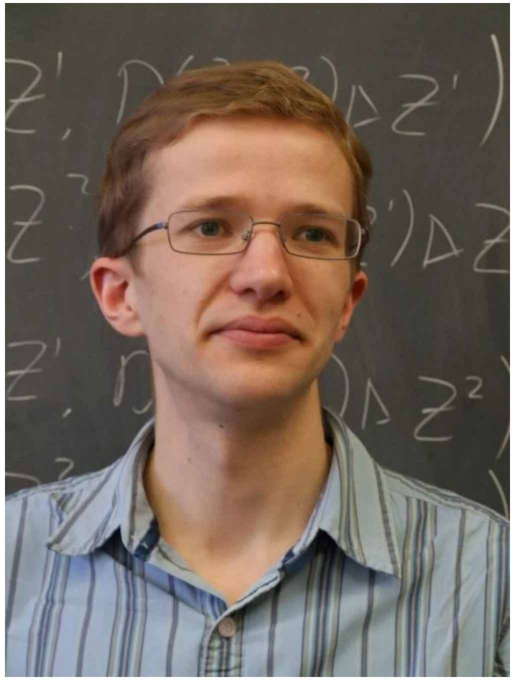
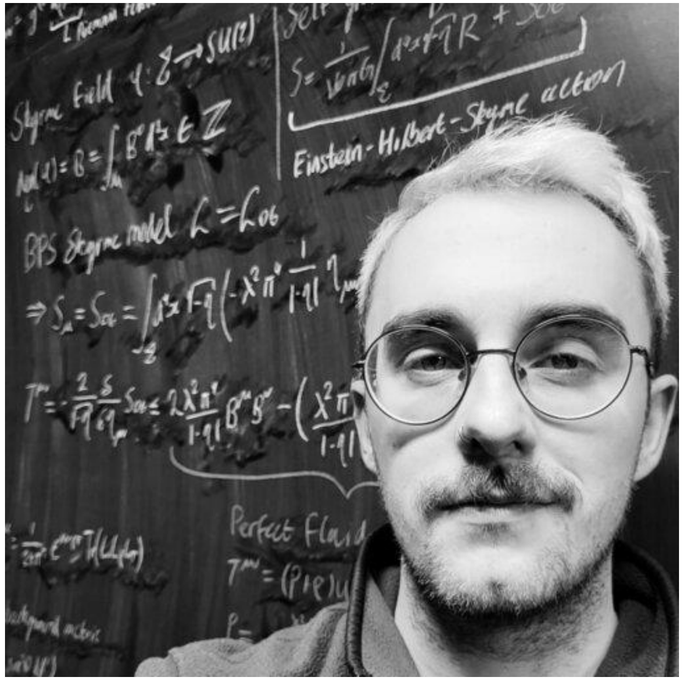
Soliton crystals

Martin Speight, University of Leeds

Joint work with Derek Harland and Paul Leask





Topological solitons

- Smooth, spatially localized lump-like solutions of classical nonlinear field theories
- Stable due to topology
- Particle like: relativistic kinematics, scattering, radiation, anti-solitons
- Naïve dream: maybe elementary particles really *are* solitons!
- Prosaic reality: probably not, but the same/similar structures are ubiquitous in condensed matter systems
- Kinks, lumps, vortices, monopoles, **Skyrmions**

Skyrmions

T.H.R. Skyrme

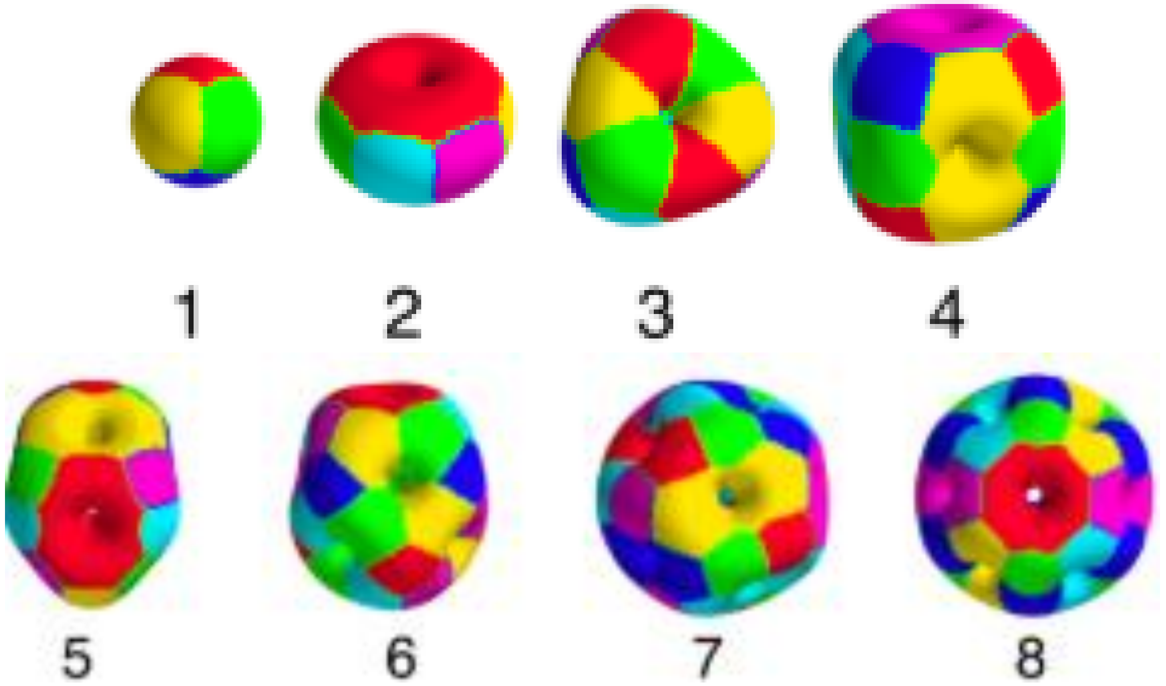
$\varphi : \mathbb{R}^3 \to S^3$

$\varphi(\infty) = (1, 0, 0, 0)$

Extension $\varphi : \mathbb{R}^3 \cup \{\infty\} \to S^3$

Topological charge $B = \deg \varphi \in \mathbb{Z}$

$$B = \int_{\mathbb{R}^3} \varphi^* \mathrm{vol}_{S^3}$$

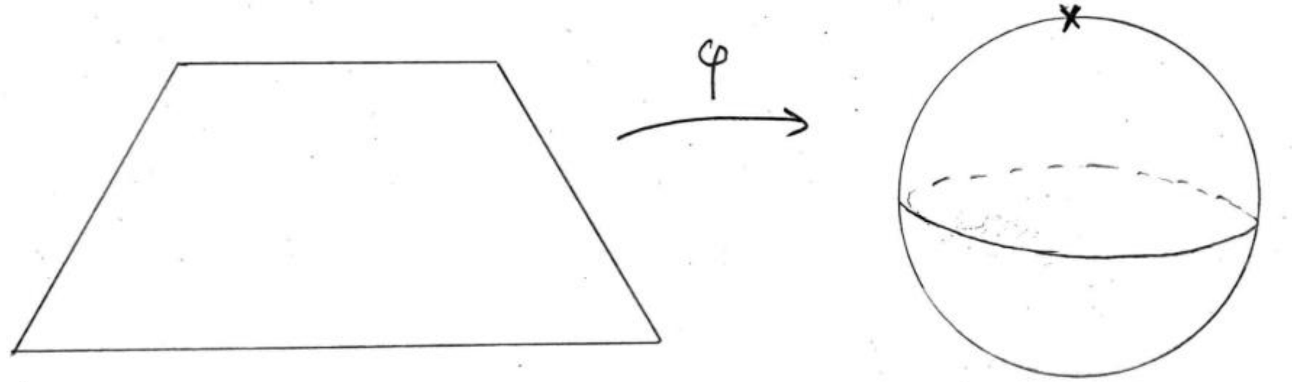


Picture credits: Skyrme portrait, Master & Fellows, Trinity College, Cambridge University
Skyrmions: Carlos Naya and Paul Sutcliffe

