Magnetic skyrmion lattices in tilted fields

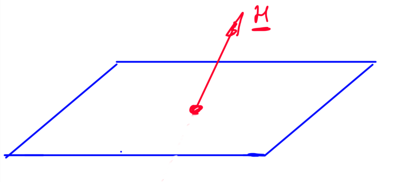
Martin Speight
https://cp1lump.github.io
Joint work with Tom Winyard, Bernd Schroers (Edinburgh)

Funded by EPSRC

22/9/25

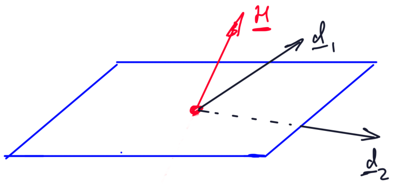
University of Leeds

The standard model



$$
\begin{aligned}
E &= \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2 x \\
&= E_{exch} + E_{DMI} + E_{pot}
\end{aligned}
$$

The standard model



$$
\begin{aligned}
E &= \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2 x \\
&= E_{exch} + E_{DMI} + E_{pot}
\end{aligned}
$$

Reduction of DMI

$$E_{DMI} = \int_{\mathbb{R}^2} \sum_{i=1,2} \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) d^2x.$$

Reduction of DMI

$$E_{DMI} = \int_{\mathbb{R}^2} \sum_{i=1,2} \boldsymbol{d}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) d^2x.$$

- Rotate coordinate system:

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Reduction of DMI

$$E_{DMI} = \int_{\mathbb{R}^2} \sum_{i=1,2} \widetilde{\boldsymbol{d}}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial \widetilde{x}_i} \right) d^2 \widetilde{x}.$$

- Rotate coordinate system:

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $\widetilde{\boldsymbol{d}}_i = R_{ij} \boldsymbol{d}_j$

Reduction of DMI

$$E_{DMI} = \int_{\mathbb{R}^2} \sum_{i=1,2} \widetilde{\boldsymbol{d}}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial \widetilde{x}_i} \right) d^2\widetilde{x}.$$

- Rotate coordinate system:

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $\widetilde{\boldsymbol{d}}_i = R_{ij}\boldsymbol{d}_j$

$$\begin{aligned} 2\widetilde{\boldsymbol{d}}_1 \cdot \widetilde{\boldsymbol{d}}_2 &= (\cos 2\alpha, \sin 2\alpha) \cdot (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2, |\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2), \\ |\widetilde{\boldsymbol{d}}_1|^2 - |\widetilde{\boldsymbol{d}}_2|^2 &= (\cos 2\alpha, \sin 2\alpha) \cdot (|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2, 2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2). \end{aligned}$$

Reduction of DMI

$$E_{DMI} = \int_{\mathbb{R}^2} \sum_{i=1,2} \widetilde{\boldsymbol{d}}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial \widetilde{x}_i} \right) d^2 \widetilde{x}.$$

- Rotate coordinate system:

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $\widetilde{\boldsymbol{d}}_i = R_{ij} \boldsymbol{d}_j$

$$\begin{aligned} 2\widetilde{\boldsymbol{d}}_1 \cdot \widetilde{\boldsymbol{d}}_2 &= (\cos 2\alpha, \sin 2\alpha) \cdot (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2, |\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2), \\ |\widetilde{\boldsymbol{d}}_1|^2 - |\widetilde{\boldsymbol{d}}_2|^2 &= (\cos 2\alpha, \sin 2\alpha) \cdot (|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2, 2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2). \end{aligned}$$

- Choose α s.t. $\widetilde{\boldsymbol{d}}_1 \cdot \widetilde{\boldsymbol{d}}_2 = 0$, $|\widetilde{\boldsymbol{d}}_1|^2 - |\widetilde{\boldsymbol{d}}_2|^2 \geq 0$

Reduction of DMI

$$E_{DMI} = \int_{\mathbb{R}^2} \sum_{i=1,2} \widetilde{\boldsymbol{d}}_i \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial \widetilde{x}_i} \right) d^2 \widetilde{x}.$$

- Rotate coordinate system:

$$\begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $\widetilde{\boldsymbol{d}}_i = R_{ij} \boldsymbol{d}_j$

$$\begin{aligned} 2\widetilde{\boldsymbol{d}}_1 \cdot \widetilde{\boldsymbol{d}}_2 &= (\cos 2\alpha, \sin 2\alpha) \cdot (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2, |\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2), \\ |\widetilde{\boldsymbol{d}}_1|^2 - |\widetilde{\boldsymbol{d}}_2|^2 &= (\cos 2\alpha, \sin 2\alpha) \cdot (|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2, 2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2). \end{aligned}$$

- Choose α s.t. $\widetilde{\boldsymbol{d}}_1 \cdot \widetilde{\boldsymbol{d}}_2 = 0$, $|\widetilde{\boldsymbol{d}}_1|^2 - |\widetilde{\boldsymbol{d}}_2|^2 \geq 0$
- Can choose x length scale s.t. $|\widetilde{\boldsymbol{d}}_1| = 1$

Reduction of DMI

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \le 1$$

Reduction of DMI

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \leq 1$$

- Exists $S \in SO(3)$ (unique if $b \neq 0$) s.t.

$$\boldsymbol{d}_1' = S\boldsymbol{d}_1 = (1, 0, 0), \qquad \boldsymbol{d}_2' = S\boldsymbol{d}_2 = (0, b, 0)$$

Reduction of DMI

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \leq 1$$

- Exists $S \in SO(3)$ (unique if $b \neq 0$) s.t.

$$\boldsymbol{d}_1' = S\boldsymbol{d}_1 = (1, 0, 0), \qquad \boldsymbol{d}_2' = S\boldsymbol{d}_2 = (0, b, 0)$$

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i S^T \boldsymbol{d}_i' \cdot \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2 x$$

Reduction of DMI

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \leq 1$$

- Exists $S \in SO(3)$ (unique if $b \neq 0$) s.t.

$$\boldsymbol{d}_1' = S\boldsymbol{d}_1 = (1,0,0), \qquad \boldsymbol{d}_2' = S\boldsymbol{d}_2 = (0,b,0)$$

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i' \cdot S \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

Reduction of DMI

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \leq 1$$

- Exists $S \in SO(3)$ (unique if $b \neq 0$) s.t.

$$\boldsymbol{d}_1' = S\boldsymbol{d}_1 = (1,0,0), \qquad \boldsymbol{d}_2' = S\boldsymbol{d}_2 = (0,b,0)$$

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i' \cdot \left(S\boldsymbol{m} \times \frac{\partial S\boldsymbol{m}}{\partial x_i} \right) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

Reduction of DMI

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \leq 1$$

- Exists $S \in SO(3)$ (unique if $b \neq 0$) s.t.

$$\boldsymbol{d}_1' = S\boldsymbol{d}_1 = (1,0,0), \qquad \boldsymbol{d}_2' = S\boldsymbol{d}_2 = (0,b,0)$$

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial S\boldsymbol{m}}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i' \cdot \left(S\boldsymbol{m} \times \frac{\partial S\boldsymbol{m}}{\partial x_i} \right) + |S\boldsymbol{H}| - S\boldsymbol{m} \cdot S\boldsymbol{H} \right\} d^2x$$

Reduction of DMI

$$\boldsymbol{d}_1 \perp \boldsymbol{d}_2, \qquad |\boldsymbol{d}_1| = 1, \qquad |\boldsymbol{d}_2| = b \leq 1$$

- Exists $S \in SO(3)$ (unique if $b \neq 0$) s.t.

$$\boldsymbol{d}_1' = S\boldsymbol{d}_1 = (1,0,0), \qquad \boldsymbol{d}_2' = S\boldsymbol{d}_2 = (0,b,0)$$

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \sum_i \left| \frac{\partial \boldsymbol{m}'}{\partial x_i} \right|^2 + \sum_i \boldsymbol{d}_i' \cdot \left(\boldsymbol{m}' \times \frac{\partial \boldsymbol{m}'}{\partial x_i} \right) + |\boldsymbol{H}'| - \boldsymbol{m}' \cdot \boldsymbol{H}' \right\} d^2x$$

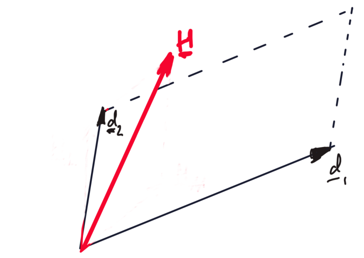
where $\boldsymbol{m}' = S\boldsymbol{m}$, $\boldsymbol{H}' = S\boldsymbol{H}$

