A curious limit of the Faddeev-Skyrme model

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The problem

• $\phi: M \to N$, (M,g) Riemannian, (N,ω) symplectic

$$E(\varphi) = \frac{1}{2} \|\varphi^* \omega\|^2 = \frac{1}{2} \int_M \varphi^* \omega \wedge * \varphi^* \omega.$$

- Existence of critical points? Stability? (Index of Hessian)
- cf Harmonic map problem: (N, h) Riemannian

$$E_D(\varphi) = \frac{1}{2} \|\mathrm{d}\varphi\|^2$$

 Joint work with Martin Svensson (Odense). Also systematically studied by Slobodeanu and some special cases by De Carli and Ferreira (M Lorentzian)

•
$$M = \mathbb{R}^3$$
, $N = S^2 \subset \mathbb{R}^3$, $V: N \to [0, \infty)$

$$E_{FS} = \int_{M} \frac{1}{2} \sum_{i} \left| \frac{\partial \varphi}{\partial x_{i}} \right|^{2} + \frac{\alpha}{2} \sum_{i < j} \varphi \cdot \left(\frac{\partial \varphi}{\partial x_{i}} \times \frac{\partial \varphi}{\partial x_{j}} \right)^{2} + V(\varphi)$$
$$= \frac{1}{2} \|d\varphi\|^{2} + \frac{\alpha}{2} \|\varphi^{*}\omega\|^{2} + \int_{M} V \circ \varphi$$

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- Conformally invariant if dim M=4, infinite dimensional symmetry group (symplectic diffeos of (N,ω))

 Minimizer in generator of $\pi_4(S^2)$? ("Pure FS instanton")



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- If φ is critical and $\omega = h(J, \cdot) = K \ddot{a}hler$ form on N

$$\left. \frac{d^2 E(\varphi_t)}{dt^2} \right|_{t=0} = \langle X, \mathscr{L}_{\varphi} X \rangle_{L^2}$$

where

$$\mathscr{L}_{\varphi}X = -J(\nabla^{\varphi}_{\sharp\delta\omega^{*}\omega}X + d\varphi(\sharp\delta d\varphi^{*}\iota_{X}\omega))$$

(Ins)stability: spectrum of \mathscr{L}_{φ}

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 Proof: Let φ be coclosed and define functional

$$F(\psi) = \langle \phi^* \omega, \psi^* \omega \rangle.$$

Then \emph{F} is a homotopy invariant of ψ (homotopy lemma). Hence, if $\psi \sim \phi$,

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$$\mathrm{d}\phi(J\nabla f)=0\Rightarrow\phi^*\omega(J\nabla f,\nabla f)=0\Rightarrow -f|\nabla f|^2=0.$$

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Physically, rather surprising



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$$\mathscr{F}X = \left\{ \begin{array}{cc} 0 & X \in \mathscr{V}_X \\ JX & X \in \mathscr{H}_X \end{array} \right.$$

where we've used isomorphism $\mathrm{d}\phi_{x}:\mathscr{H}_{x}\to T_{\phi(x)}N$



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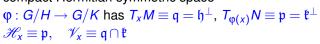
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• Example: G = SU(n), $K = S(U(k) \times U(n-k))$, $H = S(U(1)^n)$ Flag(\mathbb{C}^n) $\to Gr_k(\mathbb{C}^n)$ has div $\mathscr{F} = 0$ Coclosed, hence global minimizer

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- Example: G = SU(2), $K = S(U(1) \times U(1))$, H = eHopf: $S^3 \to S^2$ has div $\mathscr{F} \neq 0$ (of course) But it's still a global minimizer!

Energy bound $M^3 \rightarrow S^2$

• A map $\varphi: M^3 \to S^2$ is algebraically inessential if $[\varphi^*\omega] = 0$. Such maps are classified up to homotopy by

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But

$$Q(\phi) = \frac{1}{16\pi^2} \langle A, -* \, \mathrm{d}A \rangle \leq \frac{1}{16\pi^2} \|A\| \|\mathrm{d}A\| \leq \frac{1}{16\pi^2} \sqrt{\frac{2}{\lambda_1} E(\phi)} \sqrt{2E(\phi)}.$$

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- Cor: Hopf: S³ → S² is a stable critical point of the full FS energy E_D + αE for all α ≥ 1. (Conjectured by Ward, proved independently by Isobe.)

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for any area-preserving diffeo $\mathbb{R}^2 \to \mathbb{R}^2$

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- Assume $V(\varphi) = \frac{1}{2}U(\varphi)^2$ where $U: S^2 \to [0, \infty)$ is smooth.
- Bogomol'nyi type bound:

$$\begin{array}{lcl} 0 & \leq & \frac{1}{2}\|\phi^*\omega - *U\circ\phi\|^2 = E(\phi) - \int_{\mathbb{R}^2} \phi^*(U\omega) \\ \\ E(\phi) & \geq & \langle U\rangle \int_{\mathbb{R}^2} \phi^*\omega & \text{equality iff} & \phi^*\omega = *U\circ\phi \end{array}$$



• Example $U(\varphi) = 1 - \varphi_3$, for which $\langle U \rangle = 1$. Bog eqn has charge 1 solution

$$\varphi_G = (w(r)\cos\theta, w(r)\sin\theta, z(r)),$$

$$z(r) = 1 - 2e^{-r^2/2}, \quad w(r) = \sqrt{1 - z(r)^2}$$

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• Higher n solns? Let $X = \{p \in \mathbb{R}^2 : rank d\phi_p < 2\} = \phi^{-1}(0,0,1)$. Then ϕ is an area-preserving surjection

$$\left(\mathbb{R}^2\backslash X, \textit{dx} \land \textit{dy}\right) \rightarrow \left(\textit{S}^2_{\times}, \Omega\right)$$

where $\Omega = \omega/U$. Hence *X* has no bounded connected component. Weird...



...but possible! Area-preserving diffeo

$$p:(0,\infty)\times\mathbb{R}\to\mathbb{R}^2$$
 $(x,y)\mapsto(\log x,xy)$

Charge 1 solution of Bog eqn

$$\varphi(x,y) = \begin{cases} (\varphi_G \circ p)(x,y) & x > 0\\ (0,0,1) & x \le 0 \end{cases}$$

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$$q: \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2$$
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and extend. Get C^2 charge 1 solution which is constant outside strip $(0,\infty)\times(-\frac{\pi}{2},\frac{\pi}{2})$. Can patch n of these together, for any n.

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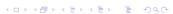
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- "Moduli space" of minimizers, even for n = 1, is very complicated.
- Cf model on S^2 with V = 0: very different.



Concluding remarks

- Variational problem for $E = \frac{1}{2} \|\phi^* \omega\|^2$ is mathematically interesting, and leads to insight into the full FS model on compact domains.
- Case $M = \Sigma^2$ finished
- Critical point in generator of $\pi_4(S^2)$? What about $\pi_3(S^2)\setminus\{-1,0,1\}$?
- Are there any unstable critical points?
- Case $M = \mathbb{R}^2$ with potential is very strange. What about $M = \mathbb{R}^3$?