Baby Skyrmions

$$\varphi : \mathbb{R}^2 \to S^2$$

$$E(\varphi) = \int_{\mathbb{R}^3} \left\{ \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} \varphi \cdot \left(\frac{\partial \varphi}{\partial x_1} \times \frac{\partial \varphi}{\partial x_2} \right)^2 + (1 - \varphi_3) \right\}$$



$$\varphi : \mathbb{R}^2 \cup \{\infty\} \to S^2$$

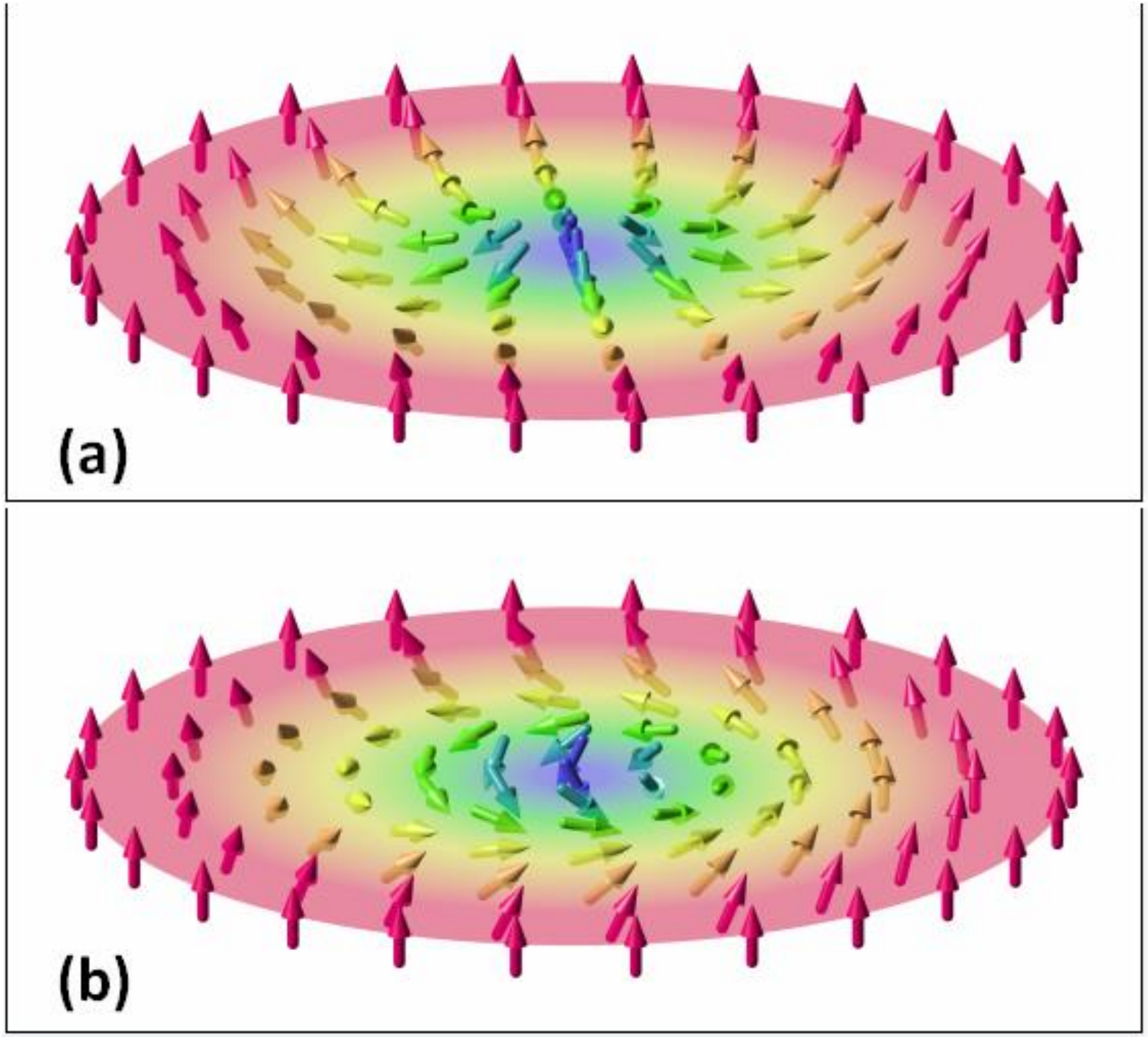
Topological charge $n = \deg \varphi \in \mathbb{Z}$

$$n = \int_{\mathbb{R}^2} \varphi \cdot \left(\frac{\partial \varphi}{\partial x_1} \times \frac{\partial \varphi}{\partial x_2} \right)$$

Baby Skyrmions

Hedgehog

Isorotated hedgehog



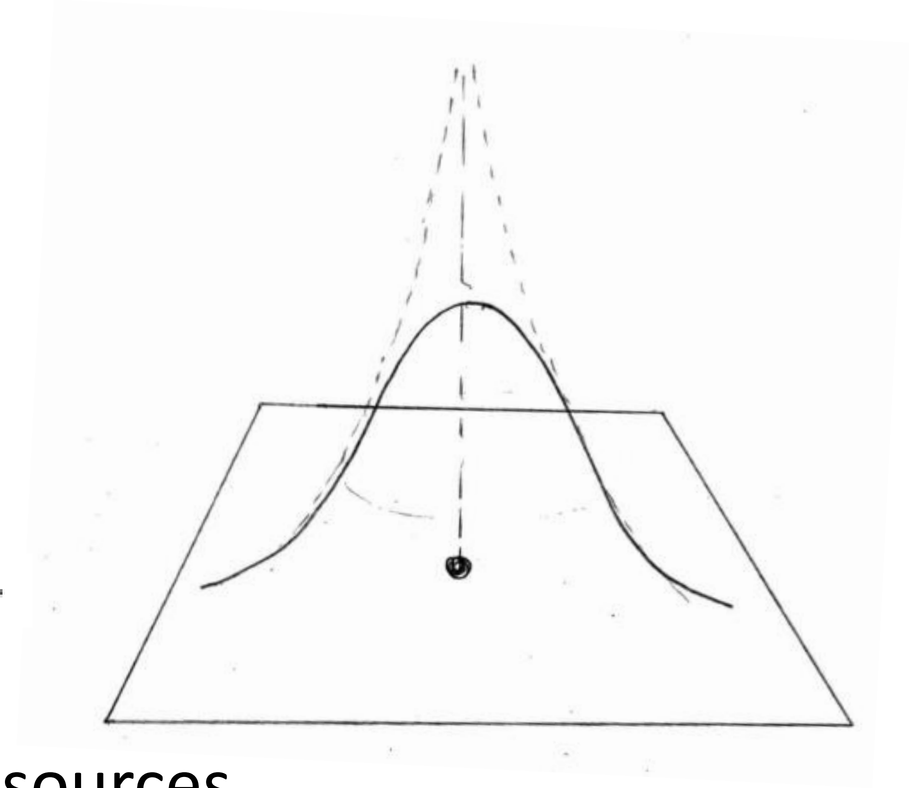
Picture credit: Karin Everschor-Sitte and Matthias Sitte

Interskyrmion forces

- Linearize model about vacuum

$$\varphi = (0,0,1) + (\varepsilon_1, \varepsilon_2, 0) + \cdots$$

$$\varepsilon_1 = -q\partial_{x_1} K_0(r) + \cdots \qquad \varepsilon_2 = -q\partial_{x_2} K_0(r) + \cdots$$



- Solution of Klein-Gordon model with static sources

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \varepsilon \cdot \partial^\mu \varepsilon - \tfrac{1}{2}|\varepsilon|^2 + \kappa_i \varepsilon_i$$

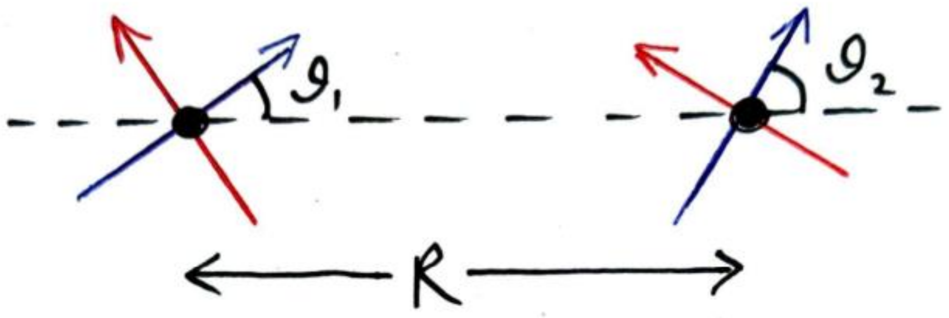
$$\kappa_1 = (q,0) \cdot \nabla\delta(x) \qquad \kappa_2 = (0,q) \cdot \nabla\delta(x)$$