Reduction of DMI

- So, WLOG $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{d}_2 = (0, b, 0)$

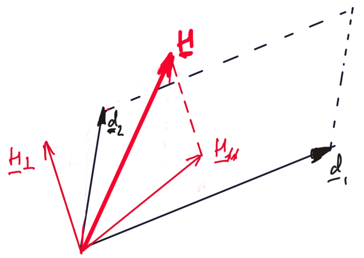
Reduction of DMI

- So, WLOG $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{d}_2 = (0, b, 0)$
- WARNING: $\boldsymbol{m}, \boldsymbol{H}$ oriented relative to plane spanned by $\boldsymbol{d}_1, \boldsymbol{d}_2$, **not** the sample plane



Reduction of DMI

- So, WLOG $\boldsymbol{d}_1 = (1, 0, 0)$, $\boldsymbol{d}_2 = (0, b, 0)$
- WARNING: $\boldsymbol{m}, \boldsymbol{H}$ oriented relative to plane spanned by $\boldsymbol{d}_1, \boldsymbol{d}_2$, **not** the sample plane



Skyrmion lattices

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m}) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

- FM state $\boldsymbol{m}(\boldsymbol{x}) = \hat{\boldsymbol{H}}$, always a solution, $E = 0$.

Skyrmion lattices

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m}) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

- FM state $\boldsymbol{m}(\boldsymbol{x}) = \hat{\boldsymbol{H}}$, always a solution, $E = 0$.
- If $\boldsymbol{m}(\boldsymbol{x}) \to \hat{\boldsymbol{H}}$ as $|\boldsymbol{x}| \to \infty$, have degree

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} \boldsymbol{m} \cdot (\partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m}) d^2x \in \mathbb{Z}$$

Skyrmion lattices

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \partial_i \boldsymbol{m}) + |\boldsymbol{H}| - \boldsymbol{m} \cdot \boldsymbol{H} \right\} d^2x$$

- FM state $\boldsymbol{m}(\boldsymbol{x}) = \hat{\boldsymbol{H}}$, always a solution, $E = 0$.
- If $\boldsymbol{m}(\boldsymbol{x}) \to \hat{\boldsymbol{H}}$ as $|\boldsymbol{x}| \to \infty$, have degree

$$Q = \frac{1}{4\pi} \int_{\mathbb{R}^2} \boldsymbol{m} \cdot (\partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m}) d^2x \in \mathbb{Z}$$

- Can seek minimizers of E of fixed Q. Skyrmions $Q = -1$ (or $Q = +1$)

Skyrmion lattices

- Derrick scaling: $\boldsymbol{m}_\lambda(\boldsymbol{x}) = \boldsymbol{m}(\lambda \boldsymbol{x})$

$$E(\lambda) = E(\boldsymbol{m}_\lambda) = E_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda^2} E_{pot}(\boldsymbol{m}).$$

Skyrmion lattices

- Derrick scaling: $\boldsymbol{m}_\lambda(\boldsymbol{x}) = \boldsymbol{m}(\lambda \boldsymbol{x})$

$$E(\lambda) = E(\boldsymbol{m}_\lambda) = E_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda^2} E_{pot}(\boldsymbol{m}).$$

$\boldsymbol{m}$ critical $\Rightarrow E'(1) = 0 \Rightarrow E_{DMI} = -2E_{pot} < 0$

Skyrmion lattices

- Derrick scaling: $\boldsymbol{m}_\lambda(\boldsymbol{x}) = \boldsymbol{m}(\lambda \boldsymbol{x})$

$$E(\lambda) = E(\boldsymbol{m}_\lambda) = E_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda^2} E_{pot}(\boldsymbol{m}).$$

  $\boldsymbol{m}$ critical $\Rightarrow E'(1) = 0 \Rightarrow E_{DMI} = -2E_{pot} < 0$

- Possible that $E(\boldsymbol{m}_{skyrmion}) < 0$: skyrmion **lattice** can then form spontaneously

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E_{exch}^{(i)} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E^{(i)}_{exch} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

$$E(\boldsymbol{m}_\lambda) = \lambda E^{(1)}_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E^{(2)}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda} E_{pot}(\boldsymbol{m})$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E^{(i)}_{exch} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

$$\begin{aligned} E(\boldsymbol{m}_\lambda) &= \lambda E^{(1)}_{exch}(\boldsymbol{m}) + \frac{1}{\lambda} E^{(2)}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda} E_{pot}(\boldsymbol{m}) \\ \left.\frac{dE(\boldsymbol{m}_\lambda)}{d\lambda}\right|_{\lambda=1} &= E^{(1)}_{exch} - E^{(2)}_{exch} - E_{pot} = 0 \end{aligned}$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E_{exch}^{(i)} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

$$\begin{aligned} E(\boldsymbol{m}_\lambda) &= \lambda E_{exch}^{(1)}(\boldsymbol{m}) + \frac{1}{\lambda} E^{(2)}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda} E_{pot}(\boldsymbol{m}) \\ \left.\frac{dE(\boldsymbol{m}_\lambda)}{d\lambda}\right|_{\lambda=1} &= E_{exch}^{(1)} - E_{exch}^{(2)} - E_{pot} = 0 \\ E &= E_{exch}^{(1)} + E_{exch}^{(2)} + E_{DMI} + E_{pot} \\ &= 2E_{exch}^{(2)} + 2E_{pot} + E_{DMI} \\ &= 2E_{exch}^{(2)} > 0 \end{aligned}$$

Skyrmion lattices: rank 1?

- $\boldsymbol{d}_1 \parallel \boldsymbol{d}_2$: $b = 0$ after reduction

$$E^{(i)}_{exch} = \frac{1}{2}\int_{\mathbb{R}^2} |\partial_i \boldsymbol{m}|^2 d^2x, \qquad E_{DMI} = \int_{\mathbb{R}^2} \boldsymbol{e}_1 \cdot (\boldsymbol{m} \times \partial_1 \boldsymbol{m}) d^2x$$

- Anisotropic Derrick scaling: $\boldsymbol{m}_\lambda(x_1, x_2) = \boldsymbol{m}(\lambda x_1, x_2)$

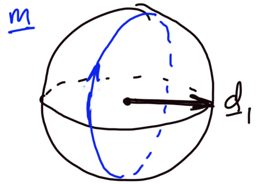
$$\begin{aligned}
E(\boldsymbol{m}_\lambda) &= \lambda E^{(1)}_{exch}(\boldsymbol{m}) + \frac{1}{\lambda}E^{(2)}(\boldsymbol{m}) + E_{DMI}(\boldsymbol{m}) + \frac{1}{\lambda}E_{pot}(\boldsymbol{m}) \\
\left.\frac{dE(\boldsymbol{m}_\lambda)}{d\lambda}\right|_{\lambda=1} &= E^{(1)}_{exch} - E^{(2)}_{exch} - E_{pot} = 0 \\
E &= E^{(1)}_{exch} + E^{(2)}_{exch} + E_{DMI} + E_{pot} \\
&= 2E^{(2)}_{exch} + 2E_{pot} + E_{DMI} \\
&= 2E^{(2)}_{exch} > 0
\end{aligned}$$

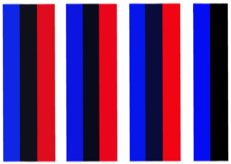
- Ground state is never a skyrmion lattice.

