Solitons on tori and soliton crystals

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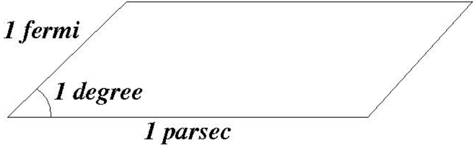
The question

- When is a spatially periodic minimizer $\varphi : \mathbb{R}^k/\Lambda \to N$ of a field-theoretic energy functional $E(\varphi)$ a crystal (periodic array of solitons in stable equilibrium)?
- E.g. Skyrme model: Castillejo et al, Kugler et al, chose $\Lambda = L\mathbb{Z}^3$, found $B = 4$ minimizer for each $L > 0$, minimized over L. Found φ with $E/B = 1.036$.
- But is this really a crystal? Given **any** Λ, B, there exists a degree B skyrme minimizer on $T^3_\Lambda = \mathbb{R}^3/\Lambda$.



For most Λ, lifted map $\mathbb{R}^3 \to N$ clearly isn't a genuine solution: artifact of bc's.

- To be a genuine crystal, E should be critical (in fact stable) under variations of Λ too.

A reformulation

- All tori are diffeomorphic through linear maps $\mathbb{R}^k \to \mathbb{R}^k$.
- Identify them all with $M = \mathbb{R}^k/\Lambda$, torus of interest.
- $\{$Vary Λ fixed $g\}$ $\longleftrightarrow$ $\{$Fix Λ vary $g\}$

$$g \in \{\sum_{ij} g_{ij} dx_i dx_j \ : \ g_{ij} \text{ const}\}$$

- Vary metric, $g_t \in \Gamma(T^*M \odot T^*M)$, g_0=Euclidean,

$$\varepsilon = \left.\frac{\partial g_t}{\partial t}\right|_{t=0} \in \mathbb{E} := \{\sum_{ij} \varepsilon_{ij} dx_i dx_j \ : \ \varepsilon_{ij} \text{ const}\}$$

- Nice $k(k+1)/2$ dimensional subspace of space of sections of rank $k(k+1)/2$ vector bundle $T^*M \odot T^*M$
  - Canonically isomorphic to any fibre: $\mathbb{E} \equiv T^*_o M \odot T^*_o M$
  - Inherits metric $\langle \widehat{\varepsilon}, \varepsilon \rangle_{\mathbb{E}} = \langle \widehat{\varepsilon}(o), \varepsilon(o) \rangle_g$

The payoff

- For **any** variation of g,

$$\left.\frac{dE(\varphi, g_t)}{dt}\right|_{t=0} =: \langle \varepsilon, S \rangle_{L^2}$$

  where $S \in \Gamma(T^*M \odot T^*M)$ is the **stress tensor** of φ.
- So E is critical for variations of Λ iff $S \perp_{L^2} \mathbb{E}$.
- Given critical g, define symmetric bilinear form

$$\text{Hess} : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$$

  as follows: for any **two-parameter** variation $g_{s,t} \in g + \mathbb{E}$, generated by $\widehat{\varepsilon} = \partial_s g_{s,t}|_{(0,0)}$, $\varepsilon = \partial_t g_{s,t}|_{(0,0)}$

$$\text{Hess}(\widehat{\varepsilon}, \varepsilon) := \left.\frac{\partial^2 E(\varphi, g_{s,t})}{\partial s \partial t}\right|_{s=t=0}.$$

- Stable under variations of Λ if $\text{Hess} > 0$.

Terminology

- Definitions:
  - An E minimizer $\varphi : M \to N$ is a **lattice** if $S \perp_{L^2} \mathbb{E}$.
  - An E minimizer $\varphi : M \to N$ is a **crystal** if $S \perp_{L^2} \mathbb{E}$ and $\text{Hess} > 0$.