- Orthogonal pair of scalar dipoles

Piette, Schroers and Zakrzewski, Z.Phys. C65 (1995) 165-174

Interskyrmion forces

- At long range, skyrmion interactions should approach forces between point particles carrying an orthogonal pair of scalar dipoles



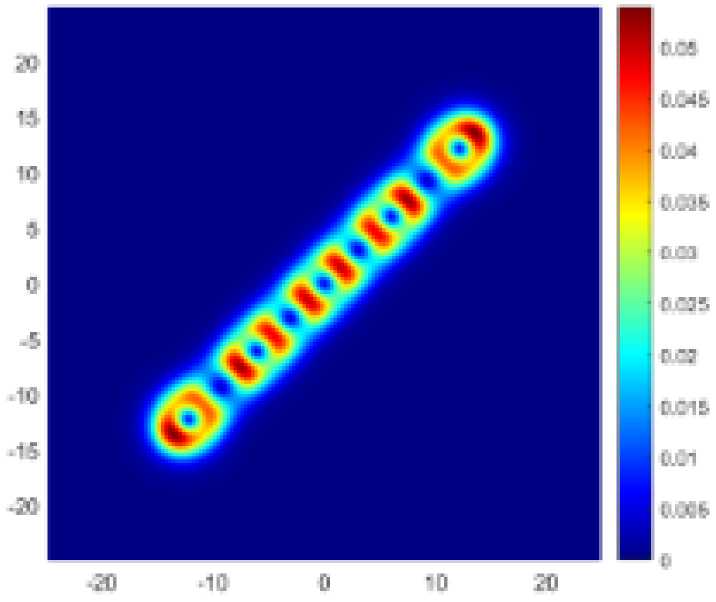
$$E_{int} = q^2 K_0(R)\cos(\theta_1 - \theta_2)$$

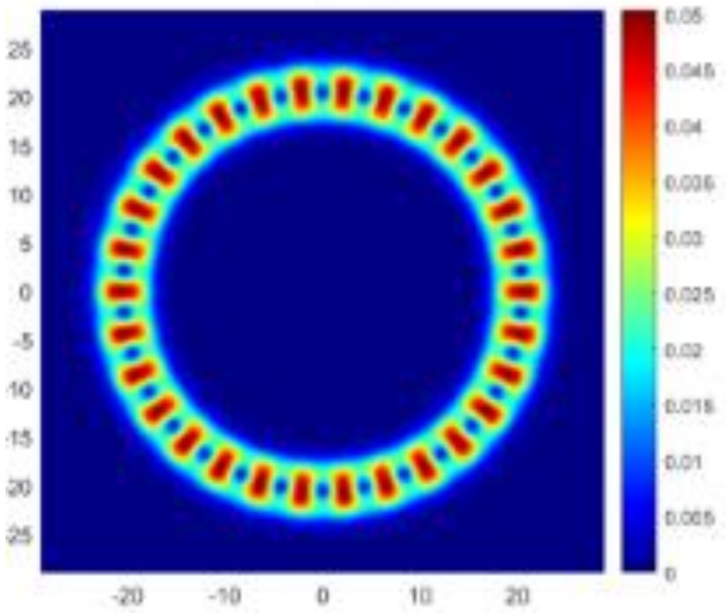
- Attractive channel:

Interskyrmion forces

- Skyrmions bind together to form “molecules”

n=10





n=30

- $n \to \infty$ Crystals?

Soliton crystals

- Usual approach: choose a plausible period lattice and topological charge per unit cell n.

$$\Lambda = \{n_1\mathbf{E}_1 + n_2\mathbf{E}_2 : (n_1, n_2) \in \mathbb{Z}^2\}$$

- Minimize E over all degree n fields with $\varphi(\mathbf{x}+\mathbf{v}) = \varphi(\mathbf{x})$ for all $\mathbf{v} \in \Lambda$

- Then minimize w.r.t. cell area

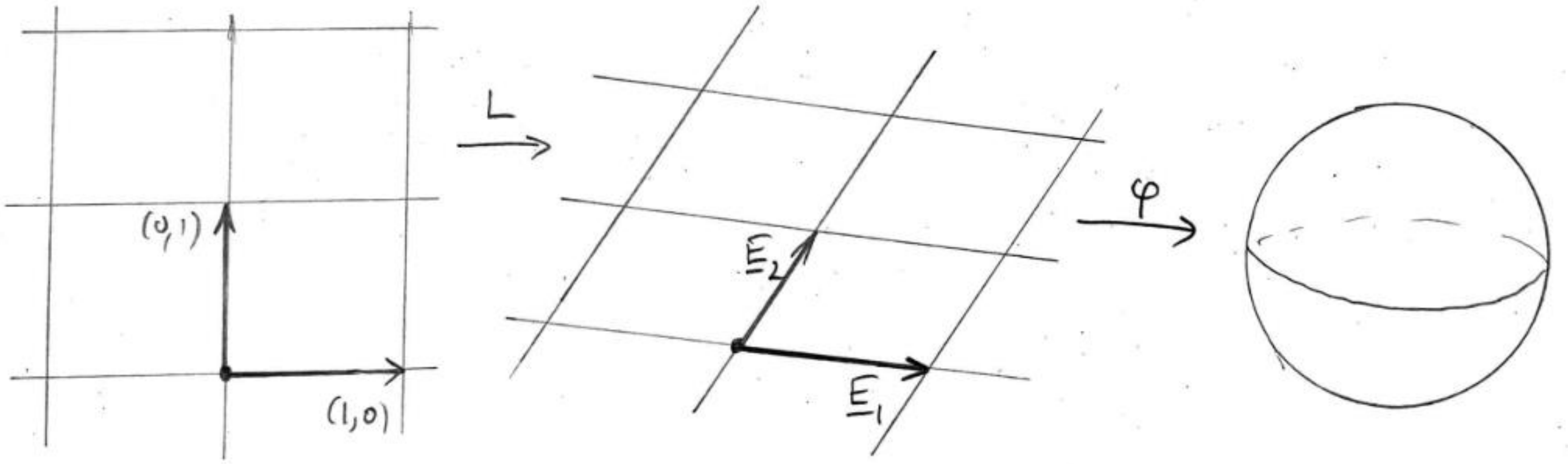
- Equivalent to putting model on (compact) torus $T^2_\Lambda = \mathbb{R}^2/\Lambda$

Soliton crystals

- Problem: once you put the model on a compact domain, every homotopy class of maps will have an energy minimizer
- Consider

- Is the $n = 2$ (say) minimizer on this torus a soliton crystal?
- Clearly not: artefact of the boundary conditions.
- We should minimize E w.r.t. field φ and period lattice Λ

Soliton crystals



- Define new coordinates $(x_1, x_2) =: X_1 E_1 + X_2 E_2$
- (T^2_Λ, g_{Euc}) is equivalent to $(\mathbb{T}^2, g)$ with metric $g = g_{ij} \mathrm{d}X_i \mathrm{d}X_j,\ g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$
- Varying Λ equivalent to fixing torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and varying metric g

Varying the lattice = varying the metric

- Given a smooth curve g_t of metrics with $\partial_t|_{t=0} g_t = \varepsilon$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi, g_t) = \langle S(\varphi, g), \varepsilon \rangle_{L^2}$$

  where $S(\varphi, g)$ is the **stress tensor**

- Condition for $\varphi : T^2_\Lambda \to S^2$ to be critical for variations of Λ:

$$\langle S(\varphi, g), \varepsilon \rangle_{L^2} = 0$$

  for all symmetric parallel bilinear forms ε

- $\varepsilon = g$: Virial constraint $E_0 = E_4$
- $\varepsilon \perp g$: φ must be "conformal on average"

