Solitons on tori and soliton crystals

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Skyrme model

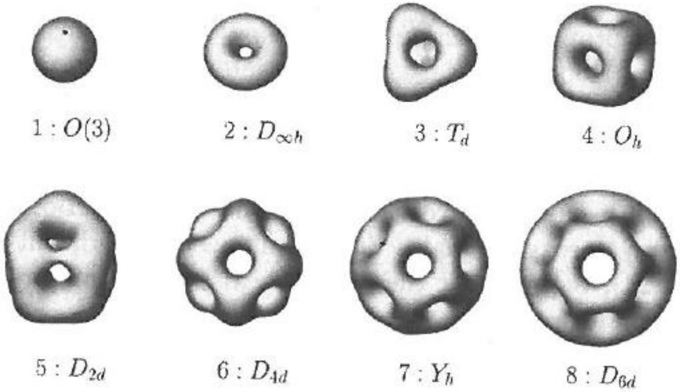
- Topological solitons: smooth, spatially localized solutions of nonlinear field theories, stable for topological reasons
- Particle-like dynamics (relativistic kinematics, anti-solitons, pair annihilation, binding, molecules etc.)
- $\varphi : (M, g) \to (G, h)$ e.g. $\mathbb{R}^3 \to SU(2)$
  - $\varphi(\infty) = e$, disjoint homotopy classes labelled by $B \in \pi_3(G)$
  - Left-invariant Maurer-Cartan form $\mu \in \Omega^1(G) \otimes \mathfrak{g}$
  - Associated two-form $\omega \in \Omega^2(G) \otimes \mathfrak{g}$, $\omega(X, Y) = [\mu(X), \mu(Y)]$
  - Skyrme energy

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 + |\varphi^*\omega|^2$$

  - Faddeev bound: $E(\varphi) \geq E_0 |B|$, unattainable
  - Degree B minimizer $\leftrightarrow$ nucleus of atomic weight B

Skyrme model

- Numerics



Battye and Sutcliffe

- E/B monotonically decreases e.g. 1.232 ($B = 1$), 1.096 ($B = 8$).

Skyrme model

- Suggests Skyrmions may be able to form a **crystal**

$$\varphi : \mathbb{R}^3/\Lambda \to G, \qquad \Lambda = \{n_1\mathbf{X}_1 + n_2\mathbf{X}_2 + n_3\mathbf{X}_3 \ : \ \mathbf{n} \in \mathbb{Z}^3\}$$

- Castillejo et al, Kugler et al, chose $\Lambda = L\mathbb{Z}^3$, found $B = 4$ minimizer for each $L > 0$, minimized over L. Found φ with $E/B = 1.036$.
- But is this really a crystal? Given any Λ, B, there exists a degree B minimizer $\varphi : \mathbb{R}^3/\Lambda \to G$.

For most Λ, lifted map $\mathbb{R}^3 \to G$ clearly isn't a genuine solution: artifact of bc's.

General question

- Given a minimizer $\varphi : \mathbb{R}^k/\Lambda \to N$ of some energy functional $E(\varphi)$, when is the lifted map $\mathbb{R}^k \to N$ a genuine crystal?
- Should be critical (in fact stable) with respect to variations of Λ too.

Change viewpoint

- All tori are diffeomorphic through linear maps $\mathbb{R}^k \to \mathbb{R}^k$.
- Identify them all with $M = \mathbb{R}^k/\Lambda_*$, the torus of interest. Now mfd is fixed, but **metric** depends on Λ

$$g_\Lambda = g_{ij}dx_i dx_j, \qquad g_{ij} \text{ const}$$

- Vary metric, $g_t \in \Gamma(T^*M \odot T^*M)$, g_0=Euclidean, $\varepsilon = \partial_t|_{t=0} g_t \in \Gamma(T^*M \odot T^*M)$
- Space of allowed variations $\mathbb{E} \subset \Gamma(T^*M \odot T^*M)$,

$$\mathbb{E} = \{\varepsilon_{ij}dx_i dx_j \,:\, \varepsilon_{ij} \text{ const}\}$$

