Solitons on tori and soliton crystals

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Skyrme model

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  - $\varphi(\infty) = e$, disjoint homotopy classes labelled by $B \in \pi_3(G)$
  - Left-invariant Maurer-Cartan form $\mu \in \Omega^1(G) \otimes \mathfrak{g}$

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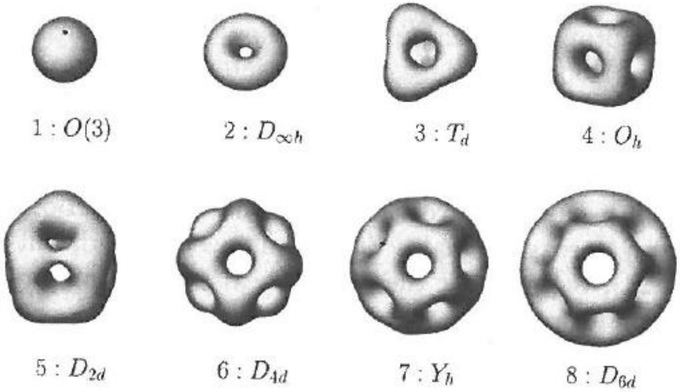
  - Faddeev bound: $E(\varphi) \geq E_0 |B|$, unattainable
  - Degree B minimizer $\leftrightarrow$ nucleus of atomic weight B

Skyrme model

- Numerics



Battye and Sutcliffe

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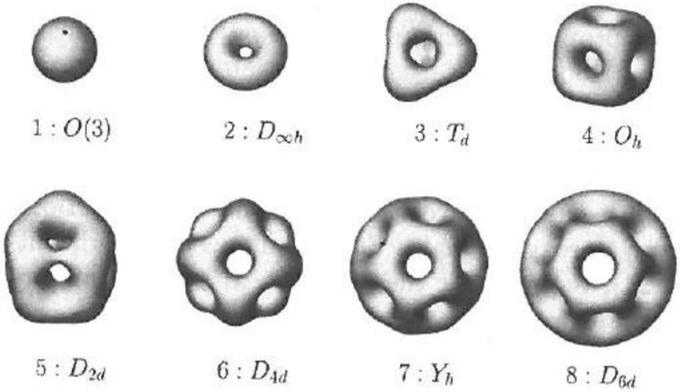

Battye and Sutcliffe

- E/B monotonically decreases e.g. 1.232 ($B = 1$), 1.096 ($B = 8$).

Skyrme model

- Suggests Skyrmions may be able to form a **crystal**

$$\varphi : \mathbb{R}^3/\Lambda \to G, \qquad \Lambda = \{n_1\mathbf{X}_1 + n_2\mathbf{X}_2 + n_3\mathbf{X}_3 \ : \ \mathbf{n} \in \mathbb{Z}^3\}$$

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- But is this really a crystal? Given any Λ, B, there exists a degree B minimizer $\varphi : \mathbb{R}^3/\Lambda \to G$.

For most Λ, lifted map $\mathbb{R}^3 \to G$ clearly isn't a genuine solution: artifact of bc's.

General question

- Given a minimizer $\varphi : \mathbb{R}^k/\Lambda \to N$ of some energy functional $E(\varphi)$, when is the lifted map $\mathbb{R}^k \to N$ a genuine crystal?

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- Should be critical (in fact stable) with respect to variations of Λ too.

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- Nice $k(k+1)/2$ dimensional subspace of space of sections of rank $k(k+1)/2$ vector bundle $T^*M \odot T^*M$
- Canonically isomorphic to any fibre: $\mathbb{E} \equiv T^*_o M \odot T^*_o M$

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$$\left.\frac{dE(\varphi, g_t)}{dt}\right|_{t=0} =: \langle \varepsilon, S \rangle_{L^2}$$

where $S \in \Gamma(T^*M \odot T^*M)$ is the stress tensor of φ.

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- Given a **two-parameter** variation $g_{s,t} \in g + \mathbb{E}$ of critical g, define

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- Definitions:
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  - A lattice φ is a **crystal** if, in addition, Hess is non-negative.

Warm-up exercise: the baby Skyrme model

- $\varphi : (M^2, g) \to (N, h, \omega)$ compact kähler (e.g. $N = S^2$)

$$E(\varphi, g) = \int_M \frac{1}{2}|\mathrm{d}\varphi|^2 + \frac{1}{2}|\varphi^*\omega|^2 + V(\varphi) = E_2 + E_4 + E_0$$

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- $\mathbb{E} = \langle g \rangle \oplus \mathbb{E}_0$, where $\mathbb{E}_0 = \langle g \rangle^\perp =$traceless SBF's, spanned by

$$\varepsilon_1 = \mathrm{d}x_1^2 - \mathrm{d}x_2^2, \qquad \varepsilon_2 = 2\mathrm{d}x_1\mathrm{d}x_2$$

- Recall φ is a **lattice** if $S \perp_{L^2} \mathbb{E}$

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$$\left.\begin{aligned} \int_M \left|d\varphi\frac{\partial}{\partial x_1}\right|^2 - \left|d\varphi\frac{\partial}{\partial x_2}\right|^2 &= 0 \\ \int_M h\Big(d\varphi\frac{\partial}{\partial x_1}, d\varphi\frac{\partial}{\partial x_2}\Big) &= 0 \end{aligned}\right\} \varphi \text{ “conformal on average”}$$

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- E.g. Jäykkä, JMS, Sutcliffe: $N = S^2$, found a degree 2 lattice with periods L, $Le^{i\pi/3}$ for potential

$$V(\varphi) = |1 - (\varphi_2 + i\varphi_2)^3|^2(1 - \varphi_3)$$

But is it a crystal?

