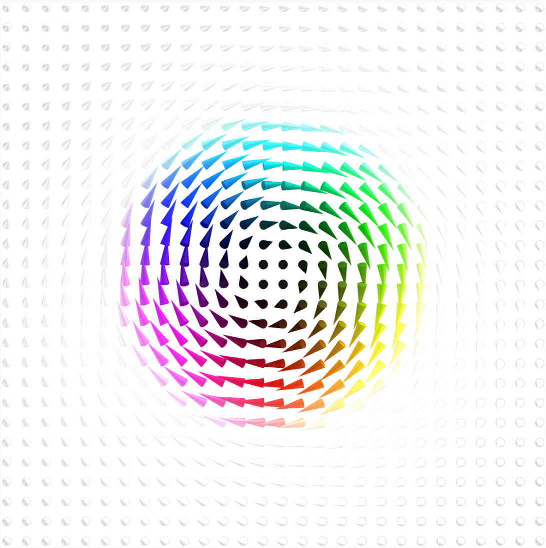
Rashba

Néel



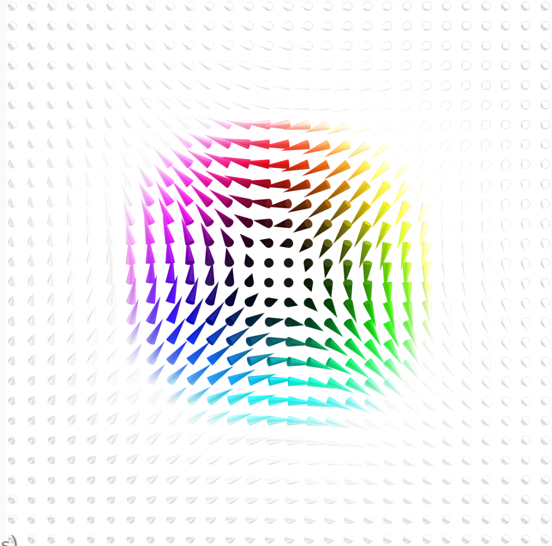
3 (equivalent) classes of DMI

Dresselhaus
Bloch

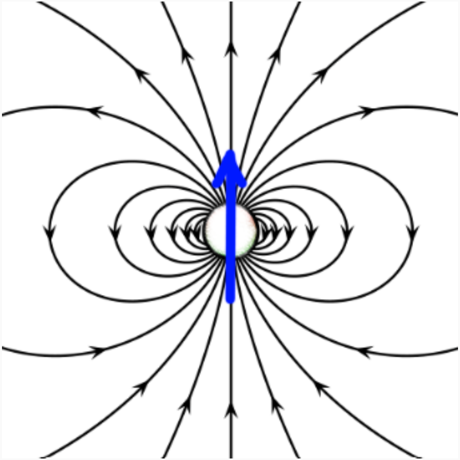


3 (equivalent) classes of DMI

Heusler
anti-Skyrmion



Dipole-dipole interactions



$$\boldsymbol{B} = -\frac{\mu_0}{4\pi r^3}\left(\boldsymbol{m} - 3\frac{\boldsymbol{m}\cdot\boldsymbol{x}}{r^2}\boldsymbol{x}\right)$$

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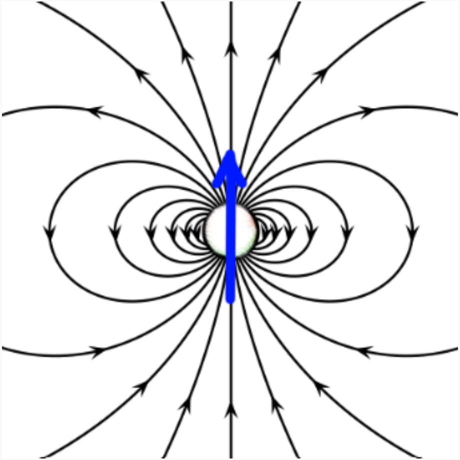