- Nice $k(k+1)/2$ dimensional subspace of space of sections of rank $k(k+1)/2$ vector bundle $T^*M \odot T^*M$
- Canonically isomorphic to any fibre: $\mathbb{E} \equiv T^*_o M \odot T^*_o M$

Change viewpoint

- For **any** variation of g,

$$\left.\frac{dE(\varphi, g_t)}{dt}\right|_{t=0} =: \langle \varepsilon, S\rangle_{L^2}$$

  where $S \in \Gamma(T^*M \odot T^*M)$ is the **stress tensor** of φ.
- So E is critical for variations of g (equivalently, Λ), if $S \perp_{L^2} \mathbb{E}$.
- Given a **two-parameter** variation $g_{s,t} \in g + \mathbb{E}$ of critical g, define

$$\mathrm{Hess}(\widehat{\varepsilon}, \varepsilon) = \left.\frac{\partial^2 E(\varphi, g_{s,t})}{\partial s \partial t}\right|_{s=t=0}$$

  where $\widehat{\varepsilon} = \partial_s g_{s,t}|_{(0,0)}$, $\varepsilon = \partial_t g_{s,t}|_{(0,0)}$.

Change viewpoint

- Definitions:
  - An E minimizer $\varphi : M \rightarrow N$ is a **lattice** if it's critical with respect to variations of g in $\mathbb{E}$, that is, if $S \perp_{L^2} \mathbb{E}$.
  - A lattice φ is a **crystal** if, in addition, Hess is non-negative.

Warm-up exercise: the baby Skyrme model

- $\varphi : (M^2, g) \to (N, h, \omega)$ compact kähler (e.g. $N = S^2$)

$$E(\varphi, g) = \int_M \frac{1}{2}|d\varphi|^2 + \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi) = E_2 + E_4 + E_0$$

- Stress tensor

$$S = \frac{1}{2}(\frac{1}{2}|d\varphi|^2 - \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi))g - \frac{1}{2}\varphi^*h$$

- $\mathbb{E} = \langle g \rangle \oplus \mathbb{E}_0$, where $\mathbb{E}_0 = \langle g \rangle^\perp =$traceless SBF's, spanned by

$$\varepsilon_1 = dx_1^2 - dx_2^2, \qquad \varepsilon_2 = 2dx_1 dx_2$$

- Recall φ is a **lattice** if $S \perp_{L^2} \mathbb{E}$

Warm-up exercise: the baby Skyrme model

$$S = \frac{1}{2}(\frac{1}{2}|d\varphi|^2 - \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi))g - \frac{1}{2}\varphi^* h$$

- $\langle S, g\rangle_{L^2} = 0$ iff $\int_M \left(-\frac{1}{2}|\varphi^*\omega|^2 + V(\varphi)\right) = 0$

$$E_0 = E_4 \qquad \text{virial constraint}$$

- $S \perp_{L^2} \mathbb{E}_0$ iff $\langle \varphi^* h, dx_1^2 - dx_2^2\rangle_{L^2} = \langle \varphi^* h, dx_1 dx_2\rangle_{L^2} = 0$

$$\left.\begin{aligned} \int_M \left|d\varphi\frac{\partial}{\partial x_1}\right|^2 - \left|d\varphi\frac{\partial}{\partial x_2}\right|^2 &= 0 \\ \int_M h(d\varphi\frac{\partial}{\partial x_1}, d\varphi\frac{\partial}{\partial x_2}) &= 0 \end{aligned}\right\} \varphi \text{ “conformal on average”}$$

Warm-up exercise: the baby Skyrme model

- These conditions are easily checked numerically (unlike varying Λ!)
- E.g. Jäykkä, JMS, Sutcliffe: $N = S^2$, found a degree 2 lattice with periods L, $Le^{i\pi/3}$ for potential

$$V(\varphi) = |1 - (\varphi_2 + i\varphi_2)^3|^2(1 - \varphi_3)$$

But is it a crystal?