Spiral phase

- Ground state for $\boldsymbol{H} = \boldsymbol{0}$?

$$\boldsymbol{m}(x_1, x_2) = (0, \cos x_1, -\sin x_1), \qquad \mathscr{E} = -\frac{1}{2}$$

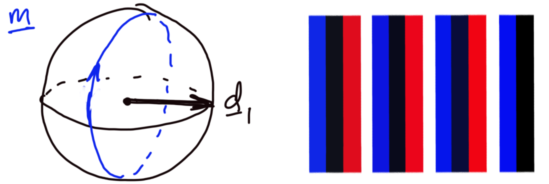




Spiral phase

- Ground state for $\boldsymbol{H} = \mathbf{0}$?

$$\boldsymbol{m}(x_1, x_2) = (0, \cos x_1, -\sin x_1), \qquad \mathscr{E} = -\frac{1}{2}$$



- Spiral phase: $\boldsymbol{m}(\boldsymbol{n}\cdot\boldsymbol{x})$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(t+T) \equiv \boldsymbol{m}(t)$

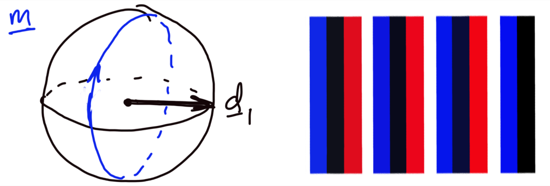
$$\langle \mathscr{E} \rangle = \frac{1}{T}\int_0^T \left\{ \frac{1}{2}|\dot{\boldsymbol{m}}|^2 + \boldsymbol{d}(\boldsymbol{n}) \cdot (\boldsymbol{m} \times \dot{\boldsymbol{m}}) + |\boldsymbol{H}| - \boldsymbol{H}\cdot\boldsymbol{m} \right\} < 0$$

where $\boldsymbol{d}(\boldsymbol{n}) = n_1 \boldsymbol{d}_1 + n_2 \boldsymbol{d}_2$

Spiral phase

- Ground state for $\boldsymbol{H} = \boldsymbol{0}$?

$$\boldsymbol{m}(x_1, x_2) = (0, \cos x_1, -\sin x_1), \qquad \mathscr{E} = -\frac{1}{2}$$



- Spiral phase: $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x})$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(t + T) \equiv \boldsymbol{m}(t)$

$$\langle \mathscr{E} \rangle = \frac{1}{T} \int_0^T \left\{ \frac{1}{2} |\dot{\boldsymbol{m}}|^2 + \boldsymbol{d}(\boldsymbol{n}) \cdot (\boldsymbol{m} \times \dot{\boldsymbol{m}}) + |\boldsymbol{H}| - \boldsymbol{H} \cdot \boldsymbol{m} \right\} < 0$$

where $\boldsymbol{d}(\boldsymbol{n}) = n_1 \boldsymbol{d}_1 + n_2 \boldsymbol{d}_2$

- Must exist for $|\boldsymbol{H}|$ small

Spiral phase

- Domain $\mathscr{H}_{SP} = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists$ spiral phase for $E_{\boldsymbol{H}}\}$

Spiral phase

- Domain $\mathscr{H}_{SP} = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H}}\}$
- Nonempty: $\mathbf{0} \in \mathscr{H}_{SP}$

Spiral phase

- Domain $\mathscr{H}_{SP} = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H}}\}$
- Nonempty: $\mathbf{0} \in \mathscr{H}_{SP}$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_{SP} \Rightarrow s\boldsymbol{H} \in \mathscr{H}_{SP}$ for all $s \in [0, 1]$

Spiral phase

- Domain $\mathscr{H}_{SP} = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H}}\}$
- Nonempty: $\mathbf{0} \in \mathscr{H}_{SP}$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_{SP} \Rightarrow s\boldsymbol{H} \in \mathscr{H}_{SP}$ for all $s \in [0, 1]$
- Inversion symmetric: $\boldsymbol{H} \in \mathscr{H}_{SP} \Rightarrow -\boldsymbol{H} \in \mathscr{H}_{SP}$

Spiral phase

- Domain $\mathscr{H}_{SP} = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H}}\}$
- Nonempty: $\mathbf{0} \in \mathscr{H}_{SP}$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_{SP} \Rightarrow s\boldsymbol{H} \in \mathscr{H}_{SP}$ for all $s \in [0, 1]$
- Inversion symmetric: $\boldsymbol{H} \in \mathscr{H}_{SP} \Rightarrow -\boldsymbol{H} \in \mathscr{H}_{SP}$
- Open

Spiral phase

- Domain $\mathscr{H}_{SP} = \{\boldsymbol{H} \in \mathbb{R}^3 : \exists \text{ spiral phase for } E_{\boldsymbol{H}}\}$
- Nonempty: $\mathbf{0} \in \mathscr{H}_{SP}$
- Star shaped: $\boldsymbol{H} \in \mathscr{H}_{SP} \Rightarrow s\boldsymbol{H} \in \mathscr{H}_{SP}$ for all $s \in [0,1]$
- Inversion symmetric: $\boldsymbol{H} \in \mathscr{H}_{SP} \Rightarrow -\boldsymbol{H} \in \mathscr{H}_{SP}$
- Open
- Bounded: $\mathscr{H}_{SP} \subset B_1(0)$

Spiral phase

- For any $\boldsymbol{m} : S^1_T \to S^2$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x})$ has

$$T\langle \mathscr{E} \rangle \quad = \quad \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 - \langle \boldsymbol{m} \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}} \rangle + \frac{1}{2}|\boldsymbol{H}|\|\hat{\boldsymbol{H}} - \boldsymbol{m}\|^2$$

Spiral phase

- For any $\boldsymbol{m} : S^1_T \to S^2$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x})$ has

$$T\langle \mathscr{E} \rangle \quad = \quad \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 - \langle \boldsymbol{m} \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}} \rangle + \frac{1}{2}|\boldsymbol{H}|\|\hat{\boldsymbol{H}} - \boldsymbol{m}\|^2$$

- $\boldsymbol{m}_0 := \boldsymbol{m} - \hat{\boldsymbol{H}}$

Spiral phase

- For any $\boldsymbol{m} : S^1_T \to S^2$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x})$ has

$$T\langle \mathscr{E} \rangle = \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 - \langle \boldsymbol{m} \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}} \rangle + \frac{1}{2}|\boldsymbol{H}|\|\hat{\boldsymbol{H}} - \boldsymbol{m}\|^2$$

- $\boldsymbol{m}_0 := \boldsymbol{m} - \hat{\boldsymbol{H}}$

$$\begin{aligned} T\langle \mathscr{E} \rangle &= \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - \langle \boldsymbol{m}_0 \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}_0 \rangle \\ &\quad -((\hat{\boldsymbol{H}} \times \boldsymbol{d}(\boldsymbol{n})) \cdot \int_{S^1_T} \dot{\boldsymbol{m}}_0 + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2 \end{aligned}$$

Spiral phase

- For any $\boldsymbol{m} : S^1_T \to S^2$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x})$ has

$$T\langle \mathscr{E} \rangle \;\;=\;\; \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 - \langle \boldsymbol{m} \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}} \rangle + \frac{1}{2}|\boldsymbol{H}|\|\hat{\boldsymbol{H}} - \boldsymbol{m}\|^2$$

- $\boldsymbol{m}_0 := \boldsymbol{m} - \hat{\boldsymbol{H}}$

$$T\langle \mathscr{E} \rangle \;\;=\;\; \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - \langle \boldsymbol{m}_0 \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}_0 \rangle + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2$$

Spiral phase

- For any $\boldsymbol{m}: S^1_T \to S^2$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(\boldsymbol{n}\cdot\boldsymbol{x})$ has

$$T\langle\mathscr{E}\rangle = \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 - \langle \boldsymbol{m}\times\boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}\rangle + \frac{1}{2}|\boldsymbol{H}|\|\hat{\boldsymbol{H}} - \boldsymbol{m}\|^2$$

- $\boldsymbol{m}_0 := \boldsymbol{m} - \hat{\boldsymbol{H}}$

$$\begin{aligned} T\langle\mathscr{E}\rangle &= \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - \langle \boldsymbol{m}_0\times\boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}_0\rangle + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2 \\ &\geq \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - |\boldsymbol{d}(\boldsymbol{n})|\|\boldsymbol{m}_0\|\|\dot{\boldsymbol{m}}_0\| + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2 \end{aligned}$$

Spiral phase

- For any $\boldsymbol{m} : S^1_T \to S^2$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x})$ has

$$T\langle \mathscr{E} \rangle \;=\; \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 - \langle \boldsymbol{m} \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}\rangle + \frac{1}{2}|\boldsymbol{H}|\|\hat{\boldsymbol{H}} - \boldsymbol{m}\|^2$$

- $\boldsymbol{m}_0 := \boldsymbol{m} - \hat{\boldsymbol{H}}$

$$\begin{aligned} T\langle \mathscr{E} \rangle &= \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - \langle \boldsymbol{m}_0 \times \boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}_0\rangle + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2 \\ &\geq \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - |\boldsymbol{d}(\boldsymbol{n})|\|\boldsymbol{m}_0\|\|\dot{\boldsymbol{m}}_0\| + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2 \\ &= \frac{1}{2}(\|\dot{\boldsymbol{m}}_0\| - |\boldsymbol{d}(\boldsymbol{n})|\|\boldsymbol{m}_0\|)^2 + \frac{1}{2}(|\boldsymbol{H}| - |\boldsymbol{d}(\boldsymbol{n})|)\|\boldsymbol{m}_0\|^2 \end{aligned}$$