More explicitly…

- Define area $A = \sqrt{\det g}$ and $s = \sqrt{\det g}\, g^{-1}$ (note that $\det s = 1$)
- For any fixed $\varphi : \mathbb{T}^2 \to S^2$,

$$E(\varphi, g) = \frac{1}{2}\operatorname{tr}(H(\varphi)s) + \frac{C_4(\varphi)}{A} + C_0(\varphi)A$$

where $$H_{ij}(\varphi) = \int_{\mathbb{T}^2} \frac{\partial\varphi}{\partial X_i}\cdot\frac{\partial\varphi}{\partial X_j}, \quad C_4(\varphi) = \frac{1}{2}\int_{\mathbb{T}_2} |\partial_1\varphi\times\partial_2\varphi|^2, \quad C_0(\varphi) = \int_{\mathbb{T}_2} V(\varphi)$$

- This has a unique global minimum at

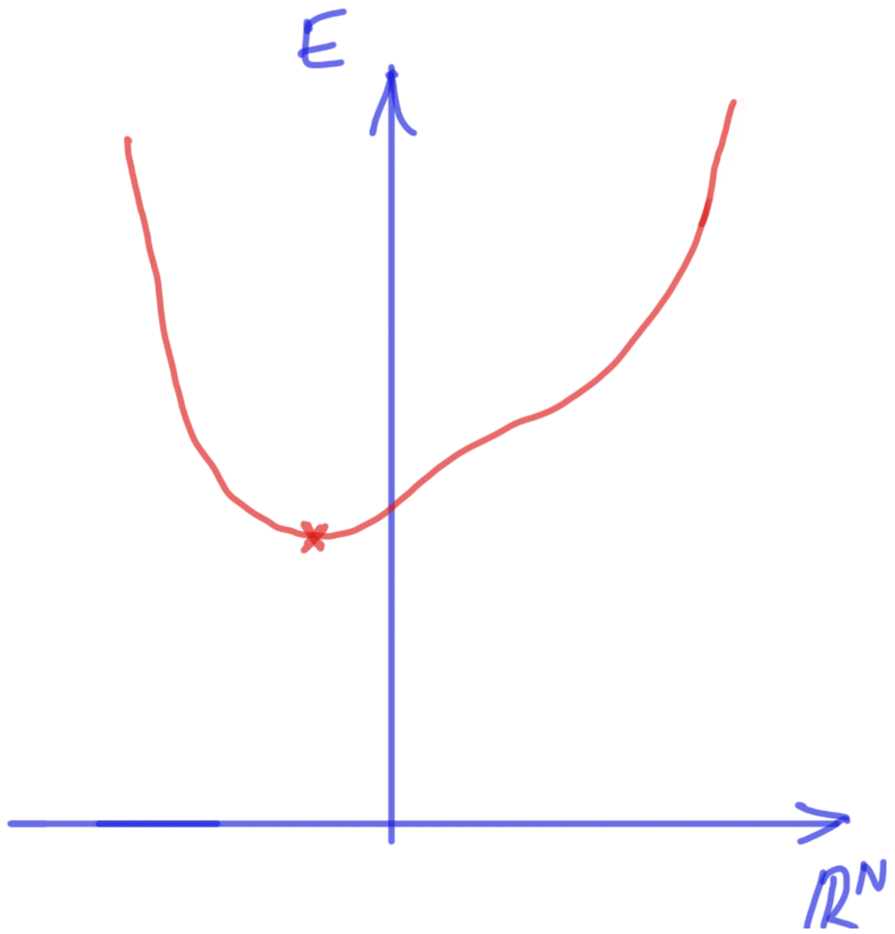
$$A = \sqrt{C_0/C_4}, \qquad s \parallel H^{-1}$$

Numerical method

1. Choose an initial guess (φ, g)
2. Minimize $E(\varphi, g)$ w.r.t. φ with g fixed by ``Arrested Newton Flow”
3. Compute $H(\varphi_{min}), C_4(\varphi_{min}), C_0(\varphi_{min})$
4. Construct $g = \lambda H$ with area $\sqrt{C_0/C_4}$
5. Go to 2

