Non-monotonic vortex interactions and type 1.5 superconductivity

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24/11/17

Ginzburg-Landau Theory

$$F = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |(\nabla - iA)\psi|^2 + \frac{1}{2} B^2 - \alpha(T)|\psi|^2 + \frac{\beta(T)}{2} |\psi|^4 \right\}$$

Ginzburg-Landau Theory

$$F = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|(\nabla - iA)\psi|^2 + \frac{1}{2}B^2 + \frac{\beta}{2}\left(|\psi|^2 - \frac{\alpha}{\beta}\right)^2 \right\}$$

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- $F < \infty$:

$$\psi \sim \sqrt{\frac{\alpha}{\beta}} e^{i\chi}, \qquad A \sim \nabla\chi \qquad \text{at large } r$$

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- $\Rightarrow$ Flux quantization:

$$\Phi = \int_{\mathbb{R}^2} B = \oint_{S^1_\infty} A = 2\pi n$$

Vortices

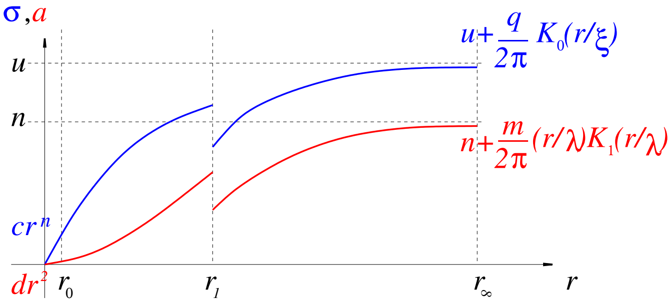
$$F = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|(\nabla - iA)\psi|^2 + \frac{1}{2}B^2 + \frac{\beta}{2}\left(|\psi|^2 - \frac{\alpha}{\beta}\right)^2 \right\}$$

- $\psi = \sigma(r)e^{in\theta}$, $A = \frac{a(r)}{r}\widehat{\theta}$

$$\begin{aligned} -\sigma'' - \frac{1}{r}\sigma' + \frac{(n-a)^2}{r^2}\sigma + 2\beta\left(\sigma^2 - \frac{\alpha}{\beta}\right)\sigma &= 0 \\ -a'' + \frac{1}{r}a' - (n-a)\sigma^2 &= 0 \end{aligned}$$

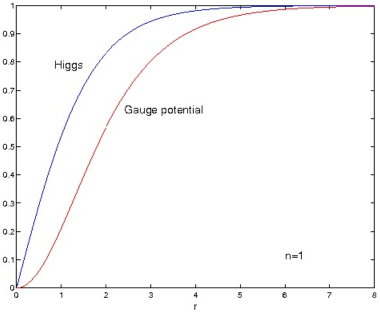
- BCs: $\sigma(0) = a(0) = 0$, $\sigma(\infty) = u := \sqrt{\alpha/\beta}$, $a(\infty) = n$
- Shooting method

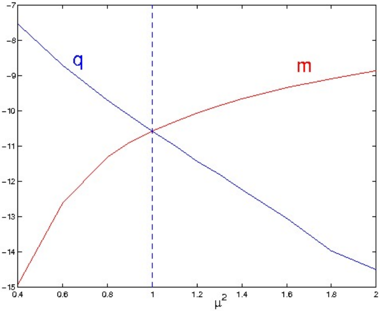
- $u = \sqrt{\alpha/\beta} = \lambda^{-1}$, $\xi^{-1} = 2\sqrt{\alpha}$



- $f : \mathbb{R}^4 \to \mathbb{R}^4$, $(c, d, q, m) \mapsto$ mismatch in σ, σ', a, a'
- Solve $f(c, d, q, m) = (0, 0, 0, 0)$ via Newton Raphson