Spiral phase

- For any $\boldsymbol{m} : S^1_T \to S^2$, $\boldsymbol{n} \in S^1$, $\boldsymbol{m}(\boldsymbol{n}\cdot\boldsymbol{x})$ has

$$T\langle\mathscr{E}\rangle \;=\; \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 - \langle \boldsymbol{m}\times\boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}\rangle + \frac{1}{2}|\boldsymbol{H}|\|\hat{\boldsymbol{H}} - \boldsymbol{m}\|^2$$

- $\boldsymbol{m}_0 := \boldsymbol{m} - \hat{\boldsymbol{H}}$

$$\begin{aligned}
T\langle\mathscr{E}\rangle &= \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - \langle \boldsymbol{m}_0\times\boldsymbol{d}(\boldsymbol{n}), \dot{\boldsymbol{m}}_0\rangle + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2 \\
&\geq \frac{1}{2}\|\dot{\boldsymbol{m}}_0\|^2 - |\boldsymbol{d}(\boldsymbol{n})|\|\boldsymbol{m}_0\|\|\dot{\boldsymbol{m}}_0\| + \frac{1}{2}|\boldsymbol{H}|\|\boldsymbol{m}_0\|^2 \\
&= \frac{1}{2}(\|\dot{\boldsymbol{m}}_0\| - |\boldsymbol{d}(\boldsymbol{n})|\|\boldsymbol{m}_0\|)^2 + \frac{1}{2}(|\boldsymbol{H}| - |\boldsymbol{d}(\boldsymbol{n})|)\|\boldsymbol{m}_0\|^2 \\
&\geq \frac{1}{2}(|\boldsymbol{H}| - 1)\|\boldsymbol{m}_0\|^2
\end{aligned}$$

Critical field $H_{SP}(\hat{\boldsymbol{H}}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

- For all $\boldsymbol{m} : S^1_1 \to S^2$, $\boldsymbol{n} \in S^1$, $T \in \mathbb{R}$, $\boldsymbol{m}(\boldsymbol{n}\cdot\boldsymbol{x}/T)$ has

$$\langle \mathscr{E} \rangle = \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 T^{-2} + \langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}} \rangle T^{-1} + |\boldsymbol{H}|(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m} \rangle)$$

Critical field $H_{SP}(\hat{H}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

- For all $\boldsymbol{m} : S^1_1 \to S^2$, $\boldsymbol{n} \in S^1$, $T \in \mathbb{R}$, $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x}/T)$ has

$$\langle \mathscr{E} \rangle = \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 T^{-2} + \langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}} \rangle T^{-1} + |\boldsymbol{H}|(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m} \rangle)$$

- Attains negative values for some T iff

$$|\boldsymbol{H}| < \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}} \rangle^2}{2\|\dot{\boldsymbol{m}}\|^2 (1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m} \rangle)}$$

Critical field $H_{SP}(\hat{\boldsymbol{H}}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

- For all $\boldsymbol{m} : S^1_1 \to S^2$, $\boldsymbol{n} \in S^1$, $T \in \mathbb{R}$, $\boldsymbol{m}(\boldsymbol{n} \cdot \boldsymbol{x}/T)$ has

$$\langle \mathscr{E} \rangle = \frac{1}{2}\|\dot{\boldsymbol{m}}\|^2 T^{-2} + \langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}} \rangle T^{-1} + |\boldsymbol{H}|(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m} \rangle)$$

- Attains negative values for some T iff

$$|\boldsymbol{H}| < \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}} \rangle^2}{2\|\dot{\boldsymbol{m}}\|^2 (1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m} \rangle)}$$

- Hence

$$H_{SP}(\hat{\boldsymbol{H}}) = \sup_{\boldsymbol{m},\boldsymbol{n}} \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}} \rangle^2}{2\|\dot{\boldsymbol{m}}\|^2 (1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m} \rangle)}$$

Critical field $H_{SP}(\hat{\boldsymbol{H}}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

$$H_{SP}(\hat{\boldsymbol{H}}) = \sup_{\boldsymbol{m},\boldsymbol{n}} \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}}\rangle^2}{2\|\dot{\boldsymbol{m}}\|^2(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m}\rangle)}$$

- Any choice of $\boldsymbol{m} \in C^\infty(S^1_1, S^2)$, $\boldsymbol{n} \in S^1$ gives a lower bound on $H_{SP}(\hat{\boldsymbol{H}})$

Critical field $H_{SP}(\hat{\boldsymbol{H}}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

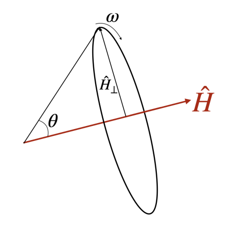
$$H_{SP}(\hat{\boldsymbol{H}}) = \sup_{\boldsymbol{m},\boldsymbol{n}} \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}}\rangle^2}{2\|\dot{\boldsymbol{m}}\|^2(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m}\rangle)}$$

- Any choice of $\boldsymbol{m} \in C^\infty(S^1_1, S^2)$, $\boldsymbol{n} \in S^1$ gives a lower bound on $H_{SP}(\hat{\boldsymbol{H}})$
- $\boldsymbol{m}(t) = (0, \cos 2\pi t, -\sin 2\pi t)$, $\boldsymbol{n} = (1, 0) \Rightarrow H_{SP}(\hat{\boldsymbol{H}}) \geq 1/2$

Critical field $H_{SP}(\hat{\boldsymbol{H}}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

$$H_{SP}(\hat{\boldsymbol{H}}) = \sup_{\boldsymbol{m},\boldsymbol{n}} \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}}\rangle^2}{2\|\dot{\boldsymbol{m}}\|^2(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m}\rangle)}$$

- Any choice of $\boldsymbol{m} \in C^\infty(S^1_1, S^2)$, $\boldsymbol{n} \in S^1$ gives a lower bound on $H_{SP}(\hat{\boldsymbol{H}})$
- $\boldsymbol{m}(t) = (0, \cos 2\pi t, -\sin 2\pi t)$, $\boldsymbol{n} = (1, 0) \Rightarrow H_{SP}(\hat{\boldsymbol{H}}) \geq 1/2$
- $\boldsymbol{m}$ = conical spiral around $\hat{\boldsymbol{H}}$, optimized for $\boldsymbol{n}$ and apex angle $\Rightarrow H_{SP}(\hat{\boldsymbol{H}}) \geq \hat{H}_1^2 + b\hat{H}_2^2$



Critical field $H_{SP}(\hat{\boldsymbol{H}}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

$$H_{SP}(\hat{\boldsymbol{H}}) = \sup_{\boldsymbol{m},\boldsymbol{n}} \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}}\rangle^2}{2\|\dot{\boldsymbol{m}}\|^2(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m}\rangle)}$$

- $\boldsymbol{m} =$ solution of pendulum equation in plane spanned by $\hat{\boldsymbol{H}}$ and $\boldsymbol{N} \perp \hat{\boldsymbol{H}}$ optimized over $\boldsymbol{n}$ and $\boldsymbol{N}$, in limit period $\to \infty$

$$H_{SP}(\hat{\boldsymbol{H}}) \geq \frac{\pi^2}{16}\delta(\hat{\boldsymbol{H}})^2$$

where

$$\delta(\hat{\boldsymbol{H}}) = \max\{\boldsymbol{d}(\boldsymbol{n}) \cdot \boldsymbol{N} : \boldsymbol{n} \in S^1, \boldsymbol{N} \in S^2, \boldsymbol{N} \cdot \hat{\boldsymbol{H}} = 0\}$$

Critical field $H_{SP}(\hat{\boldsymbol{H}}) = \sup\{h : h\hat{\boldsymbol{H}} \in \mathscr{H}_{SP}\}$

$$H_{SP}(\hat{\boldsymbol{H}}) = \sup_{\boldsymbol{m},\boldsymbol{n}} \frac{\langle \boldsymbol{d}(\boldsymbol{n}) \times \boldsymbol{m}, \dot{\boldsymbol{m}}\rangle^2}{2\|\dot{\boldsymbol{m}}\|^2(1 - \langle \hat{\boldsymbol{H}}, \boldsymbol{m}\rangle)}$$

- $\boldsymbol{m} =$ solution of pendulum equation in plane spanned by $\hat{\boldsymbol{H}}$ and $\boldsymbol{N} \perp \hat{\boldsymbol{H}}$ optimized over $\boldsymbol{n}$ and $\boldsymbol{N}$, in limit period $\to \infty$