The hessian

- Given a **two-parameter** variation $g_{s,t} \in g + \mathbb{E}$ of a lattice (φ, g), define

$$\mathrm{Hess}(\widehat{\varepsilon}, \varepsilon) = \left.\frac{\partial^2 E(\varphi, g_{s,t})}{\partial s \partial t}\right|_{s=t=0}$$

  where $\widehat{\varepsilon} = \partial_s g_{s,t}|_{(0,0)}$, $\varepsilon = \partial_t g_{s,t}|_{(0,0)}$.

- Notation:

$$\begin{aligned} (A \cdot B)(X, Y) &:= \sum_i A(X, E_i) B(E_i, Y) \\ \dot{S} &:= \partial_s S(g_{s,0})|_{s=0} \end{aligned}$$

$$\begin{aligned} \mathrm{Hess}(\widehat{\varepsilon}, \varepsilon) &= \left.\frac{d}{ds}\right|_{s=0} \int_M \langle S(g_s), \varepsilon_s \rangle_{g_s} \mathrm{vol}_{g_s} \\ &= \langle \dot{S}, \varepsilon \rangle_{L^2} - \langle \widehat{\varepsilon}, S \cdot \varepsilon \rangle_{L^2} - \langle S \cdot \widehat{\varepsilon}, \varepsilon \rangle_{L^2} + \frac{1}{2} \int_M \langle S, \varepsilon \rangle \langle g, \widehat{\varepsilon} \rangle \mathrm{vol}_g \\ &= \langle \dot{S}, \varepsilon \rangle_{L^2} - 2 \langle \widehat{\varepsilon}, S \cdot \varepsilon \rangle_{L^2} \end{aligned}$$

The hessian

$$\mathrm{Hess}(\widehat{\varepsilon},\varepsilon) = \langle \dot{S}, \varepsilon\rangle_{L^2} - 2\langle \widehat{\varepsilon}, S\cdot\varepsilon\rangle_{L^2}$$

- Baby-Skyrmion lattice:

$$\begin{aligned} S &= \frac{1}{2}\left(\frac{1}{2}|d\varphi|_g^2 - \frac{1}{2}|\varphi^*\omega|_g^2 + V(\varphi)\right)g - \frac{1}{2}\varphi^* h \\ \Rightarrow \dot{S} &= \lambda g + \frac{1}{2}\left(\frac{1}{2}|d\varphi|_g^2 - \frac{1}{2}|\varphi^*\omega|_g^2 + V(\varphi)\right)\widehat{\varepsilon} \end{aligned}$$

- $\widehat{\varepsilon}, \varepsilon \in \mathbb{E}_0$:

$$\begin{aligned} \mathrm{Hess}(\widehat{\varepsilon},\varepsilon) &= \frac{1}{2}(E_2 - E_4 - E_0)\langle \widehat{\varepsilon},\varepsilon\rangle - 2\int_M \langle \widehat{\varepsilon}, S\cdot\varepsilon\rangle \\ &= -\frac{1}{2}(E_2 - E_4 - E_0)\langle \widehat{\varepsilon},\varepsilon\rangle + \int_M \langle \widehat{\varepsilon}, \varphi^* h\cdot\varepsilon\rangle \\ &= -\frac{1}{2}E_2\langle \widehat{\varepsilon},\varepsilon\rangle + \left\langle \widehat{\varepsilon}, \left(\int_M \varphi^* h\right)\cdot\varepsilon\right\rangle \qquad \text{virial constr.} \\ &= -\frac{1}{2}E_2\langle \widehat{\varepsilon},\varepsilon\rangle + \langle \widehat{\varepsilon}, (E_2 g)\cdot\varepsilon\rangle \qquad \text{conformal on avg.} \end{aligned}$$