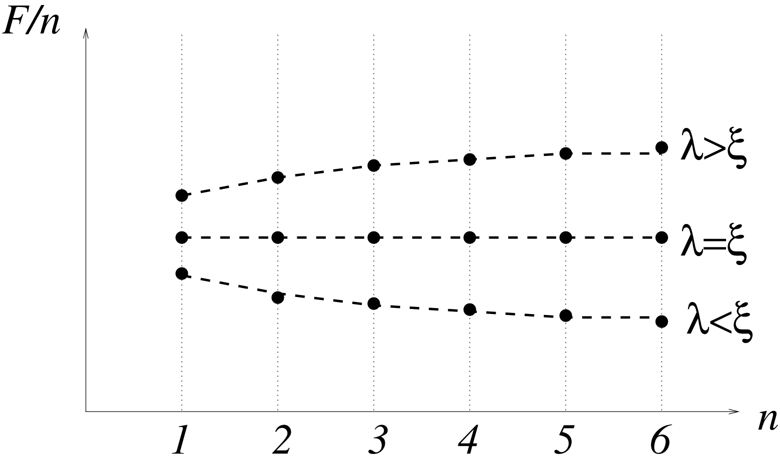
Vortices





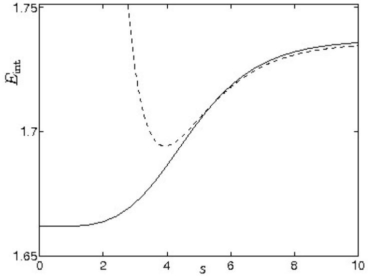
$\mu = \lambda/\xi = 2\sqrt{\beta} \qquad (= \sqrt{2}\kappa_{GL})$

Vortices

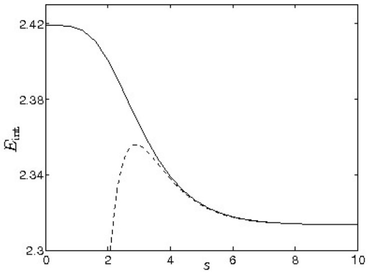


Vortices: interaction potential

$$E_{int} = \min\{F(\psi, A) \ : \ \psi(0,0) = \psi(s,0) = 0\}$$



$\lambda < \xi$



$\lambda > \xi$

Intervortex forces

$$\mathcal{L}_{AHM} = \frac{1}{2}\overline{D_\mu \psi} D^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\beta}{2}(|\psi|^2 - \alpha/\beta)^2$$

- $D_\mu = \partial_\mu - iA_\mu$, $\mu = 0, 1, 2$
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_{12} = B$
- **Static** model $\equiv$ GL
- Vortices are **topological solitons**
- Can boost them, scatter etc.

Intervortex forces

- Unwind vortex (fix gauge s.t. ψ real)

$$\psi = \sigma(r), \qquad (A^0, A) = (0, r^{-1}(a(r) - 1)\widehat{\theta})$$

- Large r behaviour: $\psi = u + \widehat{\psi}$,

$$\widehat{\psi} \sim \frac{q}{2\pi} K_0(r/\xi), \qquad (A^0, A) \sim (0, -\frac{m}{2\pi}\mathbf{k} \times \nabla K_0(r/\lambda))$$

  where $\mathbf{k} \times \nabla$ is shorthand for $(-\partial_2, \partial_1)$
- Identical to solution of **linearization** of AHM about vacuum $\psi = u$, $A_\mu = 0$

$$\mathcal{L}_{lin} = \frac{1}{2}\partial_\mu \widehat{\psi} \partial^\mu \widehat{\psi} - \frac{1}{2\xi^2}\widehat{\psi}^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\lambda^2}A_\mu A^\mu$$

  in the presence of point sources

$$\rho = q\delta(\mathbf{x}), \qquad (j^0, \mathbf{j}) = (0, -m\mathbf{k} \times \nabla\delta(\mathbf{x}))$$

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in the presence of point sources

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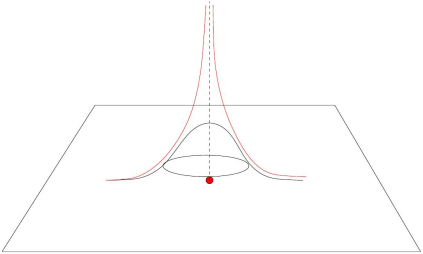
Intervortex forces

- At large r, AH vortex is indistinguishable from a point particle in massive uncoupled Klein-Gordon-Proca theory (scalar boson mass ξ^{-1}, photon mass λ^{-1}) carrying
  - scalar monopole charge q
  - magnetic dipole moment m



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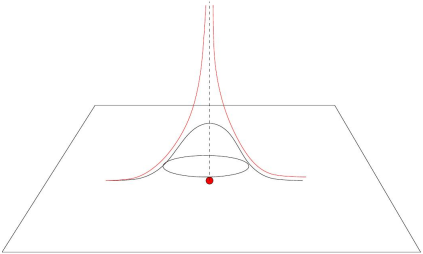
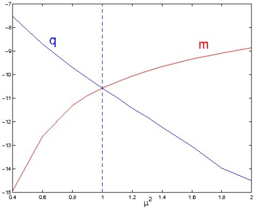
Recall these are coefficients used in our shooting scheme!