Arrested Newton Flow

- Discretize space, so $\varphi \in (S^2)^N$

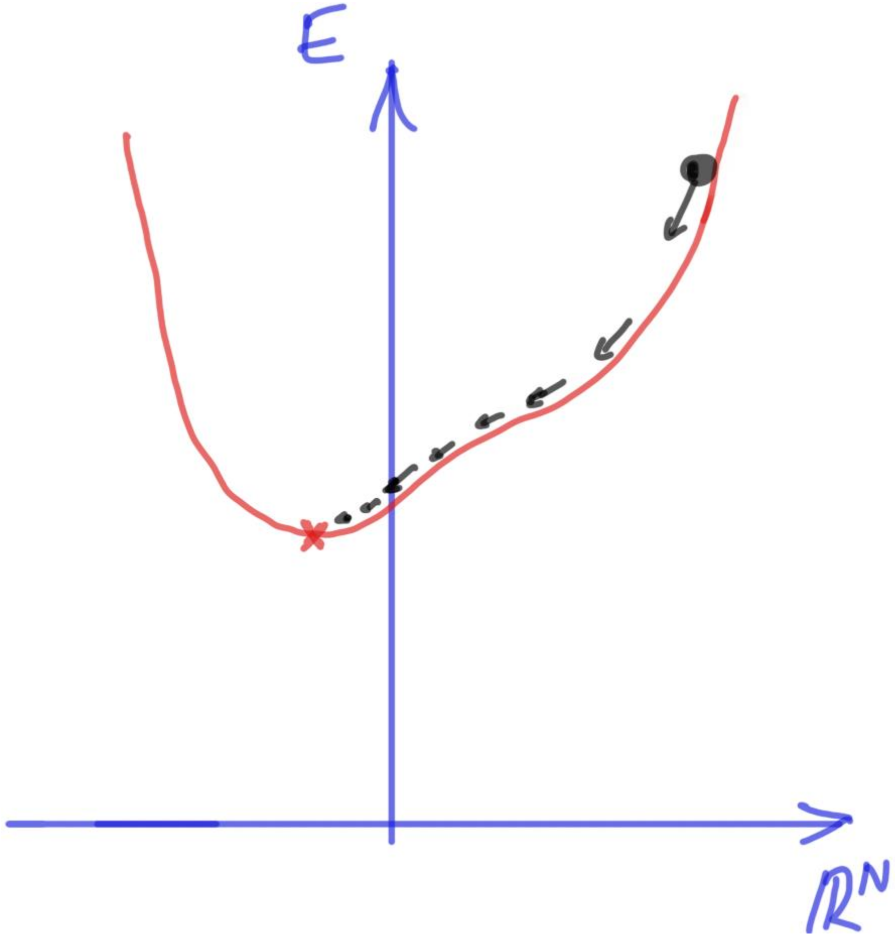


Gradient

~~Arrested Newton Flow~~

- Discretize space, so $\varphi \in (S^2)^N$

$$\varphi_t = -\nabla E(\varphi)$$



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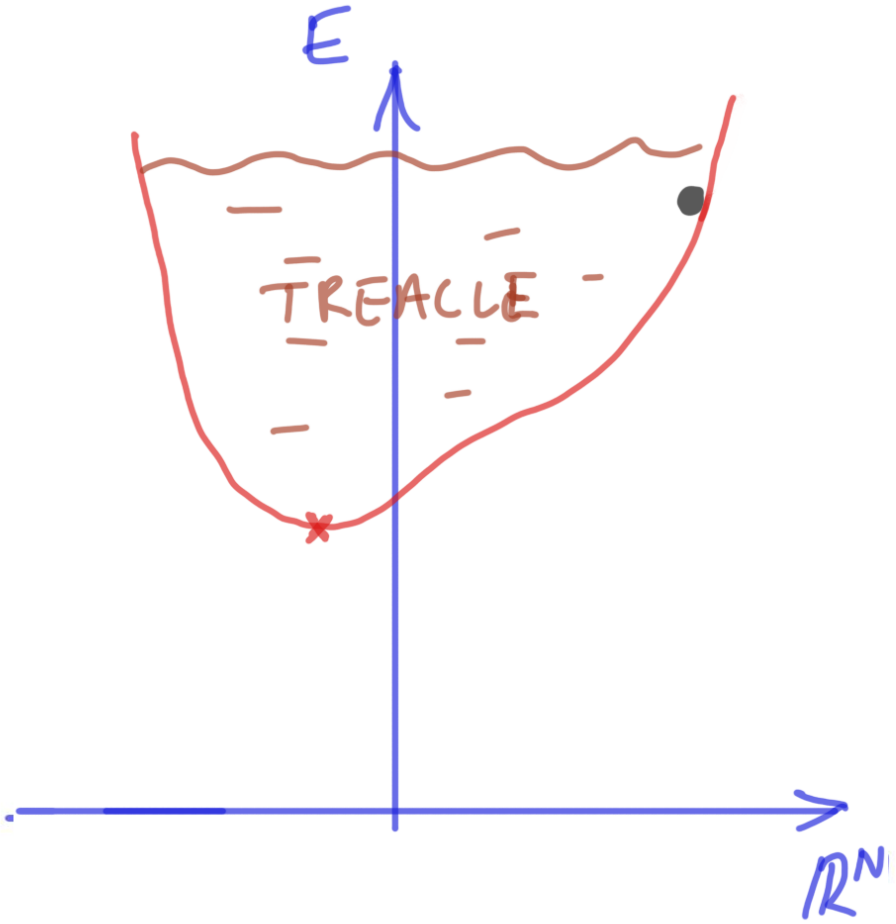
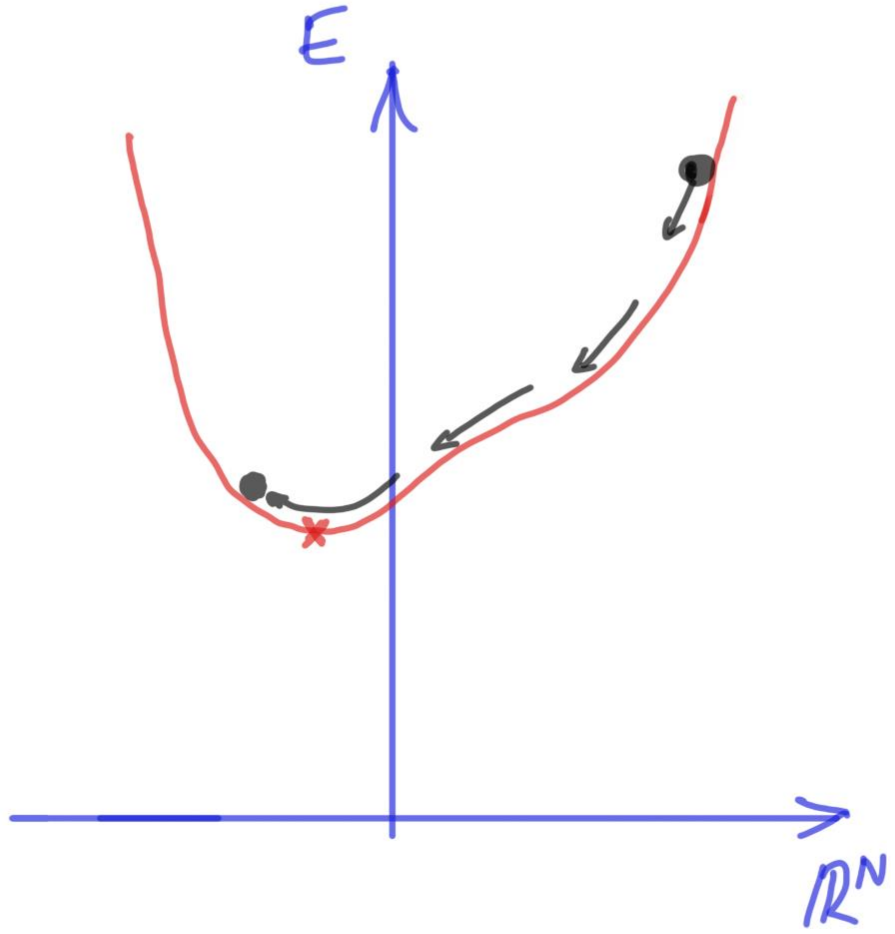