Warm-up exercise: the baby Skyrme model

- $\varphi : (M^2, g) \to (N, h, \omega)$ compact kähler (e.g. $N = S^2$)

$$E(\varphi, g) = \int_M \frac{1}{2}|d\varphi|^2 + \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi) = E_2 + E_4 + E_0$$

- Stress tensor $S = \frac{1}{2}(\frac{1}{2}|d\varphi|^2 - \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi))g - \frac{1}{2}\varphi^* h$
- $\mathbb{E} = \langle g \rangle \oplus \mathbb{E}_0$, where $\mathbb{E}_0 = \langle g \rangle^\perp =$traceless SBF's
- $\langle S, g \rangle_{L^2} = 0$ iff $\int_M \left(-\frac{1}{2}|\varphi^*\omega|^2 + V(\varphi) \right) = 0$

$$E_0 = E_4 \qquad \text{virial constraint}$$

- $S \perp_{L^2} \mathbb{E}_0$ iff $\langle \varphi^* h, dx_1^2 - dx_2^2 \rangle_{L^2} = \langle \varphi^* h, dx_1 dx_2 \rangle_{L^2} = 0$

$$\left.\begin{aligned} \int_M \left| d\varphi \tfrac{\partial}{\partial x_1} \right|^2 - \left| d\varphi \tfrac{\partial}{\partial x_2} \right|^2 &= 0 \\ \int_M h(d\varphi \tfrac{\partial}{\partial x_1}, d\varphi \tfrac{\partial}{\partial x_2}) &= 0 \end{aligned}\right\} \varphi \text{ “conformal on average”}$$

Warm-up exercise: the baby Skyrme model

- These conditions are easily checked numerically (unlike varying Λ directly, cf Karliner and Hen)
- E.g. Jäykkä, JMS, Sutcliffe: $N = S^2$, found a degree 2 lattice with periods L, $Le^{i\pi/3}$ for potential

$$V(\varphi) = |1 - (\varphi_2 + i\varphi_2)^3|^2(1 - \varphi_3)$$

  Checked [virial constraint] and [conformal on average] numerically.
- But is it a crystal (Hess > 0) ?

The hessian

- Given a **two-parameter** variation $g_{s,t} \in g + \mathbb{E}$ of a lattice, with $\widehat{\varepsilon} = \partial_s g_{s,t}|_{(0,0)}$, $\varepsilon = \partial_t g_{s,t}|_{(0,0)}$ define

$$\begin{aligned}
\mathrm{Hess}(\widehat{\varepsilon}, \varepsilon) &= \left.\frac{\partial^2 E(\varphi, g_{s,t})}{\partial s \partial t}\right|_{s=t=0} \\
&= \left.\frac{d}{ds}\right|_{s=0} \int_M \langle S(g_s), \varepsilon_s \rangle_{g_s} \mathrm{vol}_{g_s} \\
&= \langle \dot{S}, \varepsilon \rangle_{L^2} - 2\langle \widehat{\varepsilon}, S \cdot \varepsilon \rangle_{L^2} + \frac{1}{2} \int_M \langle S, \varepsilon \rangle \langle g, \widehat{\varepsilon} \rangle \mathrm{vol}_g \\
&= \langle \dot{S}, \varepsilon \rangle_{L^2} - 2\langle \widehat{\varepsilon}, S \cdot \varepsilon \rangle_{L^2}
\end{aligned}$$

where

$$\begin{aligned}
\dot{S} &:= \partial_s S(g_{s,0})|_{s=0} \\
(A \cdot B)(X, Y) &:= \sum_i A(X, E_i) B(E_i, Y) \qquad (\text{note } g \cdot A = A \cdot g = A)
\end{aligned}$$

The hessian

$$\mathrm{Hess}(\widehat{\varepsilon},\varepsilon) = \langle \dot{S},\varepsilon\rangle_{L^2} - 2\langle \widehat{\varepsilon}, S\cdot\varepsilon\rangle_{L^2}$$

- Baby-Skyrmion lattice:

$$\begin{aligned} S &= \frac{1}{2}\Big(\frac{1}{2}|d\varphi|_g^2 - \frac{1}{2}|\varphi^*\omega|_g^2 + V(\varphi)\Big)g - \frac{1}{2}\varphi^* h \\ \Rightarrow \dot{S} &= \lambda g + \frac{1}{2}\Big(\frac{1}{2}|d\varphi|_g^2 - \frac{1}{2}|\varphi^*\omega|_g^2 + V(\varphi)\Big)\widehat{\varepsilon} \end{aligned}$$