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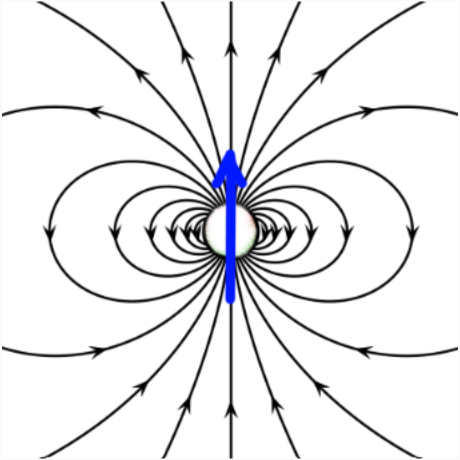

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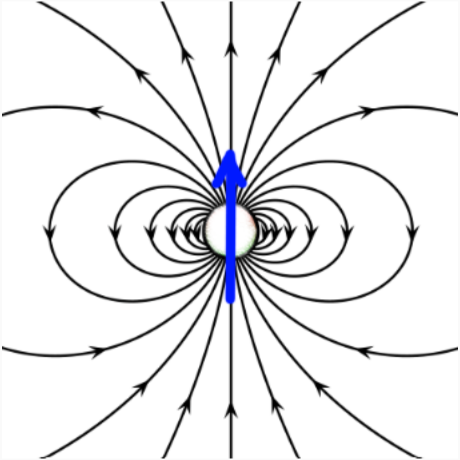
$$\boldsymbol{B} = -\boldsymbol{\nabla}\psi$$

Dipole-dipole interactions



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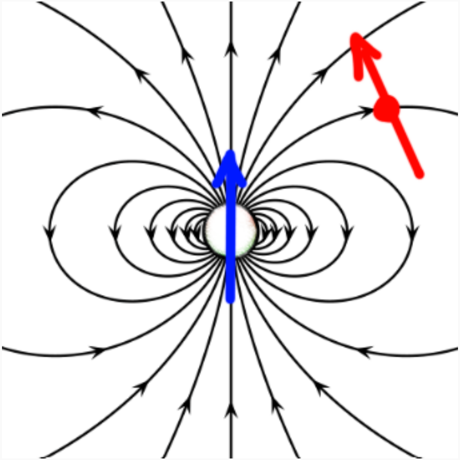
$$\boldsymbol{B} = -\boldsymbol{\nabla}\psi$$

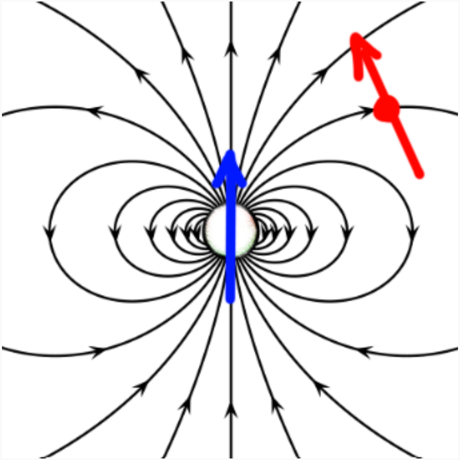
$$\psi = \mu_0 \boldsymbol{m}\cdot\boldsymbol{\nabla}\left(\frac{-1}{4\pi r}\right)$$

Dipole-dipole interactions



$$E_{int} = -\boldsymbol{B}^{(1)} \cdot \boldsymbol{m}^{(2)}$$

Dipole-dipole interactions



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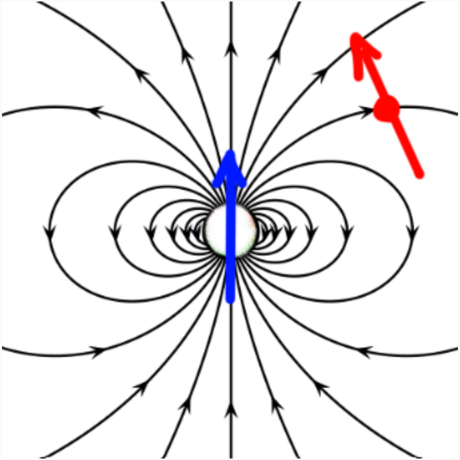
Dipole-dipole interactions