Arrested Newton Flow

- Discretize space, so $\varphi \in (S^2)^N$

$$\varphi_{tt} = -\nabla E(\varphi)$$

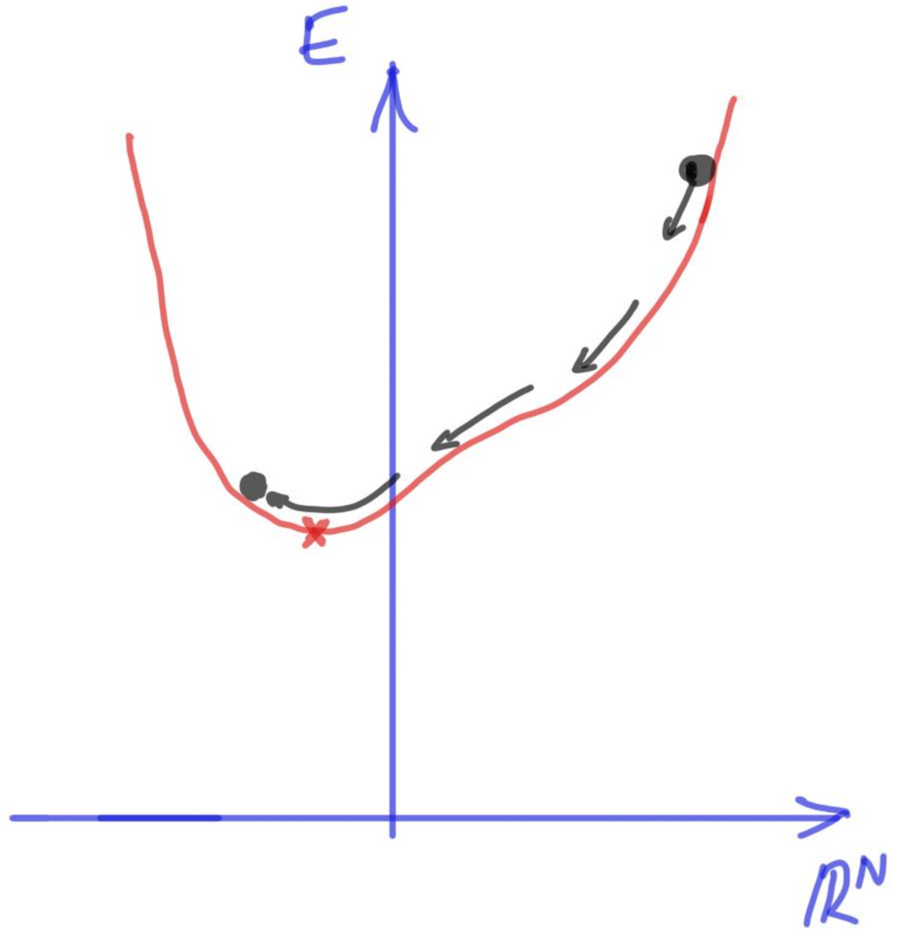


Arrested Newton Flow

- Discretize space, so $\varphi \in (S^2)^N$

$$\varphi_{tt} = -\nabla E(\varphi)$$

$$\varphi_t(t_*) = 0$$



Arrested Newton Flow

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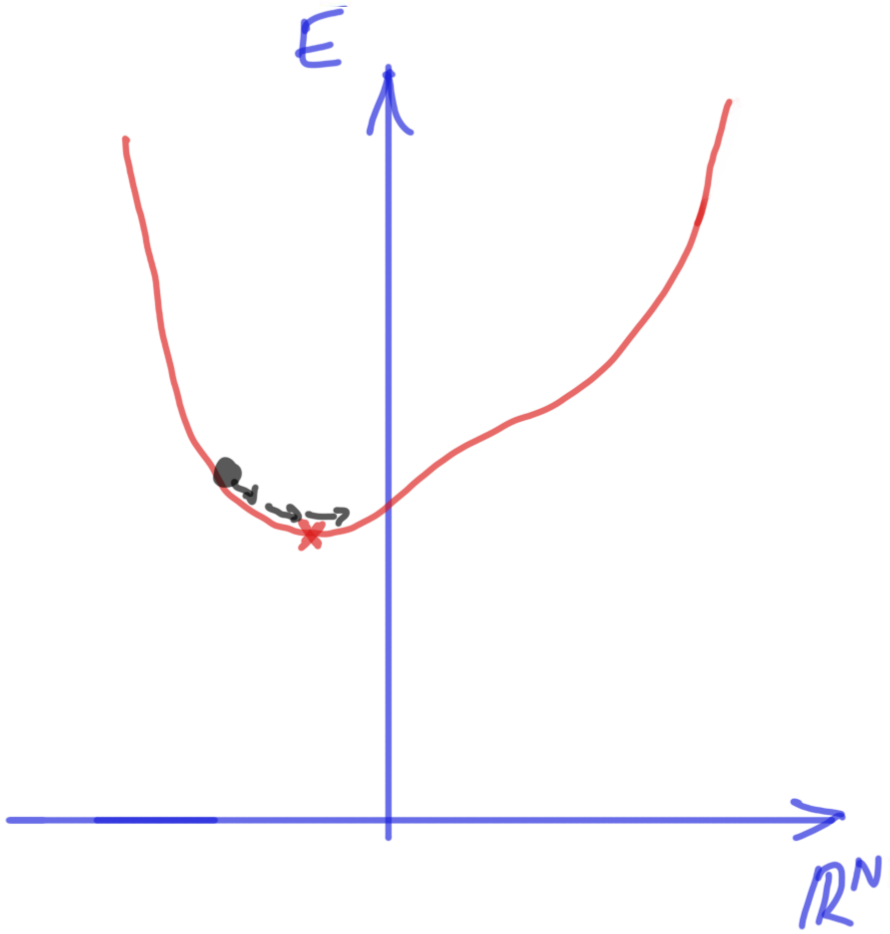
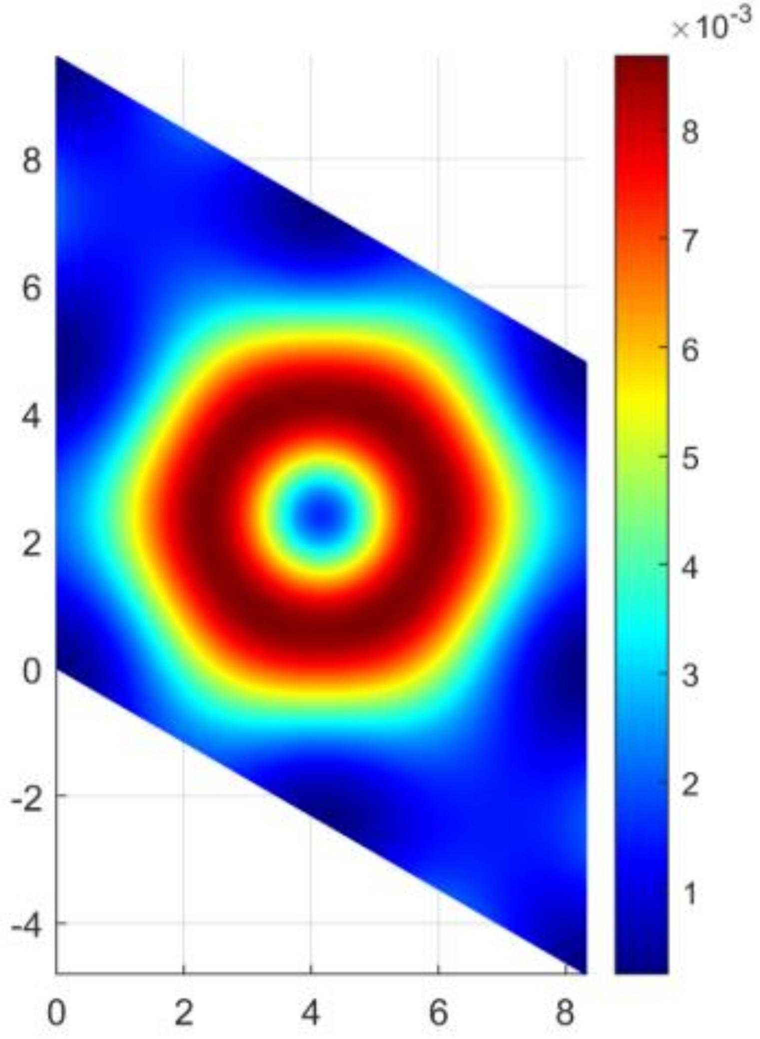


The results: $V(\varphi) = 1 - \varphi_3$

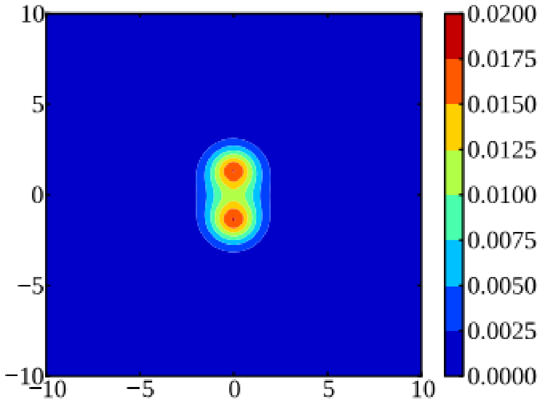


- Optimal crystal has $n = 2$ per unit cell
- Equianharmonic period lattice

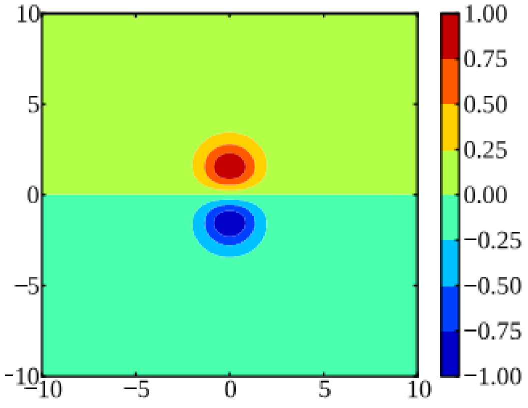
  $\Lambda = span\{L, Le^{\frac{i\pi}{3}}\}$
- Already known (Hen and Karliner 2008)
- But what if we change V?

$$V(\varphi) = \varphi_3^2$$

$n = 1$ isolated skyrmion



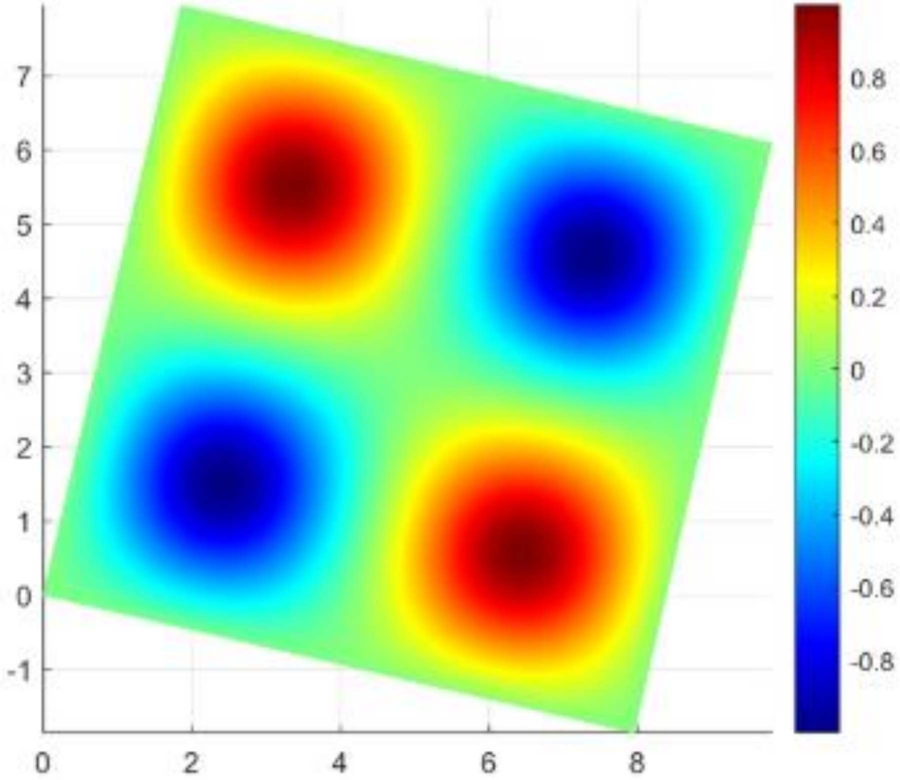
Energy density



φ_3

(J. Jaykka, J.M. Speight)

Optimal crystal



Square, $n = 2$ per unit cell

(P. Leask)