- $\widehat{\varepsilon},\varepsilon \in \mathbb{E}_0$:

$$\begin{aligned} \mathrm{Hess}(\widehat{\varepsilon},\varepsilon) &= \frac{1}{2}(E_2 - E_4 + E_0)\langle\widehat{\varepsilon},\varepsilon\rangle_{\mathbb{E}} - 2\langle\widehat{\varepsilon}, S\cdot\varepsilon\rangle_{L^2} \\ &= -\frac{1}{2}(E_2 - E_4 + E_0)\langle\widehat{\varepsilon},\varepsilon\rangle_{\mathbb{E}} + \langle\widehat{\varepsilon}, \varphi^* h\cdot\varepsilon\rangle_{L^2} \\ &= -\frac{1}{2}E_2\langle\widehat{\varepsilon},\varepsilon\rangle_{\mathbb{E}} + \langle\widehat{\varepsilon},(E_2 g)\cdot\varepsilon\rangle_{\mathbb{E}} = \frac{1}{2}E_2\langle\widehat{\varepsilon},\varepsilon\rangle_{\mathbb{E}} \end{aligned}$$

The hessian

- $\widehat{\varepsilon} = g \Rightarrow \dot{S} = \lambda g$, $\varepsilon \in \mathbb{E}_0$:

$$\text{Hess}(g, \varepsilon) = \langle \lambda g, \varepsilon \rangle_{L^2} - 2\langle S \cdot g, \varepsilon \rangle_{L^2} = 0$$

- $\text{Hess}(g, g) > 0$ (Derrick scaling)
- Hence **every** baby Skyrmion lattice is a crystal!
- Only need to check
  1. Virial constraint ($E_4 = E_0$)
  2. Conformal on average
- Weird fact (Ward?): given any Λ can construct V such that model has a **holomorphic** energy minimizer. Automatically a crystal.

The Skyrme "crystal"

- $\varphi : M \to G$ $(T^3 \to SU(2))$

$$E = E_2 + E_4 = \int_M \frac{1}{2}|d\varphi|^2 + \frac{1}{2}|\varphi^*\omega|^2$$

where

$$\omega \in \Omega^2(G) \otimes \mathfrak{g}, \qquad \omega(X,Y) = [\mu(X), \mu(Y)]$$

and μ is the left Maurer-Cartan form.

- Let $\Omega = \varphi^*\omega \in \Omega^2(M) \otimes \mathfrak{g}$.
- Stress tensor

$$S = \frac{1}{4}(|d\varphi|^2 + |\Omega|^2)g + \frac{1}{2}(\Omega \cdot \Omega - \varphi^* h)$$

- Lattice $\iff$ $S \perp_{L^2} \mathbb{E}$

The Skyrme "crystal"

- Again decompose $\mathbb{E} = \langle g \rangle \oplus \mathbb{E}_0$:

$$\langle S, g \rangle_{L^2} = 0 \iff E_2 = E_4 \qquad \text{Virial constraint}$$

$$S \perp_{L^2} \mathbb{E}_0 \iff \varphi^* h - \Omega \cdot \Omega \perp_{L^2} \mathbb{E}_0 \qquad (*)$$

- Define $\Delta \in T_o^* M \otimes T_o^* M \equiv \mathbb{E}$,

$$\Delta(X, Y) = \int_M (\varphi^* h - \Omega \cdot \Omega)(X^{ext}, Y^{ext}) \mathrm{vol}_g$$

- $(*) \iff \Delta \perp_{\mathbb{E}} \mathbb{E}_0 \iff \Delta = \lambda g$

The Skyrme "crystal"

- Skyrme "crystal" of [Castillejo/Kugler] et al, has $\Lambda = L\mathbb{Z}^3$ and is invariant under

$$s_1 : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3), \quad (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_0, -\varphi_1, \varphi_2, \varphi_3)$$
$$s_2 : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1), \quad (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_0, \varphi_2, \varphi_3, \varphi_1)$$

  They checked it satisfies virial constraint
- s_1, s_2 generate group K of order 24
- Δ invariant under induced action of K on $\mathbb{E}$
- Representation theory: $\dim(\mathbb{E}^K) = 1$:

conj. class	e	$(s_1 s_2)^3$	s_1	$(s_1 s_2)^3 s_1$	s_2	$s_1 s_2$	s_2^2	$s_1 s_2^2$
size	1	1	3	3	4	4	4	4
$\chi^{\mathbb{E}}$	6	6	2	2	0	0	0	0

$$\langle \chi^{\mathbb{E}}, \chi^{triv} \rangle = \frac{1}{|K|} \sum_{k \in K} \chi^{\mathbb{E}}(k) \times 1 = 1$$

- Certainly g is K invariant. Hence $\Delta = \lambda g$. Skyrme "crystal" is (at least) a lattice.(Same reason Ohm's law holds for copper!)