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Dipole-dipole interactions in the continuum limit

- Field induced by $\boldsymbol{m} : \mathbb{R}^3 \to S^2$

$$\boldsymbol{B}(\boldsymbol{x}) = -\frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\left\{\boldsymbol{m}(\boldsymbol{y}) - 3\frac{\boldsymbol{m}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}(\boldsymbol{x}-\boldsymbol{y})\right\} d^3\boldsymbol{y}$$

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- Interaction energy

$$E_{DDI} = \frac{\mu_0}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}\left\{\frac{\boldsymbol{m}(\boldsymbol{x})\cdot\boldsymbol{m}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3} - \frac{3\boldsymbol{m}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{m}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^5}\right\} d^3\boldsymbol{x}\, d^3\boldsymbol{y}$$

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- Nonlocal!

Dipole-dipole interactions in the continuum limit

- Field induced by $\boldsymbol{m} : \mathbb{R}^3 \to S^2$: $\boldsymbol{B} = -\nabla \psi$

$$\psi(\boldsymbol{x}) = -\mu_0 \int_{\mathbb{R}^3} \boldsymbol{m}(\boldsymbol{y}) \cdot \nabla_x \left(\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \right) \mathrm{d}^3\boldsymbol{y}$$

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- ψ satisfies Poisson's equation

$$\Delta\psi = \mu_0(-\boldsymbol{\nabla}\cdot\boldsymbol{m})$$

$\psi(\infty) = 0$

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- Interaction energy

$$E_{DDI}=\frac{1}{2}\int_{\mathbb{R}^3}\boldsymbol{m}\cdot\boldsymbol{\nabla}\psi=\frac{1}{2\mu_0}\int_{\mathbb{R}^3}\psi\Delta\psi.$$

Still nonlocal

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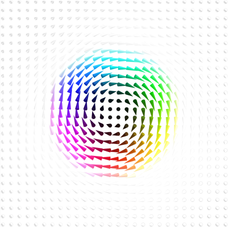
  Still nonlocal

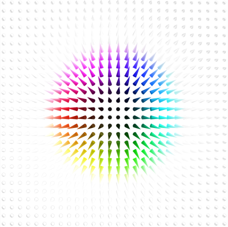
- "Coulomb energy" of "charge distribution" $\rho = -\nabla \cdot \boldsymbol{m}$

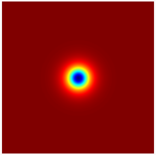
D (Bloch)



No effect

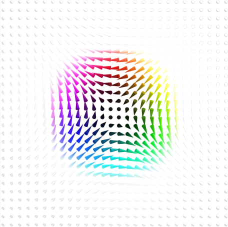
R (Néel)

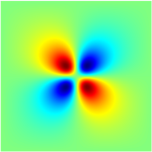




Raises energy

H ($\overline{\text{Skyrmion}}$)





Raises energy
Breaks symmetry

Numerical problem

- Minimize

$$E(\boldsymbol{m}) = \int_{\mathbb{R}^2} \frac{1}{2}|\mathrm{d}\boldsymbol{m}|^2 + \sum_{i=1}^{2} \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m}) + V(\boldsymbol{m}) + \frac{1}{2\mu_0}\psi\Delta\psi$$

where

$$\Delta\psi = \mu_0(-\boldsymbol{\nabla} \cdot \boldsymbol{m})$$

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where

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- Gradient flow? Pick $\boldsymbol{m}(0) : \mathbb{R}^2 \to S^2$, solve

$$\dot{\boldsymbol{m}}(t) = -\operatorname{grad}_{L^2} E(\boldsymbol{m}(t))$$

let $t \to \infty$.

