A curious limit of the Faddeev-Skyrme model

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May 15, 2010

The problem

- $\varphi : M \to N$, (M, g) Riemannian, (N, ω) symplectic

$$E(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^2 = \frac{1}{2}\int_M \varphi^*\omega \wedge *\varphi^*\omega.$$

- Existence of critical points? Stability? (Index of Hessian)
- cf Harmonic map problem: (N, h) Riemannian

$$E_D(\varphi) = \frac{1}{2}\|d\varphi\|^2$$

- Joint work with Martin Svensson (Odense). Also systematically studied by Slobodeanu and some special cases by De Carli and Ferreira (M Lorentzian)

Motivation: Faddeev-Skyrme model

- $M = \mathbb{R}^3$, $N = S^2 \subset \mathbb{R}^3$, $V : N \to [0, \infty)$

$$\begin{aligned} E_{FS} &= \int_M \frac{1}{2}\sum_i \left|\frac{\partial\varphi}{\partial x_i}\right|^2 + \frac{\alpha}{2}\sum_{i<j} \varphi \cdot \left(\frac{\partial\varphi}{\partial x_i} \times \frac{\partial\varphi}{\partial x_j}\right)^2 + V(\varphi) \\ &= \frac{1}{2}\|\mathrm{d}\varphi\|^2 + \frac{\alpha}{2}\|\varphi^*\omega\|^2 + \int_M V \circ \varphi \end{aligned}$$

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- Conformally invariant if $\dim M = 4$, infinite dimensional symmetry group (symplectic diffeos of (N, ω))
  Minimizer in generator of $\pi_4(S^2)$? ("Pure FS instanton")

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- If φ is critical and $\omega = h(J\cdot,\cdot) =$ Kähler form on N

$$\frac{d^2E(\varphi_t)}{dt^2}\bigg|_{t=0} = \langle X, \mathscr{L}_\varphi X\rangle_{L^2}$$

where

$$\mathscr{L}_\varphi X = -J(\nabla^\varphi_{\sharp\delta\varphi^*\omega}X + \mathrm{d}\varphi(\sharp\delta\mathrm{d}\varphi^*\iota_X\omega))$$

(Ins)stability: spectrum of $\mathscr{L}_\varphi$

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  Proof: Let φ be coclosed and define functional

$$F(\psi) = \langle \varphi^*\omega, \psi^*\omega \rangle.$$

Then F is a homotopy invariant of ψ (homotopy lemma). Hence, if $\psi \sim \varphi$,

$$E(\varphi) = \frac{1}{2}F(\varphi) = \frac{1}{2}F(\psi) \leq \frac{1}{2}\|\varphi^*\omega\|\|\psi^*\omega\|.$$

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- E.g. $\mathrm{Id} : N \to N$, $p_1 : N \times P \to N$ are global minimimizers
- **Cor:** If φ is critical and a.e. immersive, then φ minimizes E in its htpy class

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$$\mathrm{d}\varphi(J\nabla f) = 0 \Rightarrow \varphi^*\omega(J\nabla f, \nabla f) = 0 \Rightarrow -f|\nabla f|^2 = 0.$$

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  Proof: f is constant. If $f = 0$, $E(\varphi) = 0$. Otherwise, φ has no critical points, so is a covering map. But $\pi_1(S^2) = 0$, so $\Sigma = S^2$ and φ is an (area preserving) diffeomorphism. $\square$

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- Physically, rather surprising

Critical submersions

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(In fact, iff harmonic, hence iff a harmonic morphism)

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- Example: $G = SU(n)$, $K = S(U(k) \times U(n-k))$, $H = S(U(1)^n)$
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- Example: $G = SU(2)$, $K = S(U(1) \times U(1))$, $H = e$
  $\mathrm{Hopf} : S^3 \to S^2$ has $\mathrm{div}\,\mathscr{F} \neq 0$ (of course)
  But it's still a global minimizer!

Energy bound $M^3 \to S^2$

- A map $\varphi : M^3 \to S^2$ is **algebraically inessential** if $[\varphi^*\omega] = 0$. Such maps are classified up to homotopy by

$$Q(\varphi) = \frac{1}{16\pi^2}\int_M A \wedge \mathrm{d}A \qquad \text{where} \quad \mathrm{d}A = \varphi^*\omega$$

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But

$$Q(\varphi) = \frac{1}{16\pi^2}\langle A, -*\mathrm{d}A\rangle \leq \frac{1}{16\pi^2}\|A\|\|\mathrm{d}A\| \leq \frac{1}{16\pi^2}\sqrt{\frac{2}{\lambda_1}E(\varphi)}\sqrt{2E(\varphi)}.$$

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- **Cor:** $\mathrm{Hopf} : S^3 \to S^2$ is a stable critical point of the full FS energy $E_D + \alpha E$ for all $\alpha \geq 1$. (Conjectured by Ward, proved independently by Isobe.)

Baby Skyrmions $\varphi : \mathbb{R}^2 \to S^2$

$$E(\varphi) = \frac{1}{2}\hat{\alpha}\|\mathrm{d}\varphi\|^2 + \frac{1}{2}\|\varphi^*\omega\|^2 + \int_{\mathbb{R}^2} V \circ \varphi$$

- $\varphi(\infty) = \varphi_0 \in V^{-1}(0)$, so fields labelled by integer charge $n = (4\pi)^{-1} \int \varphi^*\omega$

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- Assume $V(\varphi) = \frac{1}{2}U(\varphi)^2$ where $U : S^2 \to [0, \infty)$ is smooth.
- Bogomol'nyi type bound:

$$\begin{aligned} 0 &\leq \frac{1}{2}\|\varphi^*\omega - *U \circ \varphi\|^2 = E(\varphi) - \int_{\mathbb{R}^2} \varphi^*(U\omega) \\ E(\varphi) &\geq \langle U \rangle \int_{\mathbb{R}^2} \varphi^*\omega \qquad \text{equality iff} \qquad \varphi^*\omega = *U \circ \varphi \end{aligned}$$

- Example $U(\varphi) = 1 - \varphi_3$, for which $\langle U \rangle = 1$. Bog eqn has charge 1 solution

$$\varphi_G = (w(r)\cos\theta, w(r)\sin\theta, z(r)),$$

$$z(r) = 1 - 2e^{-r^2/2}, \quad w(r) = \sqrt{1 - z(r)^2}$$

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and "plastic" deformations
- Higher n solns? Let $X = \{p \in \mathbb{R}^2 \,:\, rank\,\mathrm{d}\varphi_p < 2\} = \varphi^{-1}(0,0,1)$. Then φ is an area-preserving surjection
$$(\mathbb{R}^2 \backslash X, dx \wedge dy) \to (S^2_\times, \Omega)$$
where $\Omega = \omega / U$. Hence X has no bounded connected component. Weird...

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and extend. Get C^2 charge 1 solution which is constant outside strip $(0, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Can patch n of these together, for any n.

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- Cf model on S^2 with $V = 0$: very different.

Concluding remarks

- Variational problem for $E = \frac{1}{2}\|\varphi^*\omega\|^2$ is mathematically interesting, and leads to insight into the full FS model on compact domains.
- Case $M = \Sigma^2$ finished
- Critical point in generator of $\pi_4(S^2)$? What about $\pi_3(S^2)\backslash\{-1,0,1\}$?
- Are there *any* unstable critical points?
- Case $M = \mathbb{R}^2$ with potential is very strange. What about $M = \mathbb{R}^3$?