The hessian

- Given a **two-parameter** variation $g_{s,t} \in g + \mathbb{E}$ of a lattice (φ, g), define

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- Notation:

$$\begin{aligned} (A \cdot B)(X, Y) &:= \sum_i A(X, E_i) B(E_i, Y) \\ \dot{S} &:= \partial_s S(g_{s,0})|_{s=0} \end{aligned}$$

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&= -\frac{1}{2}E_2\langle\widehat{\varepsilon},\varepsilon\rangle + \langle \widehat{\varepsilon}, (E_2 g)\cdot\varepsilon\rangle \qquad \text{conformal on avg.}
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- $\widehat{\varepsilon}\in\mathbb{E}_0 \Rightarrow \dot{S} = \lambda g + \frac{1}{2}(\frac{1}{2}|\mathrm{d}\varphi|_g^2 - \frac{1}{2}|\varphi^*\omega|_g^2 + V(\varphi))\widehat{\varepsilon}$
- $\varepsilon\in\mathbb{E}_0$ also:

$$\begin{aligned}
\mathrm{Hess}(\widehat{\varepsilon},\varepsilon) &= \frac{1}{2}(E_2 - E_4 - E_0)\langle\widehat{\varepsilon},\varepsilon\rangle - 2\int_M \langle\widehat{\varepsilon}, S\cdot\varepsilon\rangle \\
&= -\frac{1}{2}(E_2 - E_4 - E_0)\langle\widehat{\varepsilon},\varepsilon\rangle + \int_M \langle\widehat{\varepsilon}, \varphi^* h\cdot\varepsilon\rangle \\
&= -\frac{1}{2}E_2\langle\widehat{\varepsilon},\varepsilon\rangle + \langle\widehat{\varepsilon}, \left(\int_M \varphi^* h\right)\cdot\varepsilon\rangle \qquad \text{virial constr.} \\
&= -\frac{1}{2}E_2\langle\widehat{\varepsilon},\varepsilon\rangle + \langle\widehat{\varepsilon}, (E_2 g)\cdot\varepsilon\rangle \qquad \text{conformal on avg.} \\
&= \frac{1}{2}E_2\langle\widehat{\varepsilon},\varepsilon\rangle
\end{aligned}$$

The hessian

- $\widehat{\varepsilon} = g \Rightarrow \dot{S} = \lambda g$, $\varepsilon \in \mathbb{E}_0$:

$$\mathrm{Hess}(g, \varepsilon) = \langle \lambda g, \varepsilon \rangle_{L^2} - 2\langle S \cdot g, \varepsilon \rangle_{L^2} = 0$$

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- Only need to check
  1. Virial constraint ($E_4 = E_0$)
  2. Conformal on average

The Skyrme “crystal”

- $\varphi : M \to G = SU(2)$

$$E = E_2 + E_4 = \int_M \frac{1}{2}|\mathrm{d}\varphi|^2 + \frac{1}{2}|\Omega|^2$$

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- Skyrme "crystal" of [Castillejo/Kugler] et al, has $\Lambda = L\mathbb{Z}^3$ and is invariant under

$$s_1 : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3), \quad (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_0, -\varphi_1, \varphi_2, \varphi_3)$$
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- s_1, s_2 generate group K of order 24
- Δ invariant under induced action of K on $T_o^*M \odot T_o^*M$

$$\widehat{s}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \widehat{s}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

w.r.t. $dx_1^2, dx_2^2, dx_3^3, dx_1 dx_2, dx_1 dx_3, dx_2, dx_3$

The Skyrme "crystal"

- Decompose K-rep on $T_o^* M \odot T_o^* M$ into irreps, count copies of trivial rep

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- Hess > 0? Hess $\in \mathbb{E}^* \odot \mathbb{E}^*$ also invariant under induced K action

$$\langle \chi^{\mathbb{E}^* \odot \mathbb{E}^*}, \chi^{triv} \rangle = 2$$

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$$\begin{aligned} H_1(g,g) &= 1, \qquad H_1(\widehat{\varepsilon},\varepsilon) = 0 \quad \text{if } \varepsilon \in \mathbb{E}_0 \\ H_2(\widehat{\varepsilon},\varepsilon) &= \mathrm{tr}(\widehat{\varepsilon}\cdot\varepsilon) \end{aligned}$$

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  so it suffices to check

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- Fairly long calculation:

$$\mathrm{Hess}(2\mathrm{d}x_1\mathrm{d}x_2, 2\mathrm{d}x_1\mathrm{d}x_2) = \|\Omega_{23}\|_{L^2}^2 + \|\Omega_{31}\|_{L^2}^2 > 0$$

- So the Skyrme “crystal” is a crystal!

Concluding remarks

- Answered the question “when is a spatially periodic soliton solution a soliton crystal?”

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- Skyrme "crystal":
  - Lattice iff virial constraint and $\Delta = \int_M (\varphi^* h - \Omega \cdot \Omega) = c_0 g$
  - Numerical work already showed virial constraint holds
  - Symmetry implies $\Delta = c_0 g$
  - Symmetry also implies $\mathrm{Hess} > 0$
  - Skyrme "crystal" is a crystal.

Concluding remarks

- Conditions are numerically accessible. E.g. for a periodic Skyrme field $\Delta(\partial_1, \partial_2) = 0$ iff

$$\int_{T^3} (\mathrm{tr}\, L_1 L_2 + \mathrm{tr}[L_1, L_3][L_2, L_3]) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = 0$$

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- Classify Λ such that K_Λ equivariance and Virial $\Rightarrow$ crystal?
- Other possibilities: partial periodicity $T^2 \times \mathbb{R}$?
  - Hexagonal Skyrmion sheet (Battye and Sutcliffe)
  - Generalized Skyrmion multisheets (Silva Lobo and Ward)