Numerical problem

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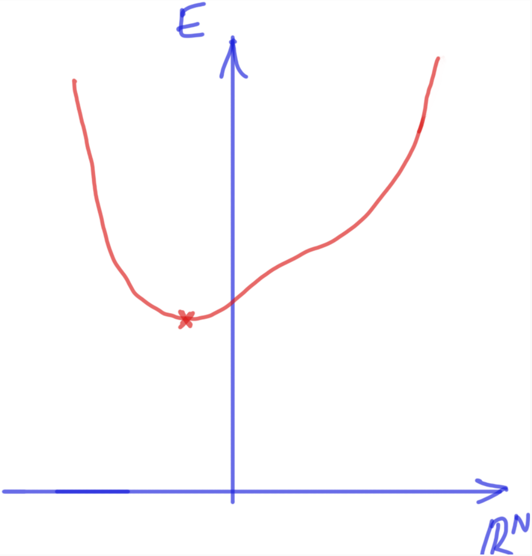
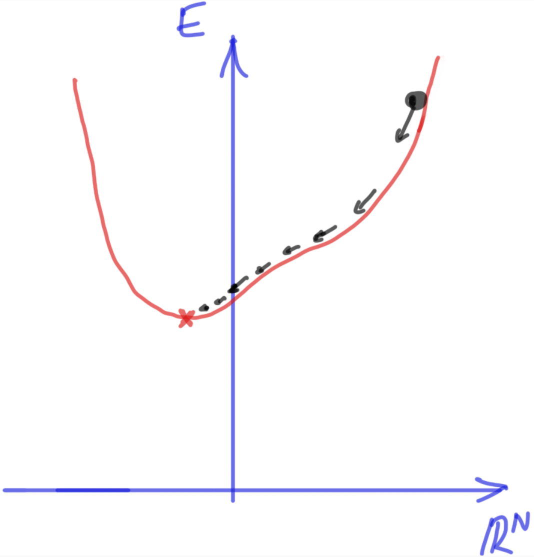
  let $t \to \infty$.

