Wave-map flow and the geometry of the space of holomorphic maps

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Harmonic maps

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- Conformally invariant in dimension 2.

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- $\dim\Sigma = 2$ most interesting

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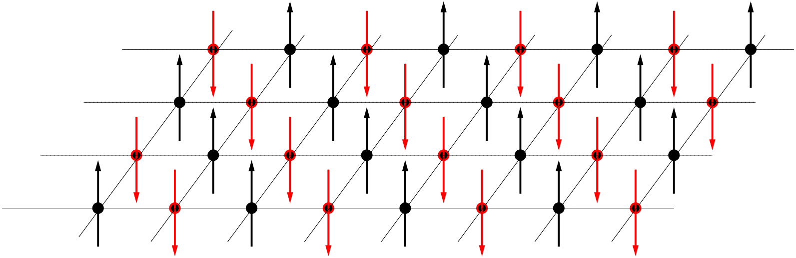

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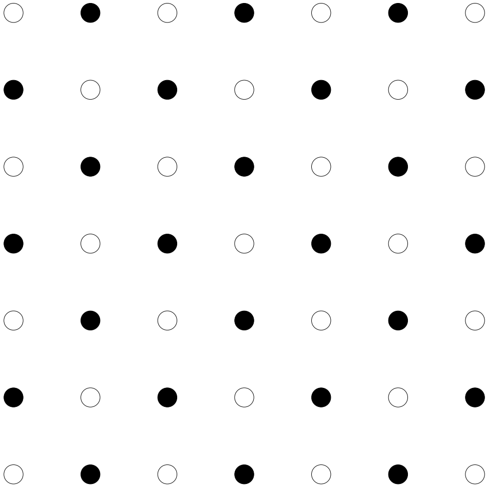

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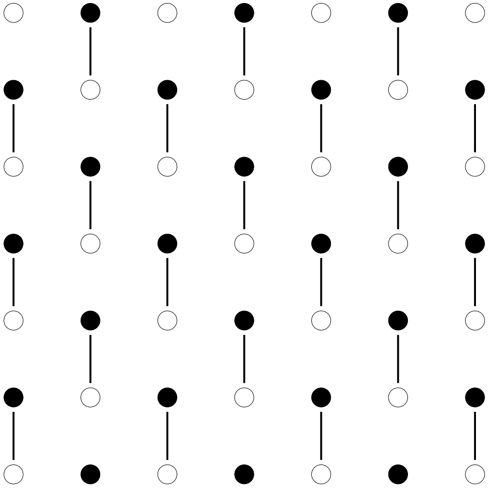

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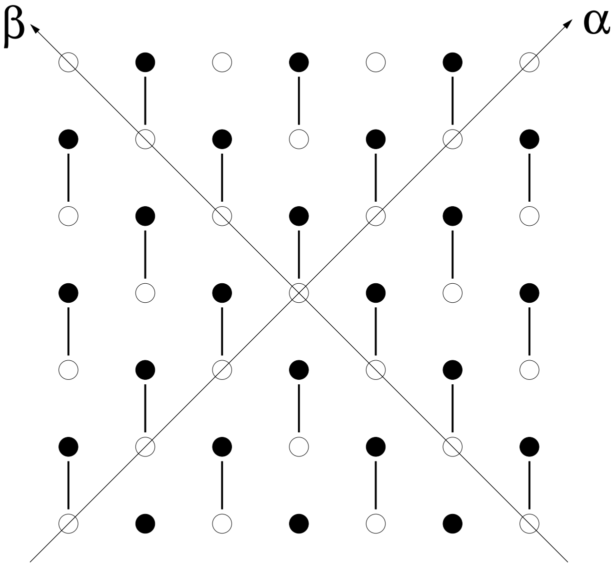