The Skyrme "crystal"

- Hess > 0? Hess $\in \mathbb{E}^* \odot \mathbb{E}^*$ also invariant under induced K action

$$\langle \chi^{\mathbb{E}^* \odot \mathbb{E}^*}, \chi^{triv} \rangle = 2$$

- Define $H_1, H_2 \in \mathbb{E}^* \odot \mathbb{E}^*$

$$\begin{aligned} H_1(g,g) &= 1, \qquad H_1(\widehat{\varepsilon}, \varepsilon) = 0 \quad \text{if } \varepsilon \in \mathbb{E}_0 \\ H_2(\widehat{\varepsilon}, \varepsilon) &= \operatorname{tr}(\widehat{\varepsilon} \cdot \varepsilon) \end{aligned}$$

  These are invariant under K (actually, under all isometries of M)
- Hence K-invariance implies Hess $= c_1 H_1 + c_2 H_2$so it suffices to check

$$\begin{aligned} \text{Hess}(g,g) &> 0 \qquad \text{[true, by Derrick scaling]} \\ \text{Hess}(\varepsilon, \varepsilon) &> 0 \qquad \text{for any single } \varepsilon \in \mathbb{E}_0 \end{aligned}$$

- Fairly long calculation:

$$\text{Hess}(2dx_1 dx_2, 2dx_1 dx_2) = \|\Omega_{23}\|^2_{L^2} + \|\Omega_{31}\|^2_{L^2} > 0$$

  The Skyrme "crystal" is a crystal!

Concluding remarks

- Gave necessary conditions for a spatially periodic soliton solution to be a soliton crystal
- Conditions formulated in terms of stress tensor S (first variation of E w.r.t. g) and hessian Hess (second variation)
  - Lattice (critical) if $S \perp_{L^2} \mathbb{E} \subset \Gamma(T^*M \odot T^*M)$
  - Crystal (stable) if $S \perp_{L^2} \mathbb{E}$ and $\text{Hess} > 0$
- Baby Skyrmions:
  - Lattice iff satisfies virial constraint and is "conformal on average"
  - Lattice $\Rightarrow$ crystal (stability is automatic)!
- Skyrme "crystal":
  - Lattice iff virial constraint and $\Delta = \int_M (\varphi^* h - \Omega \cdot \Omega) = \lambda g$
  - Numerical work already showed virial constraint holds
  - Symmetry implies $\Delta = \lambda g$ and $\text{Hess} > 0$
  - Skyrme "crystal" is a crystal.

Concluding remarks

- Conditions are numerically accessible. E.g. for a periodic Skyrme field $\Delta(\partial_1, \partial_2) = 0$ iff

$$\int_{T^3} (\operatorname{tr} L_1 L_2 + \operatorname{tr}[L_1, L_3][L_2, L_3]) dx_1 dx_2 dx_3 = 0$$

  where $L_i = \varphi^{-1} \partial_i \varphi$
- Looks familiar...

Scaling Identities for Solitons beyond Derrick's Theorem

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Abstract

New integral identities satisfied by topological solitons in a range of classical field theories are presented. They are derived by considering independent length rescalings in orthogonal directions, or equivalently, from the conservation of the stress tensor. These identities are refinements of Derrick's theorem.

Concluding remarks

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and rotating this by 45° in the (x^1, x^2) plane gives

$$\int \left\{ -\frac{1}{2} \,\mathrm{Tr}\,(R_1 R_2) - \frac{1}{8} \,\mathrm{Tr}\,([R_1, R_3][R_2, R_3]) \right\} d^3x = 0 \,, \qquad (2.10)$$

which is a further novel identity. Two more identities are obtained by permuting the

- ...even when I think I've thought up something Nick hasn't, it turns out he (kind of) has!