- LLG with $\gamma_{gyro} = 0$

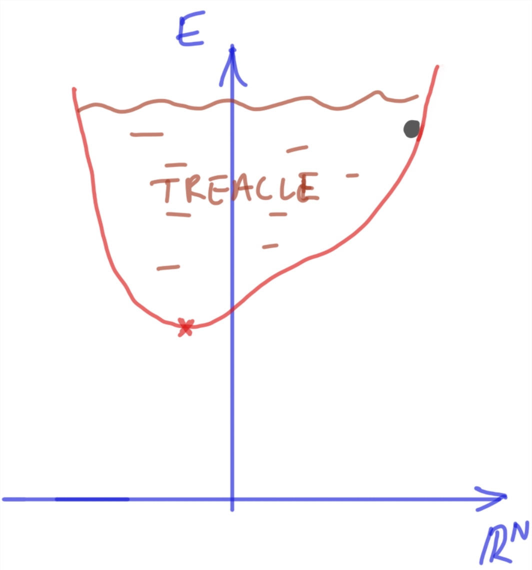
Gradient flow



Gradient flow

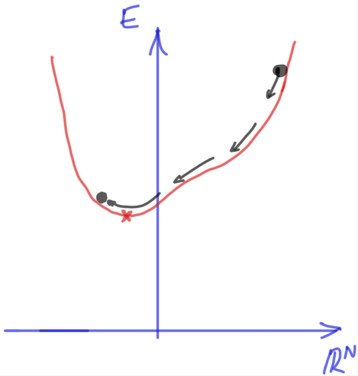


Gradient flow



Arrested Newton flow

$$\ddot{\boldsymbol{m}}(t) = -\,\mathrm{grad}_{L^2} E(\boldsymbol{m}(t))$$



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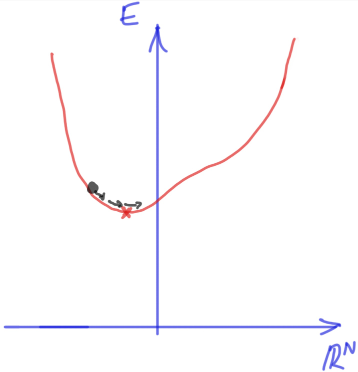

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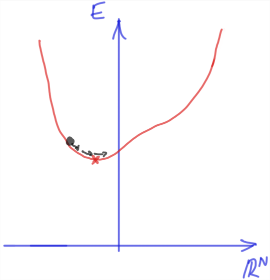

If

$$\langle \dot{\boldsymbol{m}}(t_*), \operatorname{grad}_{L^2} E(\boldsymbol{m}(t_*)) \rangle_{L^2} > 0,$$

set $\dot{\boldsymbol{m}}(t_*) = 0$, restart flow at $\boldsymbol{m}(t_*)$.

The gradient of E

- Smooth curve of maps $\boldsymbol{m}_t : \mathbb{R}^2 \to S^2$
- $\varepsilon = \partial_t \boldsymbol{m}_t|_{t=0}$

$$\left.\frac{d}{dt}\right|_{t=0} E(\boldsymbol{m}_t) = \cdots = \int_{\mathbb{R}^2} \varepsilon \cdot G(\boldsymbol{m})$$

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$\mathrm{grad}_{L^2} E(\boldsymbol{m}) = P_{\boldsymbol{m}} G(\boldsymbol{m})$ where

$$P_{\boldsymbol{m}}(\boldsymbol{u}) = \boldsymbol{u} - (\boldsymbol{m} \cdot \boldsymbol{u})\boldsymbol{m}.$$

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- In our case

$$\mathrm{grad}_{L^2} E(\boldsymbol{m}) = P_{\boldsymbol{m}} (\Delta \boldsymbol{m} - 2\boldsymbol{d}_i \times \partial_i \boldsymbol{m} + \mathrm{grad}\, V(\boldsymbol{m}) + \boldsymbol{\nabla}\, \psi)$$

where $\Delta\psi = \mu_0(-\boldsymbol{\nabla} \cdot \boldsymbol{m})$.

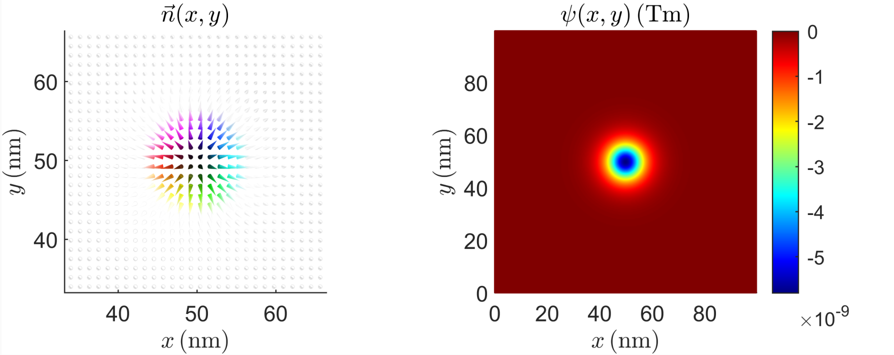
Arrested Newton flow

- Solve $\ddot{\boldsymbol{m}} = -\,\mathrm{grad}_{L^2} E(\boldsymbol{m})$ using RK4
- A each time t_i, need $\psi(t_i)$ to compute grad E
- Construct $\psi(t_i)$ by minimizing

$$F(\psi) = \int_{\mathbb{R}^2} \frac{1}{2}|\mathrm{d}\psi|^2 + \mu_0\psi(\boldsymbol{\nabla}\cdot\boldsymbol{m}(t_i))$$