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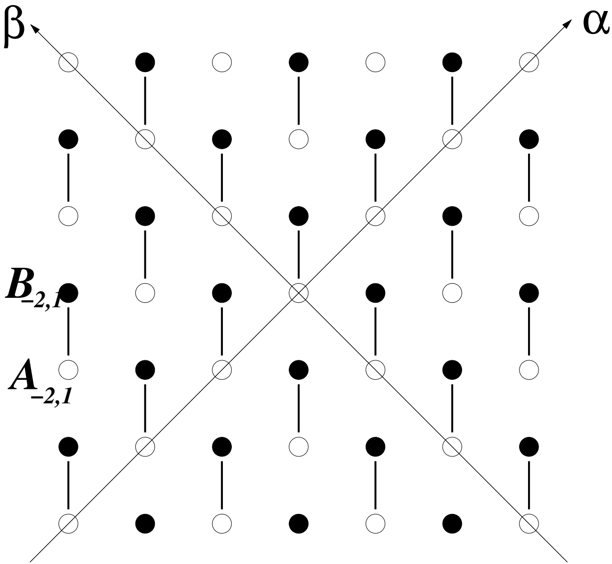

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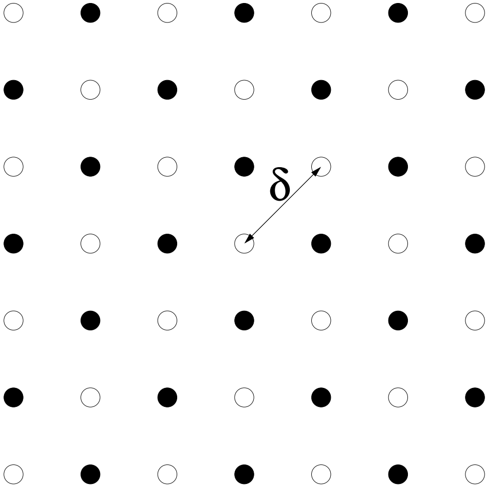

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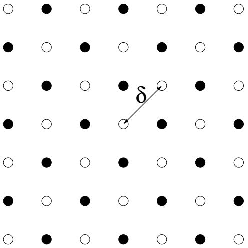

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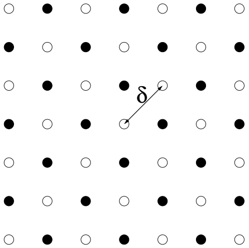

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$$\left.\begin{array}{c} \mathbf{A}_{\alpha,\beta} \\ \mathbf{B}_{\alpha,\beta} \\ J_{ij} \end{array}\right\} \xrightarrow{\delta\to 0} \left\{\begin{array}{l} \mathrm{A}(x,y) \\ \mathrm{B}(x,y) \\ \mathrm{J}(x,y) \end{array}\right.$$

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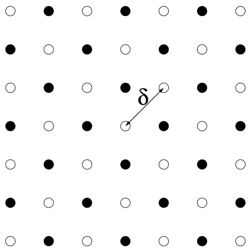

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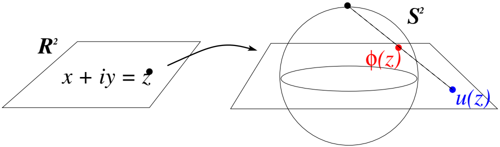
- $\varphi : \Sigma \to S^2$: $E \geq 4\pi n$, equality iff holomorphic

Lumps



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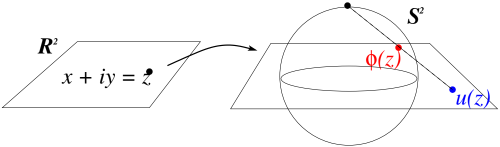

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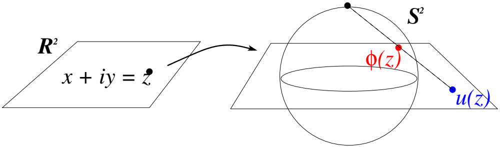

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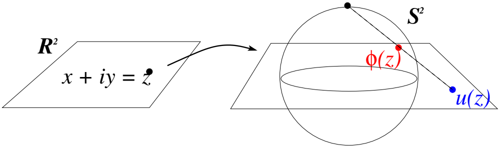

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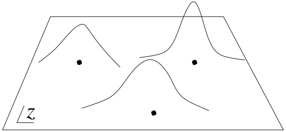