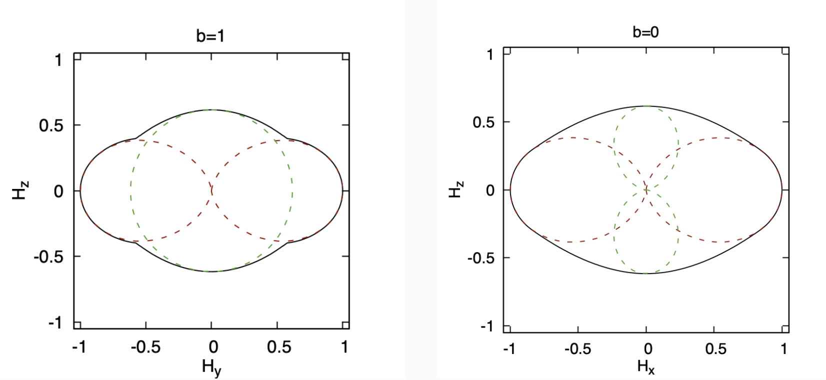
  $$H_{SP}(\hat{\boldsymbol{H}}) \geq \frac{\pi^2}{16}\delta(\hat{\boldsymbol{H}})^2$$

  where

  $$\delta(\hat{\boldsymbol{H}}) = \max\{\boldsymbol{d}(\boldsymbol{n}) \cdot \boldsymbol{N} : \boldsymbol{n} \in S^1, \boldsymbol{N} \in S^2, \boldsymbol{N} \cdot \hat{\boldsymbol{H}} = 0\}$$

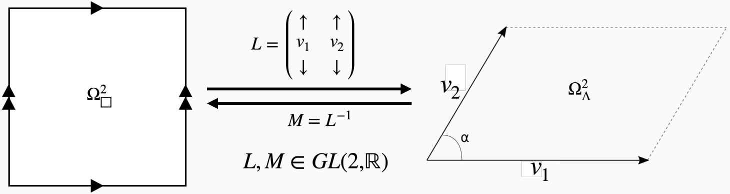
- E.g. for $b = 1$, $\delta(\hat{\boldsymbol{H}}) \equiv 1$

Critical field



green dashed: pendulum bound, red dashed: conical bound

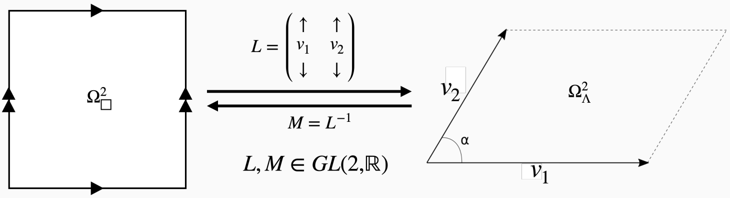
Skyrmion lattices



- Assume $\boldsymbol{m} : \mathbb{R}^2 \to S^2$ is periodic w.r.t. lattice Λ generated by $\boldsymbol{v}_1, \boldsymbol{v}_2$

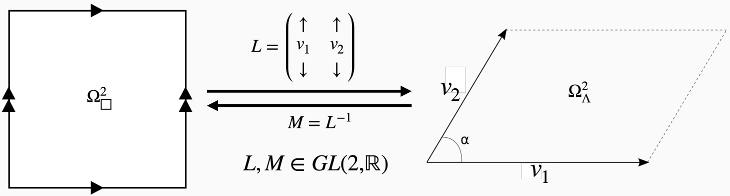
$$\boldsymbol{m}(\boldsymbol{x} + n_1 \boldsymbol{v}_1 + n_2 \boldsymbol{v}_2) = \boldsymbol{m}(\boldsymbol{x})$$

for all $(n_1, n_2) \in \mathbb{Z}^2$.



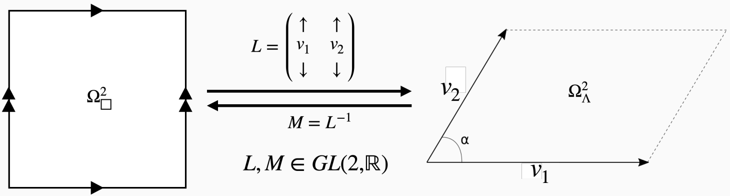
- Assume $\boldsymbol{m} : \mathbb{R}^2 \to S^2$ is periodic w.r.t. lattice Λ generated by $\boldsymbol{v}_1, \boldsymbol{v}_2$
$$\boldsymbol{m}(\boldsymbol{x} + n_1\boldsymbol{v}_1 + n_2\boldsymbol{v}_2) = \boldsymbol{m}(\boldsymbol{x})$$
for all $(n_1, n_2) \in \mathbb{Z}^2$.
- It lives on torus $\Omega_\Lambda = \mathbb{R}^2/\Lambda$

Skyrmion lattices



- Energy per unit area is

$$\langle \mathscr{E} \rangle = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \left\{ \frac{1}{2} |d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i}) + V(\boldsymbol{m}) \right\} d^2x$$



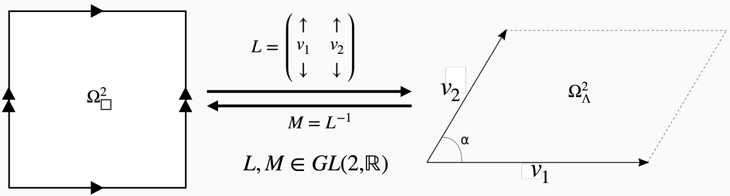
- Energy per unit area is

$$\langle \mathscr{E} \rangle = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \left\{ \frac{1}{2} |d\boldsymbol{m}|^2 + \sum_i \boldsymbol{d}_i \cdot (\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i}) + V(\boldsymbol{m}) \right\} d^2x$$

- Define new coords X_1, X_2,

$$\boldsymbol{x} = X_1 \boldsymbol{v}_1 + X_2 \boldsymbol{v}_2 = L\boldsymbol{X}$$

Skyrmion lattices



$$\begin{aligned}
\langle \mathscr{E} \rangle &= \frac{1}{2}\mathrm{tr}(M^T G(\boldsymbol{m})M) + \mathrm{tr}(\mathscr{D}(\boldsymbol{m})M) + C(\boldsymbol{m}) \\
G_{ij}(\boldsymbol{m}) &= \int_\square \frac{\partial \boldsymbol{m}}{\partial X_i} \cdot \frac{\partial \boldsymbol{m}}{\partial X_j} dX_1 dX_2 \\
\mathscr{D}_{ij}(\boldsymbol{m}) &= \boldsymbol{d}_i \cdot \int_\square \boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial X_j} dX_1 dX_2 \\
C(\boldsymbol{m}) &= \int_\square V(\boldsymbol{m}) dX_1 dX_2
\end{aligned}$$

Skyrmion lattices

- Switch viewpoint: $\boldsymbol{m}$ is a field on the fixed torus $\Omega_{\square} = \mathbb{R}^2/\mathbb{Z}^2$, but it minimizes a functional that depends parametrically on $M \in GL(2, \mathbb{R})$ (the lattice geometry)

Skyrmion lattices

- Switch viewpoint: $\boldsymbol{m}$ is a field on the fixed torus $\Omega_\square = \mathbb{R}^2/\mathbb{Z}^2$, but it minimizes a functional that depends parametrically on $M \in GL(2, \mathbb{R})$ (the lattice geometry)
- Optimal lattice minimizes $\langle \mathscr{E} \rangle$ with respect to **both** $\boldsymbol{m}$ and M!