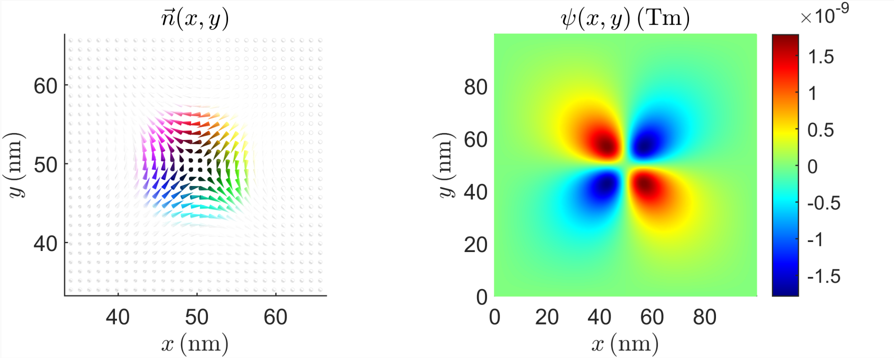
using a conjugate gradient algorithm.

Rashba



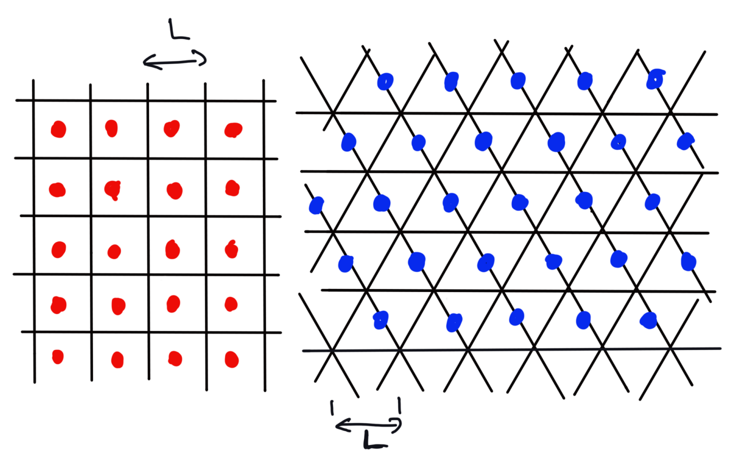
$\vec{n}(x,y)$
60
50
40
y (nm)
40
50
60
x (nm)
$\psi(x,y)$ (Tm)
80
60
40
20
0
y (nm)
0
20
40
60
80
x (nm)
0
-1
-2
-3
-4
-5
$\times 10^{-9}$

Heusler


$\vec{n}(x,y)$
60
50
40
y (nm)
40
50
60
x (nm)
$\psi(x,y)$ (Tm)
$\times 10^{-9}$
1.5
1
0.5
0
-0.5
-1
-1.5
80
60
40
20
0
y (nm)
0
20
40
60
80
x (nm)