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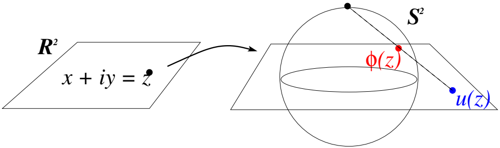

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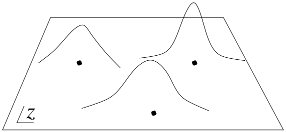
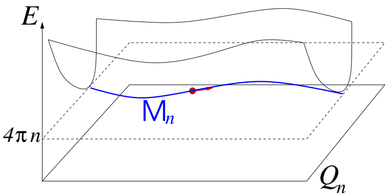


- Moduli space $M_n \subset \mathbb{C}^{2n}$

Geodesic approximation (Ward, after Manton)



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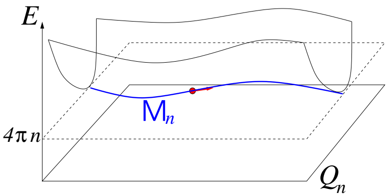

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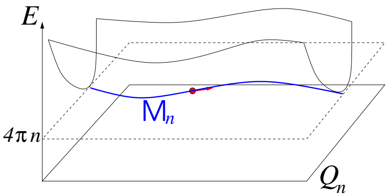

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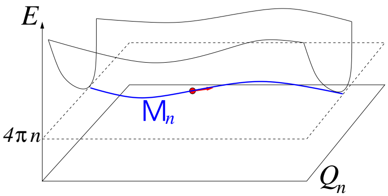

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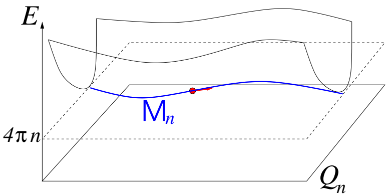

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Geodesic motion on (M_n, γ) where $\gamma = L^2$ metric.

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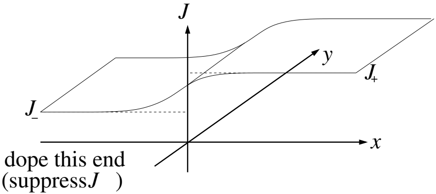

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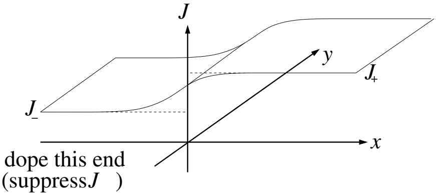

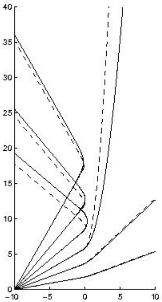

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  - Impose periodicity, $J =$constant, $\Sigma = T^2$

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Theorem (Sadun,JMS)

Let Σ be a compact Riemann surface with metric g_Σ, M_n be the (smooth locus) of the space of degree n holomorphic maps $\Sigma \longrightarrow S^2$, and γ be the L^2 metric on M_n. Then (M_n, γ) is geodesically incomplete.

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- As $k \to \infty$, energy localizes around zeros of u_0, i.e. lumps shrink and collapse

$\mathrm{M}_1(S^2)$ (JMS, Baptista)

- $\mathrm{M}_1 = \mathrm{Rat}_1 = PL(2, \mathbb{C}) = SO(3) \times \mathbb{R}^3 = TSO(3) = \ldots \subset \mathbb{C}P^3$

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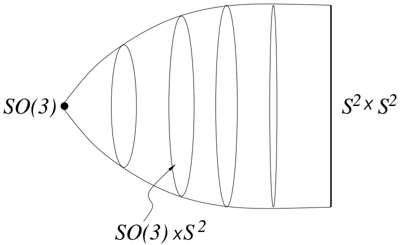

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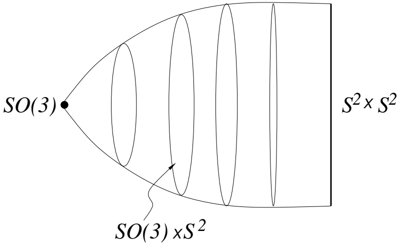