Intervortex forces

- Deep principle / leap of faith:
  PHYSICS IS MODEL INDEPENDENT!

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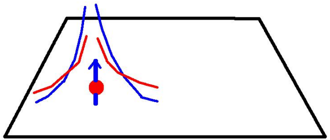

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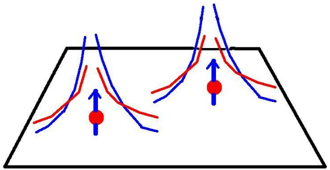

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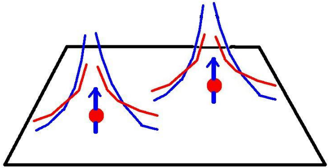

$$L_{int} = \int_{\mathbb{R}^2} (\rho_{(1)} \widehat{\psi}_{(2)} - j^{\mu}_{(1)} A^{(2)}_{\mu})$$

Intervortex forces

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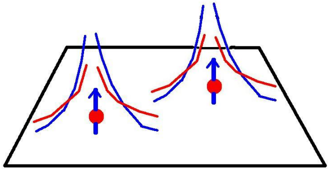

$$L_{int} = \frac{1}{2\pi}\left[q^2 K_0(s/\xi) - m^2 K_0(s/\lambda)\right]$$

Intervortex forces

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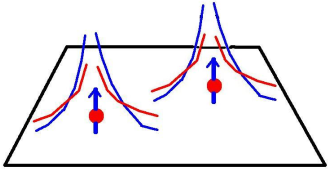

$$E_{int} = \frac{1}{2\pi}\left[-q^2 K_0(s/\xi) + m^2 K_0(s/\lambda)\right]$$

Repulsive (at large s) if $\lambda > \xi$, attractive if $\lambda < \xi$ (vanishes identically if $\lambda = \xi$)

Intervortex forces

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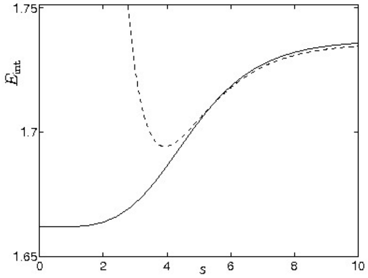
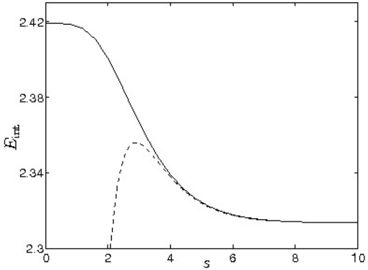


$\lambda < \xi$



$\lambda > \xi$

Magnetic response

$$G = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|D\psi|^2 + V(|\psi|) + \frac{1}{2}(B-H)^2 \right\} = F - H \int_{\mathbb{R}^2} B$$

- Compare
  - normal state ($\psi = 0,\ B = H$)
  - Meissner state ($\psi = u,\ B = 0$)
  - vortex
- $G_{vortex} = F_{vortex} - 2\pi H,\ G_{Meissner} = 0$