The hessian

- $\widehat{\varepsilon}, \varepsilon \in \mathbb{E}_0$: $\mathrm{Hess}(\widehat{\varepsilon}, \varepsilon) = \frac{1}{2} E_2 \langle \widehat{\varepsilon}, \varepsilon \rangle$
- $\widehat{\varepsilon} = g \Rightarrow \dot{S} = \lambda g$, $\varepsilon \in \mathbb{E}_0$:

$$\mathrm{Hess}(g, \varepsilon) = \langle \lambda g, \varepsilon \rangle_{L^2} - 2 \langle S \cdot g, \varepsilon \rangle_{L^2} = 0$$

- $\mathrm{Hess}(g, g) > 0$ (Derrick scaling)
- Hence **every** baby Skyrmion lattice is a crystal!
- Only need to check
  1. Virial constraint ($E_4 = E_0$)
  2. Conformal on average

Exact baby Skyrme crystals

- Consider baby Skyrme model with $V(\varphi) = \frac{1}{2}U(\varphi)^2$
- Recall Bogomol'nyi bound**s** for $\varphi : M^2 \to S^2$,

$$E_2 = \frac{1}{2}\|d\varphi\|^2 \geq 4\pi n,$$

equality iff φ holomorphic

$$E_4 + E_0 = \frac{1}{2}(\|\varphi^*\omega\|^2 + \|U\circ\varphi\|^2) \geq 4\pi\langle U\rangle n,$$

equality iff $\varphi^*\omega = *U\circ\varphi$

- Given any lattice Λ, there is a degree 2 holomorphic map $\wp : \mathbb{C}/\Lambda \to S^2 = \mathbb{C}\cup\{\infty\}$ satisfying

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = P_3(\wp)$$

- For any holomorphic map $\varphi = W(z)$,

$$\varphi^*\omega = \frac{4|W'(z)|^2}{(1+|W(z)|^2)^2}\frac{i}{2}dz\wedge d\bar{z}$$

Exact baby Skyrme crystals

- Hence, if we choose

$$V(W) = \frac{8|P_3(W)|^2}{(1+|W|^2)^4},$$

 model has an exact solution $\varphi = \wp(z)$ on $\mathbb{C}/\Lambda$.

- Automatically a crystal
- Lattice $\Lambda \Rightarrow$ four vacua: ∞, $P_3^{-1}(0)$.
- Question: Given V with four vacua, can we use the corresponding elliptic function to predict Λ?

The Skyrme "crystal"

- $\varphi : M \to G = SU(2)$

$$E = E_2 + E_4 = \int_M \frac{1}{2}|d\varphi|^2 + \frac{1}{2}|\Omega|^2$$

where $\Omega = \varphi^*\omega \in \Omega^2(M) \otimes \mathfrak{g}$.

- Stress tensor $S = \frac{1}{4}(|d\varphi|^2 + |\Omega|^2)g + \frac{1}{2}(\Omega \cdot \Omega - \varphi^* h)$
- Lattice $\iff$ $S \perp_{L^2} \mathbb{E}$

$$\langle S, g\rangle_{L^2} = 0 \iff E_2 = E_4 \qquad \text{Virial constraint}$$
$$\varphi^* h - \Omega \cdot \Omega \perp_{L^2} \mathbb{E}_0$$

- Define SBF $\Delta : T_oM \times T_oM \to \mathbb{R}$,

$$\Delta(X, Y) = \int_M (\varphi^* h - \Omega \cdot \Omega)(X, Y)\mathrm{vol}_g$$

- Lattice $\iff$ Virial, $\Delta = c_0 g$

The Skyrme "crystal"

- Skyrme "crystal" of [Castillejo/Kugler] et al, has $\Lambda = L\mathbb{Z}^3$ and is invariant under

$$s_1 : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3), \quad (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_0, -\varphi_1, \varphi_2, \varphi_3)$$
$$s_2 : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1), \quad (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_0, \varphi_2, \varphi_3, \varphi_1)$$

  They checked it satisfies virial constraint

- s_1, s_2 generate group K of order 24
- Δ invariant under induced action of K on $T_o^*M \odot T_o^*M$

$$\widehat{s}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \widehat{s}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

w.r.t. $dx_1^2, dx_2^2, dx_3^2, dx_1dx_2, dx_1dx_3, dx_2dx_3$

The Skyrme "crystal"