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_{\Box} \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2} \mathrm{tr}(M^T G M) + \mathrm{tr}(\mathscr{D} M) + C$$

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_\square \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2}\,\mathrm{tr}(M^T G M) + \mathrm{tr}(\mathscr{D} M) + C$$

attains global min at

$$M_* = -G^{-1}\mathscr{D}^T$$

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_\square \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2} \mathrm{tr}(M^T G M) + \mathrm{tr}(\mathscr{D} M) + C$$

  attains global min at

$$M_* = -G^{-1} \mathscr{D}^T$$

- G symmetric pos def (if $Q \neq 0$), certainly invertible

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_\square \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2}\operatorname{tr}(M^T G M) + \operatorname{tr}(\mathscr{D} M) + C$$

attains global min at

$$M_* = -G^{-1}\mathscr{D}^T$$

- G symmetric pos def (if $Q \neq 0$), certainly invertible
- $\det M_* = \det \mathscr{D} / \det G = 0$ if $\mathscr{D}$ not invertible!
  - $b \neq 0$: $\det \mathscr{D} \neq 0$ for a.e. $\boldsymbol{m}$
  - $b = 0$: $\det \mathscr{D} = 0$ for a.e. $\boldsymbol{m}$ [no stable lattices for rank 1]
- $\det M_* < 0$ if $\det \mathscr{D} < 0$

Skyrmion lattices: optimizing the period lattice

- Fix $\boldsymbol{m} : \Omega_\square \to S^2$

$$\langle \mathscr{E} \rangle = \frac{1}{2}\mathrm{tr}(M^T G M) + \mathrm{tr}(\mathscr{D} M) + C$$

attains global min at

$$M_* = -G^{-1}\mathscr{D}^T$$

- G symmetric pos def (if $Q \neq 0$), certainly invertible
- $\det M_* = \det \mathscr{D} / \det G = 0$ if $\mathscr{D}$ not invertible!
  - $b \neq 0$: $\det \mathscr{D} \neq 0$ for a.e. $\boldsymbol{m}$
  - $b = 0$: $\det \mathscr{D} = 0$ for a.e. $\boldsymbol{m}$ [no stable lattices for rank 1]
- $\det M_* < 0$ if $\det \mathscr{D} < 0$
  - diffeo flips orientation, $Q \mapsto -Q$
  - Feature not bug!

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2,\mathbb{R})$

$$\langle\mathscr{E}\rangle : C^\infty(\Omega_\square, S^2) \to \mathbb{R}.$$

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2, \mathbb{R})$

$$\langle \mathscr{E} \rangle : C^\infty(\Omega_\square, S^2) \to \mathbb{R}.$$

- Minimize by **arrested Newton flow**

$$\ddot{\boldsymbol{m}} = -\,\mathrm{grad}_{L^2} \langle \mathscr{E} \rangle$$

$\boldsymbol{m}(0)$ some initial guess, $\dot{\boldsymbol{m}}(0) = \boldsymbol{0}$

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2,\mathbb{R})$

$$\langle \mathscr{E} \rangle : C^\infty(\Omega_\square, S^2) \to \mathbb{R}.$$

- Minimize by **arrested Newton flow**

$$\ddot{\boldsymbol{m}} = -\operatorname{grad}_{L^2}\langle \mathscr{E} \rangle$$

  $\boldsymbol{m}(0)$ some initial guess, $\dot{\boldsymbol{m}}(0) = \mathbf{0}$

- Arrest (set $\dot{\boldsymbol{m}}(t) = \mathbf{0}$) if

$$\langle \dot{\boldsymbol{m}}(t), \operatorname{grad}_{L^2}\langle \mathscr{E} \rangle \rangle_{L^2} > 0$$

Skyrmion lattices: optimizing the field

- Fix $M \in GL(2, \mathbb{R})$

$$\langle \mathscr{E} \rangle : C^\infty(\Omega_\Box, S^2) \to \mathbb{R}.$$

- Minimize by **arrested Newton flow**

$$\ddot{\boldsymbol{m}} = -\operatorname{grad}_{L^2} \langle \mathscr{E} \rangle$$

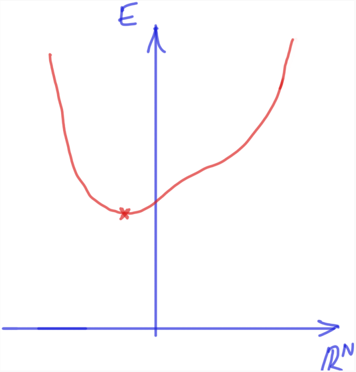
  $\boldsymbol{m}(0)$ some initial guess, $\dot{\boldsymbol{m}}(0) = \mathbf{0}$

- Arrest (set $\dot{\boldsymbol{m}}(t) = \mathbf{0}$) if

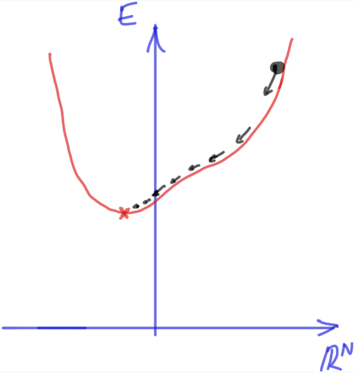
$$\langle \dot{\boldsymbol{m}}(t), \operatorname{grad}_{L^2} \langle \mathscr{E} \rangle \rangle_{L^2} > 0$$

- Continue until $\| \operatorname{grad}_{L^2} \langle \mathscr{E} \rangle \|_{C^0} < tol$

Skyrmion lattices: optimizing the field

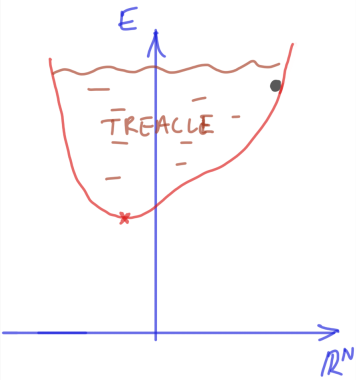


Skyrmion lattices: optimizing the field



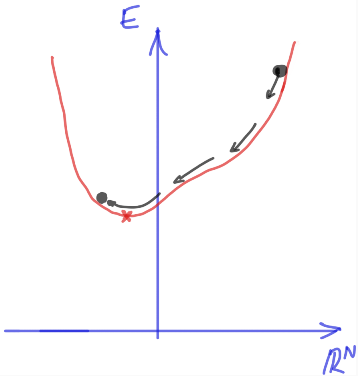
Gradient flow: $\dot{\boldsymbol{m}} = -\,\mathrm{grad}\, E$

Skyrmion lattices: optimizing the field



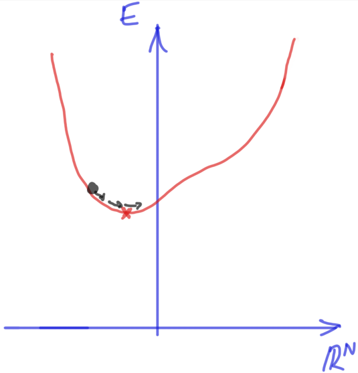
Gradient flow: $\dot{\boldsymbol{m}} = -\,\mathrm{grad}\, E$

Skyrmion lattices: optimizing the field



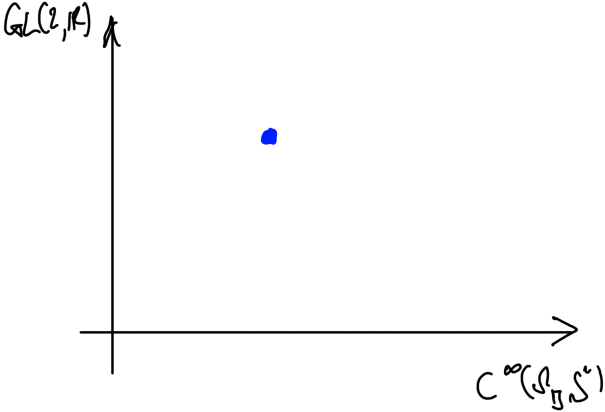
Arrested Newton flow: $\ddot{\boldsymbol{m}} = -\operatorname{grad} E$