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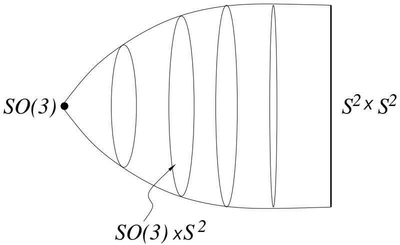

- Kähler, finite diameter and volume, Ricci positive, unbded scalar curvature
- Geodesic flow complicated. Generically lumps do **not** travel on great circles

Geometry of $M_n(\Sigma)$

- $M_2(T^2) = [T^2 \times \mathrm{Rat}_1]/[\mathbb{Z}_2 \times \mathbb{Z}_2]$, finite volume and diameter (JMS), numerics (Cova)

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2nd strand: Prove conjecture (geodesic flow in M_n approximates wave map flow)

Precise conjecture

Consider one-parameter family of Cauchy problems for wave map flow $\mathbb{R} \times \Sigma \to S^2$:

$$\varphi(0) = \varphi_0, \qquad \varphi_t(0) = \varepsilon\varphi_1$$

where $\varphi_0 \in \mathsf{M}_n$, $\varphi_1 \in T_{\varphi_0}\mathsf{M}_n$ and $\varepsilon > 0$.

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Furthermore, the time re-scaled solution

$$\varphi_\varepsilon : [0, T] \times \Sigma \to S^2, \qquad \varphi_\varepsilon(\tau, x) = \varphi(\tau/\varepsilon, x)$$

converges uniformly to $\psi : [0, T] \times \Sigma \to S^2$, the geodesic in M_n with the same initial data.

Stuart's method (joint work with Mark Haskins)

We'll sketch the proof in the case $\Sigma = T^2$. Ingredients:

1. Wave map eqn for $\varphi \leftrightarrow$ coupled ODE/PDE system for $\varphi = \psi + \varepsilon^2 Y$

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5. A priori bound

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- Choose and fix real local coords $q : \mathbb{R}^{4n} \supset U \to \mathsf{M}_n$
  Denote by $\psi(q)$ the h-map corresponding to q.
  Convenient to demand that $\varphi_0 = \psi(0)$ and $U = \mathbb{R}^{4n}$.

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$$\langle Y, Z \rangle_{L^2} = 0, \qquad \forall Z \in T_{\psi(q)} \mathsf{M}_n.$$

PDE for Y

$$Y_{tt} + L_\psi Y = k + \varepsilon j$$

where

$$\begin{aligned} L_\psi Y &= -\Delta Y - (|\psi_x|^2 + |\psi_y|^2) Y - 2(\psi_x \cdot Y_x + \psi_y \cdot Y_y)\psi \\ &\quad +2\{(\psi \cdot Y)\Delta\psi + (\psi \cdot Y)_x \psi_x + (\psi \cdot Y)_y \psi_y\} \end{aligned}$$

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k &= -\psi_{\tau\tau} - |\psi_\tau|^2 \psi \\
j &= j(\psi, \psi_\tau, Y, Y_t, Y_x, Y_y)
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Evolution of $q(\tau)$

- Recall L^2 orthogonality constraint

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Geodesic flow (with $O(\varepsilon)$ correction).

Summary: the ODE/PDE system

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Y_{tt} + LY &= k + \varepsilon j \\
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Short time existence theorem
There exist $\varepsilon_*, T > 0$, depending only on Γ, such that, for all $\varepsilon \in (0, \varepsilon_*]$ and any initial data

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Proof: Picard's method.

Coercivity, Energy Estimates

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- $\|Y\|_3$ remains bounded for $t \in [0, T/\varepsilon]$, and

$$\|Y\|_{C^0} \leq c\|Y\|_2 \leq c\|Y\|_3,$$

so $\varphi_\varepsilon(\tau)$ converges uniformly on $[0, T]$ to $\psi(q_0(\tau))$

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- T depends on how close φ_0 is to $\partial\mathsf{M}_n$.
- Geodesic approx. certainly **fails** very close to blow-up (Numerics: Linhart-Sadun, Bizoń-Chmaj-Tabor, Analysis: Rodnianski and Sterbenz)