- Decompose K-rep on $T_o^*M \odot T_o^*M$ into irreps, count copies of trivial rep

conj. class	e	$(s_1 s_2)^3$	s_1	$(s_1 s_2)^3 s_1$	s_2	$s_1 s_2$	s_2^2	$s_1 s_2^2$
size	1	1	3	3	4	4	4	4
$\widehat{\chi}$	6	6	2	2	0	0	0	0

- Hence

$$\langle \widehat{\chi}, \chi^{triv} \rangle = \frac{1}{|K|} \sum_{k \in K} \widehat{\chi}(k) \times 1 = 1$$

- Certainly g is K invariant. Hence $\Delta = c_0 g$. Skyrme "crystal" is (at least) a lattice.(Same reason Ohm's law holds for copper!)
- Hess > 0? Hess $\in \mathbb{E}^* \odot \mathbb{E}^*$ also invariant under induced K action

$$\langle \chi^{\mathbb{E}^* \odot \mathbb{E}^*}, \chi^{triv} \rangle = 2$$

The Skyrme "crystal"

- Define $H_1, H_2 \in \mathbb{E}^* \odot \mathbb{E}^*$

$$\begin{aligned} H_1(g,g) &= 1, \qquad H_1(\widehat{\varepsilon},\varepsilon) = 0 \quad \text{if } \varepsilon \in \mathbb{E}_0 \\ H_2(\widehat{\varepsilon},\varepsilon) &= \mathrm{tr}(\widehat{\varepsilon}\cdot\varepsilon) \end{aligned}$$

  These are invariant under K (actually, under all isometries of M)

- Hence K-invariance implies

$$\mathrm{Hess} = c_1 H_1 + c_2 H_2$$

  so it suffices to check

$$\begin{aligned} \mathrm{Hess}(g,g) &> 0 \qquad \text{[true, by Derrick scaling]} \\ \mathrm{Hess}(\varepsilon,\varepsilon) &> 0 \qquad \text{for any single } \varepsilon \in \mathbb{E}_0 \end{aligned}$$

- Fairly long calculation:

$$\mathrm{Hess}(2dx_1dx_2, 2dx_1dx_2) = \|\Omega_{23}\|^2_{L^2} + \|\Omega_{31}\|^2_{L^2} > 0$$

- So the Skyrme "crystal" is a crystal!

Concluding remarks

- Gave necessary conditions for a spatially periodic soliton solution to be a soliton crystal
- Conditions formulated in terms of stress tensor S (first variation of E w.r.t. g) and hessian Hess (second variation)
  - Lattice (critical) if $S \perp_{L^2} \mathbb{E} \subset \Gamma(T^*M \odot T^*M)$
  - Crystal (stable) if $\text{Hess} > 0$
- Baby Skyrmions:
  - Lattice iff satisfies virial constraint and is "conformal on average"
  - Lattice $\Rightarrow$ crystal (stability is automatic)!
- Skyrme "crystal":
  - Lattice iff virial constraint and $\Delta = \int_M (\varphi^* h - \Omega \cdot \Omega) = c_0 g$
  - Numerical work already showed virial constraint holds
  - Symmetry implies $\Delta = c_0 g$ and $\text{Hess} > 0$
  - Skyrme "crystal" is a crystal.

Concluding remarks

- Conditions are numerically accessible. E.g. for a periodic Skyrme field $\Delta(\partial_1,\partial_2)=0$ iff

$$\int_{T^3}(\mathrm{tr}\, L_1 L_2 + \mathrm{tr}[L_1,L_3][L_2,L_3])dx_1 dx_2 dx_3 = 0$$

  where $L_i = \varphi^{-1}\partial_i\varphi$
- Classify Λ such that K_Λ equivariance and Virial $\Rightarrow$ crystal?
- Other possibilities: partial periodicity $T^2 \times \mathbb{R}$?
  - Hexagonal Skyrmion sheet (Battye and Sutcliffe)
  - Generalized Skyrmion multisheets (Silva Lobo and Ward)