Skyrmion lattices: optimizing the field

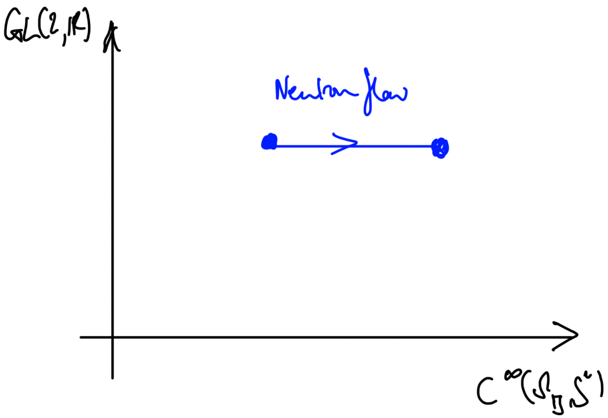


Arrested Newton flow: $\ddot{\boldsymbol{m}} = -\,\mathrm{grad}\, E$

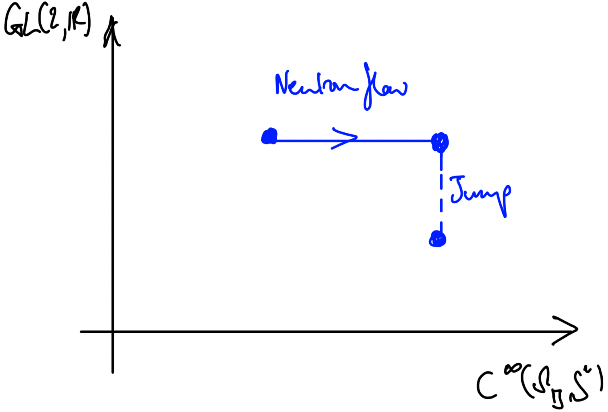
Skyrmion lattices: alternate



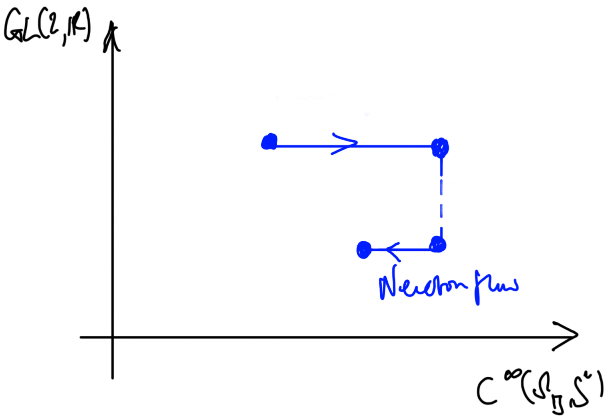
Skyrmion lattices: alternate



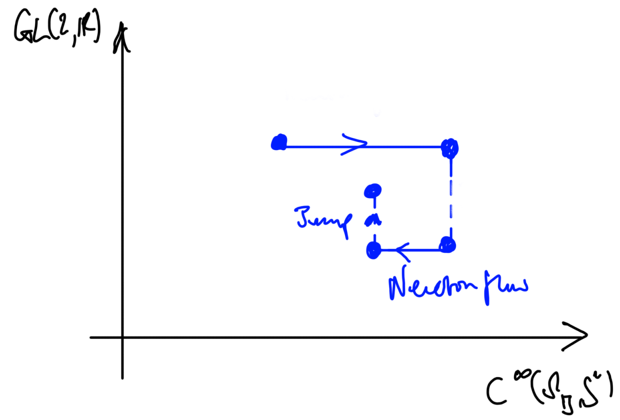
Skyrmion lattices: alternate



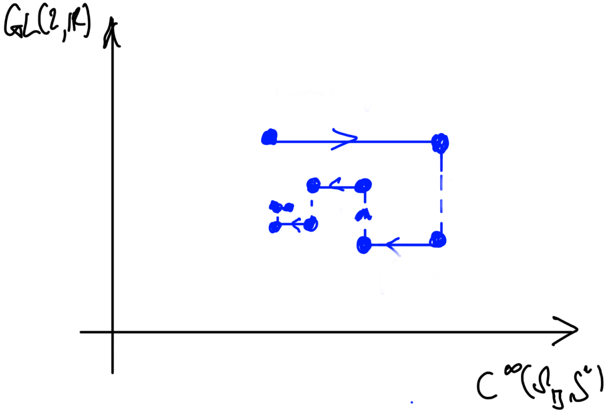
Skyrmion lattices: alternate



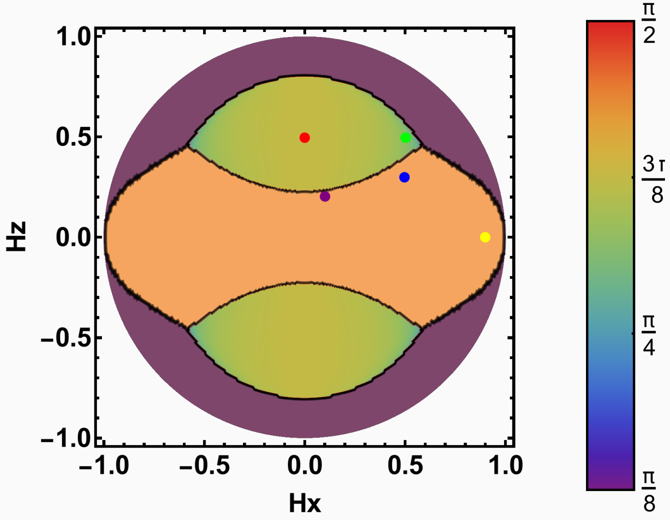
Skyrmion lattices: alternate



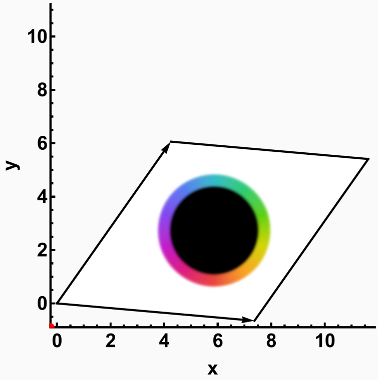
Skyrmion lattices: alternate

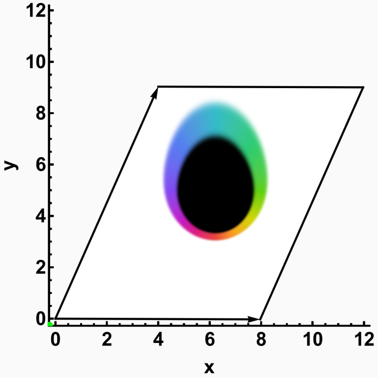


Skyrmion lattices: $b = 1$



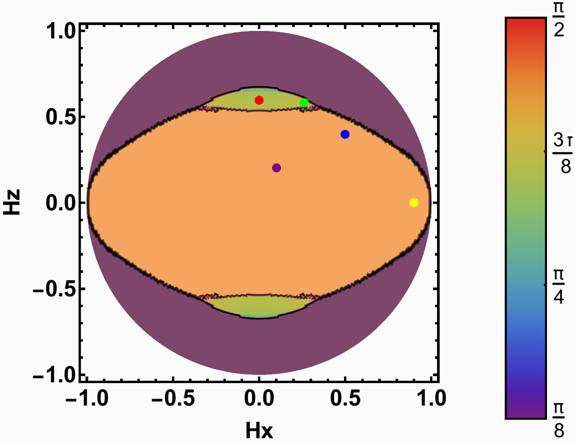
Skyrmion lattices: $b = 1$





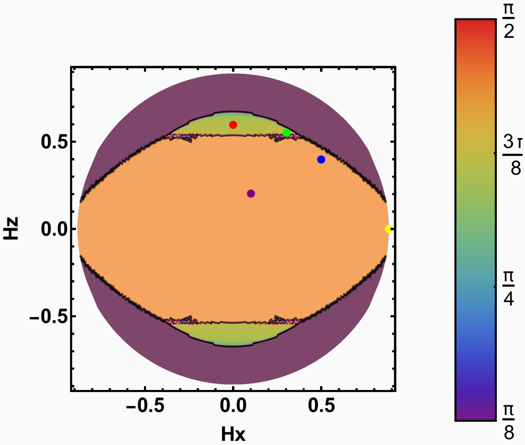
Skyrmion lattices: $b = 0.8$

$$\varphi = 0$$



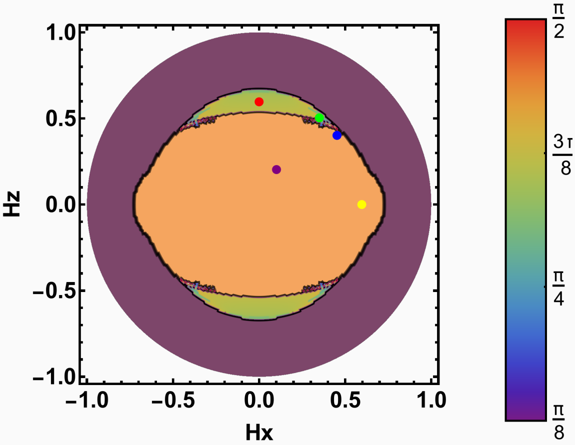
Skyrmion lattices: $b = 0.8$

$$\varphi = \pi/6$$



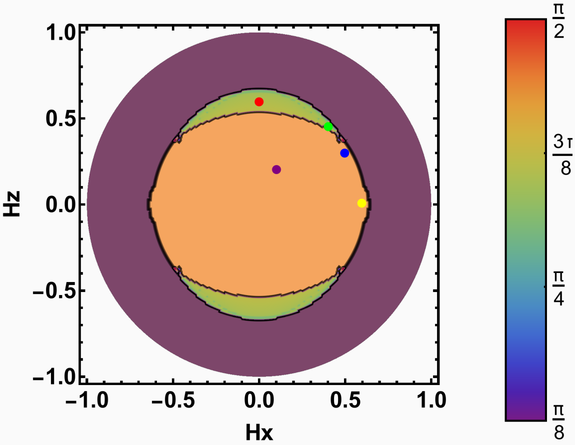
Skyrmion lattices: $b = 0.8$

$$\varphi = \pi/3$$



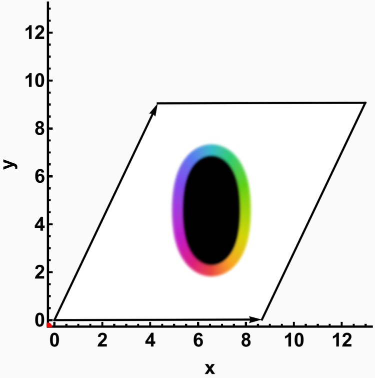
Skyrmion lattices: $b = 0.8$

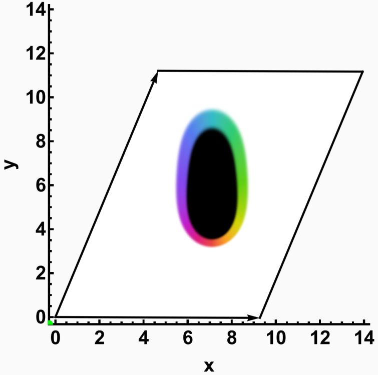
$$\varphi = \pi/2$$



Skyrmion lattices: $b = 0.8$

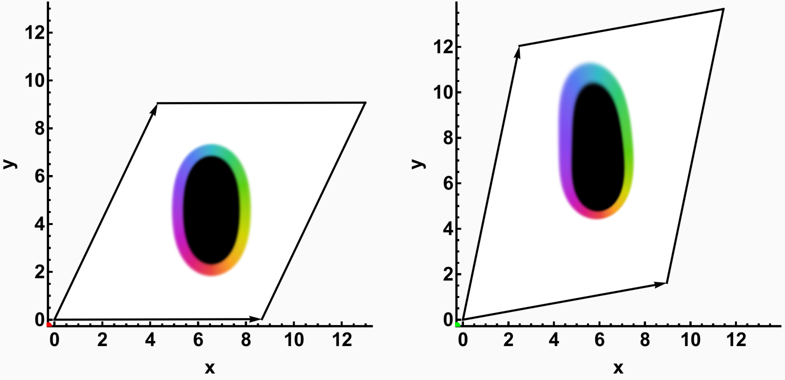
$$\varphi = 0$$





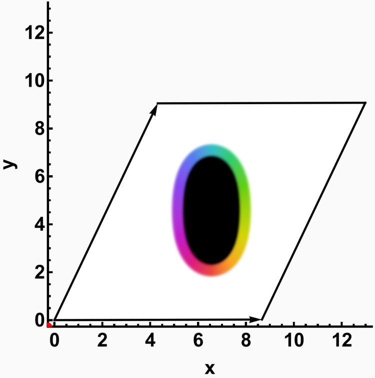
Skyrmion lattices: $b = 0.8$

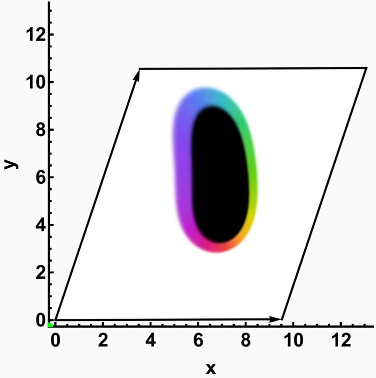
$$\varphi = \pi/6$$



Skyrmion lattices: $b = 0.8$

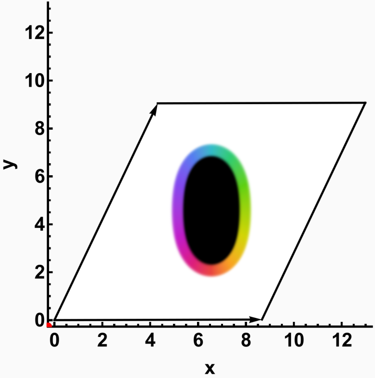
$$\varphi = \pi/3$$

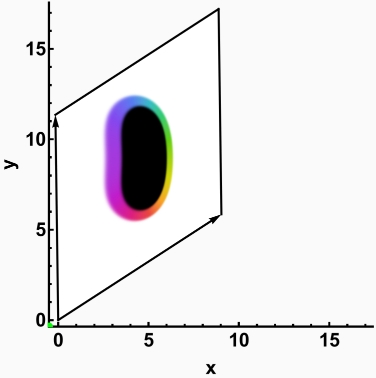




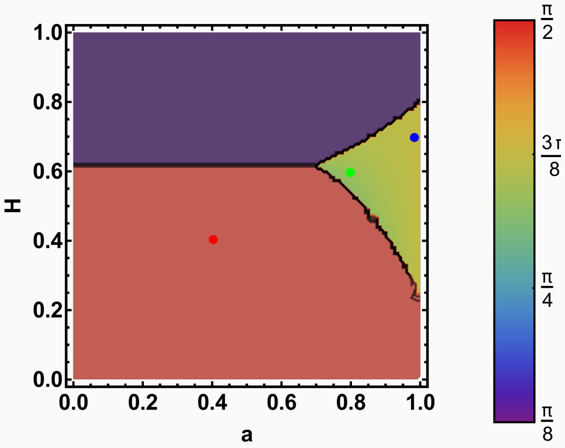
Skyrmion lattices: $b = 0.8$

$$\varphi = \pi/2$$

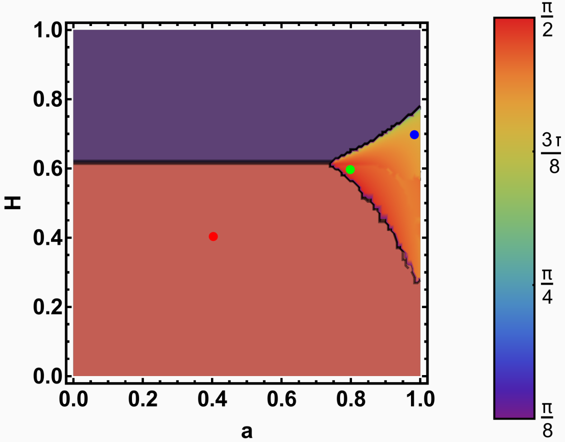




Skyrmion lattices: $\hat{H} = (0, 0, 1)$



Skyrmion lattices: $\hat{H} = (0, 1/2, \sqrt{3}/2)$



Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

- At low $|\boldsymbol{H}|$, ground state is a spiral phase