Two topological energy bounds:

$$
\begin{aligned}
E_2(\varphi) &= \frac{1}{2}\int_\Sigma |\varphi_x|^2 + |\varphi_y|^2 \\
&= \frac{1}{2}\int_\Sigma |\varphi_y - \varphi \times \varphi_x|^2 + \int_\Sigma \varphi \cdot (\varphi_x \times \varphi_y) \\
&= \frac{1}{2}\int_\Sigma |\varphi_y - \varphi \times \varphi_x|^2 + 2\pi n
\end{aligned}
$$

Hence $E_2(\varphi) \geq 2\pi n$ with equality iff

$$
\begin{aligned}
\varphi_y &= \varphi \times \varphi_x \\
d\varphi \circ J_\Sigma &= J_{S^2} \circ d\varphi
\end{aligned}
$$

that is, $\varphi : \Sigma \to S^2$ is **holomorphic**

1

(Lichnerowicz)

Two topological bounds

$$E_4(\varphi) + E_0(\varphi) = \frac{1}{2}\int_\Sigma |\varphi^*\mathrm{vol}_{S^2}|^2 + U(\varphi)^2$$
$$= \frac{1}{2}\int_\Sigma (*\varphi^*\mathrm{vol}_{S^2} - U(\varphi))^2 + \int_\Sigma \varphi^*(U\mathrm{vol}_{S^2})$$
$$= \frac{1}{2}\int_\Sigma (*\varphi^*\mathrm{vol}_{S^2} - U(\varphi))^2 + 4\pi\langle U\rangle n$$

Hence, $(E_4 + E_0)(\varphi) \geq 4\pi\langle U\rangle n$ with equality iff

$$\varphi^*\mathrm{vol}_{S^2} = U(\varphi)\mathrm{vol}_\Sigma$$

$$\varphi^*(\mathrm{vol}_{S^2}/U) = \mathrm{vol}_\Sigma$$

that is, $\varphi : (\Sigma, \mathrm{vol}_\Sigma) \to (S^2, \mathrm{vol}_{S^2}/U)$ is **volume preserving**

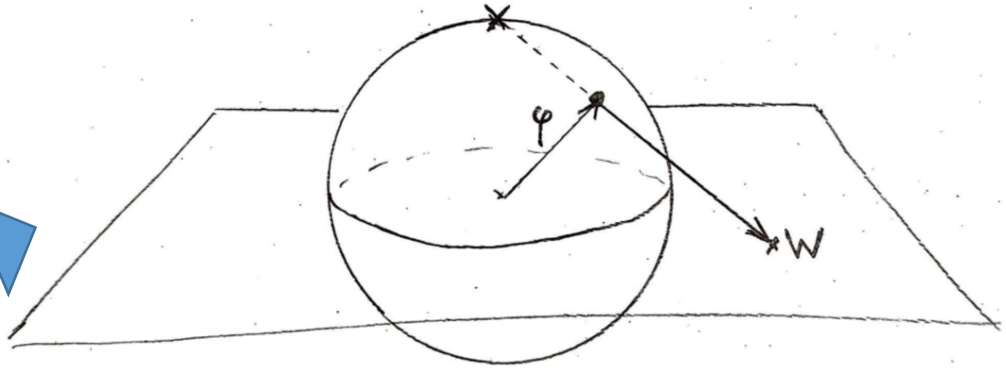
2

(C. Adam, T. Romanczukiewicz, J. Sanchez-Guillen and A. Wereszczynski, J.M. Izquierdo, M.S. Rashi d, B. Piette and W.J. Zakrzewski, D.H. Tchrakian, M. de Innocentis and R.S. Ward…

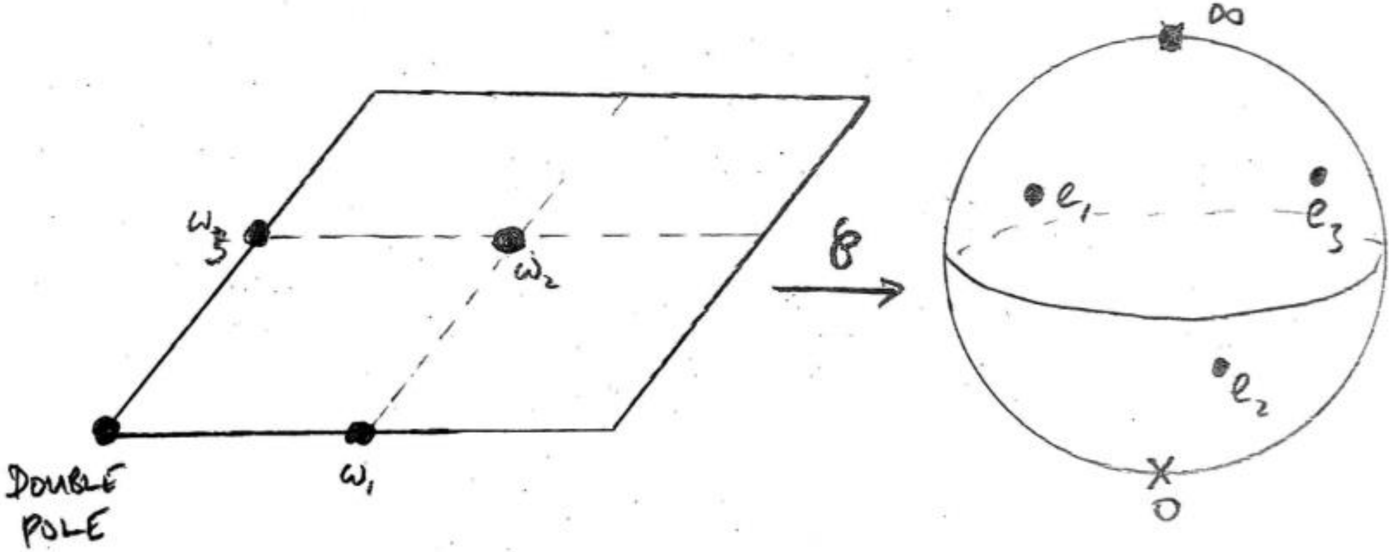
…..and me)

Cooking up a potential

Stereographic coordinate on S^2



Weierstrass P-function:



Degree $n = 2$ holomorphic map

$$\varphi_\wp : \Sigma \to S^2$$

$$z \mapsto W(z) = \wp(z)$$

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

Cooking up a potential

- For any holomorphic map $\varphi : z \mapsto W(z)$,

$$\varphi^* \mathrm{vol}_{S^2} = \frac{4|W'(z)|^2}{(1+|W(z)|^2)^2} \frac{i}{2} dz \wedge d\bar{z}$$

- Define

$$U(W) = \frac{16|(W-e_1)(W-e_2)(W-e_3)|}{(1+|W|^2)^2}$$

  $V = U^2/2$ is a smooth potential on S^2 with exactly 4 vacua

- $\varphi_\wp$ is **holomorphic** and satisfies $\varphi_\wp^* \mathrm{vol}_{S^2} = (U \circ \varphi_\wp) \mathrm{vol}_\Sigma$

- So $E(\varphi_\wp, g_\Lambda) = 2\pi n(1 + 2\langle U \rangle)$ the least possible energy among all degree 2 maps $\mathbb{T}^2 \to S^2$ **and all metrics** on $\mathbb{T}^2$!

- $\varphi_\wp$ is certainly a soliton crystal for the model with potential $V = U^2/2$

Cooking up a potential

- This holds **on any torus** $\mathbb{C}/\Lambda$!