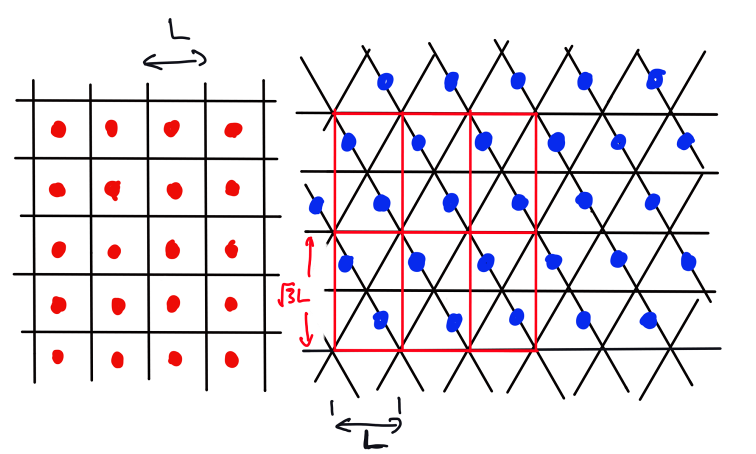
Numerical results: skyrmion lattices

$$\boldsymbol{m} : \mathbb{R}^2/\Lambda \to S^2$$

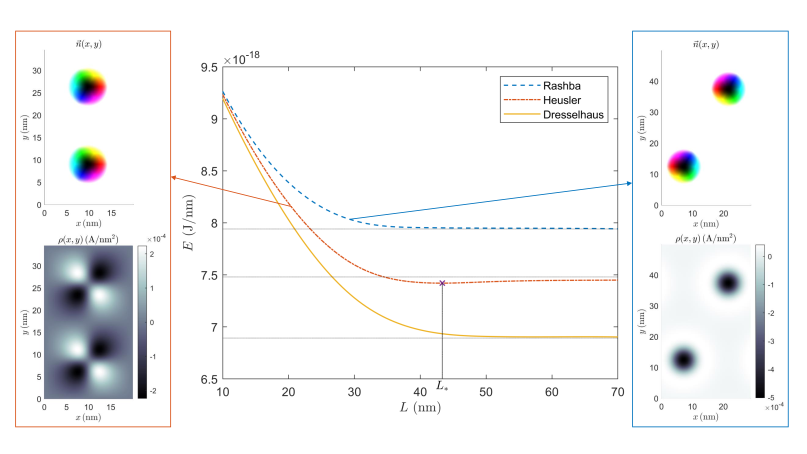


Numerical results: skyrmion lattices

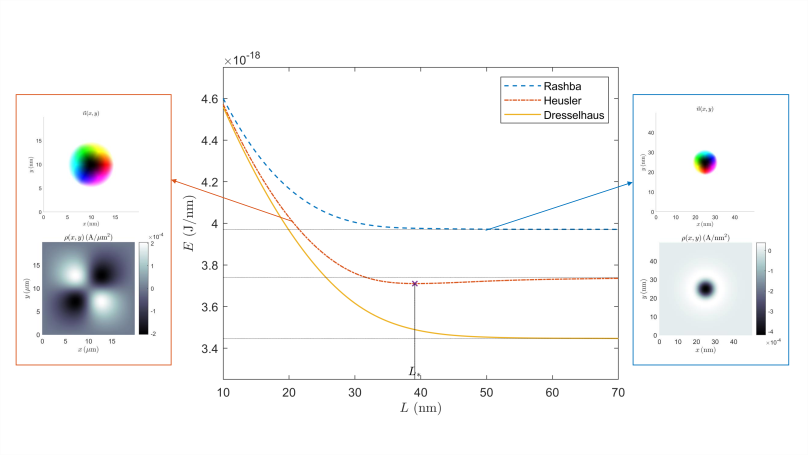
$$\boldsymbol{m} : \mathbb{R}^2/\Lambda \to S^2$$



Numerical results: skyrmion lattices (triangular)



Numerical results: skyrmion lattices (square)



Numerical results: skyrmion lattices

But minimizer will (probably) exist on **any** torus!

Numerical results: skyrmion lattices

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Should really minimize over period lattice as well as field (see Tom's talk...)

Concluding remarks

- Dipole-dipole interaction energy is nonlocal
- Coincides with Coulomb energy of charge distribution $\rho = -\nabla \cdot \boldsymbol{m}$
- Can minimize total energy using Arrested Newton Flow
  - To compute grad E need to solve for ψ at each step
  - Conjugate gradient method
- Effect depends strongly on DMI

DMI	isolated skyrmions	skyrmion lattice
Dresselhaus	none	weak
Rashba	weak	weak
Heusler	strong	strong

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Picture credit