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0, 1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1

- At low $|\boldsymbol{H}|$, ground state is a spiral phase
- Spiral phase domain depends nontrivially on b

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0,1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1
- At low $|\boldsymbol{H}|$, ground state is a spiral phase
- Spiral phase domain depends nontrivially on b
- Minimal model with $b < b_* \approx 0.75$ never (i.e. for no $\boldsymbol{H}$) has skyrmion lattice as ground state

Conclusions

- Can reduce most general DMI term to a 1 parameter family, parametrized by $b \in [0,1]$

$$b^2 = \frac{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 - \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}{|\boldsymbol{d}_1|^2 + |\boldsymbol{d}_2|^2 + \sqrt{(|\boldsymbol{d}_1|^2 - |\boldsymbol{d}_2|^2)^2 + (2\boldsymbol{d}_1 \cdot \boldsymbol{d}_2)^2}}$$

  $b = 1$ standard, $b = 0$ rank 1
- At low $|\boldsymbol{H}|$, ground state is a spiral phase
- Spiral phase domain depends nontrivially on b
- Minimal model with $b < b_* \approx 0.75$ never (i.e. for no $\boldsymbol{H}$) has skyrmion lattice as ground state
- Minimal model with $\boldsymbol{H}$ to close to the $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ plane never (i.e. for no b) has skyrmion lattice as ground state

Conclusions

- Skyrmion lattices (when they exist) always have $Q = \pm 1$ per unit cell

Conclusions

- Skyrmion lattices (when they exist) always have $Q = \pm 1$ per unit cell
- Cell geometry gets distorted if $\boldsymbol{H}$ not perpendicular to $\{\boldsymbol{d}_1, \boldsymbol{d}_2\}$ plane, but nothing really weird happens.

Open questions

- Could skyrmion lattices have technological applications? Is the ability to manipulate their geometry by changing $\boldsymbol{H}$ useful?
- Can we **prove** existence of skyrmion lattices?