- There is a choice of potential $V(\varphi)$ for which the baby Skyrme model has a skyrmion crystal with the above period lattice

The adult Skyrme model

- $\varphi : \mathbb{R}^3 \to S^3 = SU(2)$

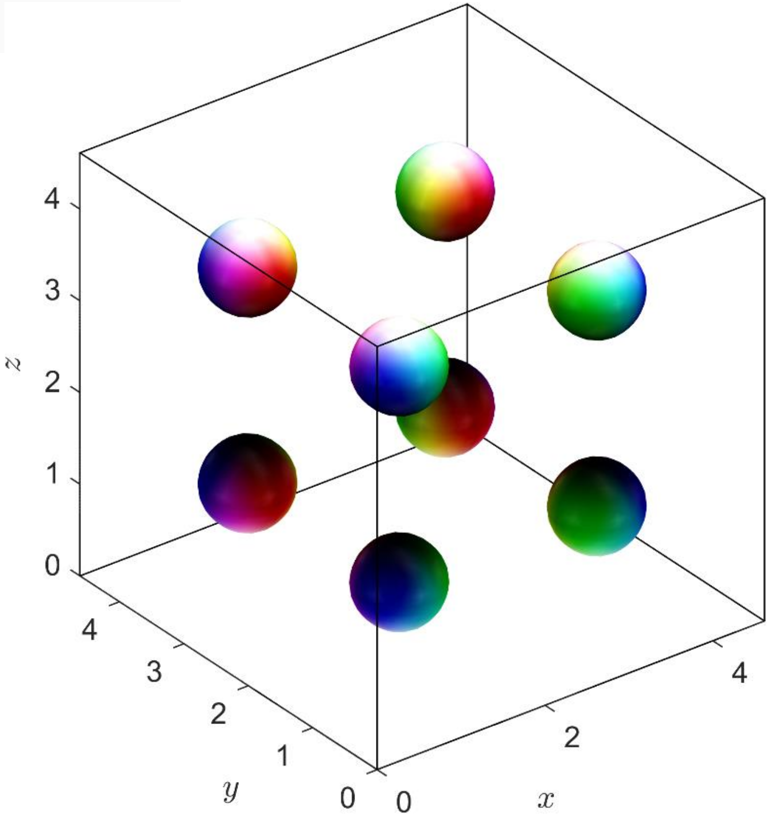
$$E = \int_{\mathbb{R}^3} \frac{1}{2}|d\varphi|^2 + \frac{1}{2}|\varphi^*\Omega|^2 + m_\pi^2(1 - \varphi_0)$$

$$= \mathrm{Tr}(H\Sigma^{-1}) + \mathrm{Tr}(F\Sigma) + \frac{C_0}{\det \Sigma}$$

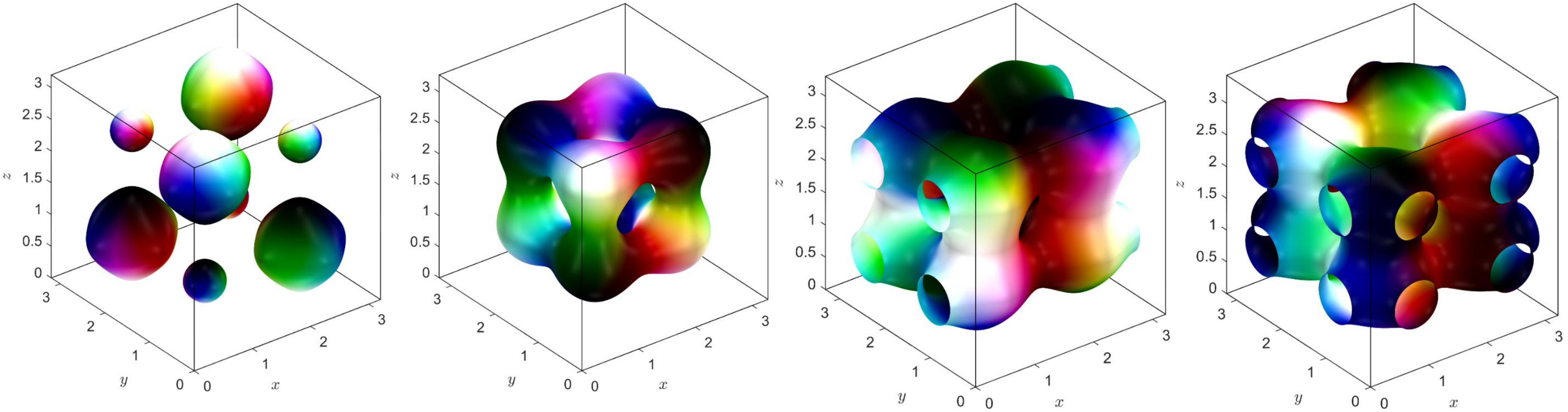
where $\Sigma = \frac{g}{\sqrt{\det g}}$.

- $m_\pi = 0$: Kugler-Shtrikman crystal of half-skyrmions ($B = 4$)

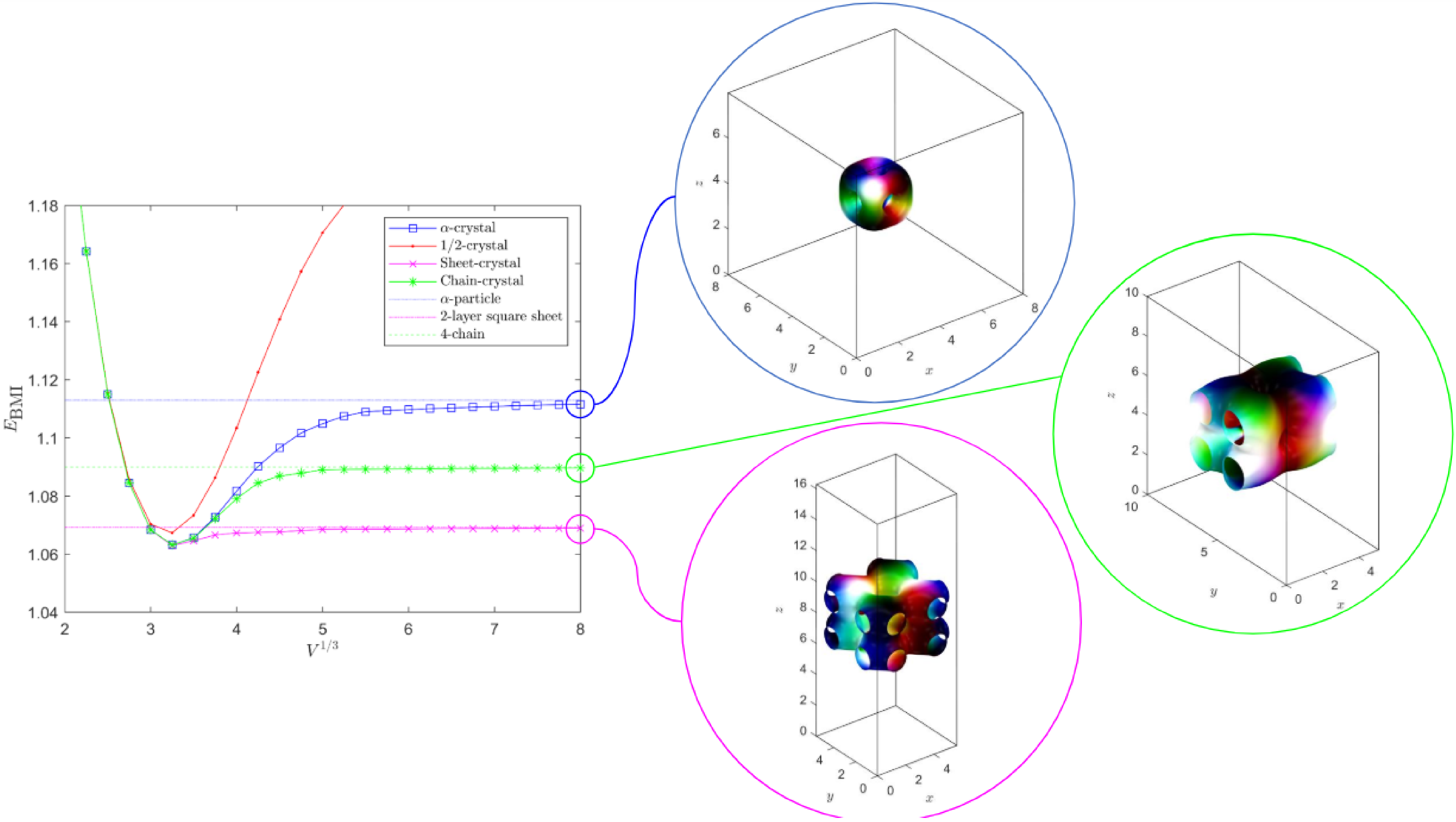
- What if $m_\pi > 0$?



Bifurcation!



½ crystal > α crystal > chain > sheet



α-crystal
1/2-crystal
Sheet-crystal
Chain-crystal
α-particle
2-layer square sheet
4-chain
E_{BMI}
$V^{1/3}$