$$H_{c1} = \frac{F_{vortex}}{2\pi}$$

- $g_{normal} = V(0) - \frac{1}{2}H^2,\ g_{Meissner} = 0$

$$H_c = 2\sqrt{V(0)}$$

Magnetic response

$$G = F - H \int_{\mathbb{R}^2} B$$

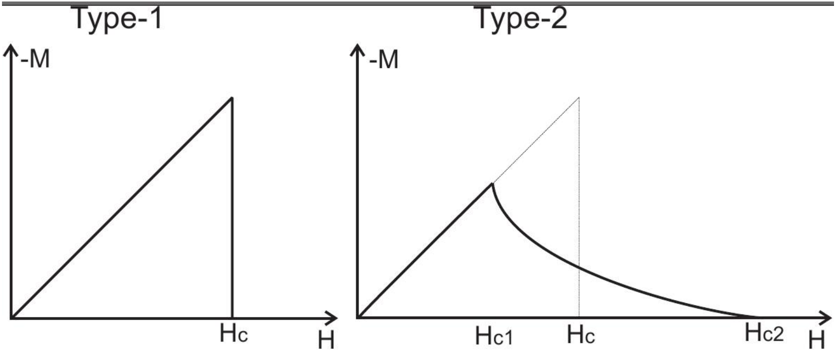
- Normal state is linearly stable $\iff$ lowest eigenvalue of

$$(\nabla - iA_H)^2$$

  exceeds $|V''(0)|$, i.e. $H > |V''(0)| = H_{c2}$
- Type I, $\xi > \lambda$: $H_c < H_{c2} < H_{c1}$
- Type II, $\xi < \lambda$: $H_{c1} < H_c < H_{c2}$

Magnetic response

$$M = \langle B - H \rangle$$



Two-component GL theory

$$F = \int_{\mathbb{R}^2} \left\{ \frac{1}{2}|D\psi_1|^2 + \frac{1}{2}|D\psi_2|^2 + V(\psi_1, \psi_2) + \frac{1}{2}B^2 \right\}$$

Wide range of possibilities for V

- $V_{\text{simple}} = \sum_a (-\alpha_a |\psi_a|^2 + \frac{\beta_a}{2}|\psi_a|^4)$
- $V = V_{\text{simple}} + \frac{\eta}{2}(\overline{\psi_1}\psi_2 + \psi_1\overline{\psi_2})$
- $V_{\text{passive band}} = -\alpha_1|\psi_1|^2 + \frac{\beta_1}{2}|\psi_1|^4 + \alpha_2|\psi_2|^2$

All have

- min at $(\psi_1, \psi_2) = (u_1, u_2) \neq (0, 0)$
- $F < \infty$ implies $(\psi_1, \psi_2) \sim (u_1, u_2)e^{i\chi}$, $A \sim \nabla\chi$, hence flux quantization
- vortices $\psi_a = \sigma_a(r)e^{ni\theta}$, $A = r^{-1}a(r)\widehat{\theta}$

Vortex asymptotics

$$\mathcal{L}_{AHM} = \frac{1}{2}\sum_a \overline{D_\mu \psi_a} D^\mu \psi_a - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\psi_1, \psi_2)$$

- Unwind vortex, linearize about vacuum: $\psi_a = u_a + \widehat{\psi}_a$ real,

$$\mathcal{L}_{lin} = \tfrac{1}{2}\sum_a \partial_\mu \widehat{\psi}_a \partial^\mu \widehat{\psi}_a - \tfrac{1}{2}\sum_{a,b} \widehat{\psi}_a \mathscr{H}_{ab} \widehat{\psi}_b - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(u_1^2 + u_2^2) A_\mu A^\mu$$

  where

$$\mathscr{H}_{ab} = \left. \frac{\partial^2 V}{\partial|\psi_a|\partial|\psi_b|} \right|_{(u_1, u_2)}$$

- In general, $\widehat{\psi}_{1,2}$ are directly coupled ($\mathscr{H}$ isn't diagonal). Let $\mathscr{H}$ have eigenvalues ξ_i^{-2}, eigenvectors v_i,

$$\begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix} = \chi_1 v_1 + \chi_2 v_2$$

Vortex asymptotics

$$\mathcal{L}_{lin} = \frac{1}{2}\sum_a \left(\partial_\mu \chi_a \partial^\mu \chi_a - \xi_a^{-2}\chi_a^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(u_1^2 + u_2^2)A_\mu A^\mu$$

- Pair of uncoupled Klein-Gordon fields of masses ξ_1^{-1}, ξ_1^{-1}, Proca field of mass $\lambda^{-1} = \sqrt{u_1^2 + u_2^2}$.
- vortex looks (at large r) like solution of $\mathcal{L}_{lin} + \rho_1\chi_1 + \rho_2\chi_2 - j_\mu A^\mu$ with sources

$$\rho_a = q_a\delta(\mathbf{x}), \qquad (j^0, \mathbf{j}) = (0, -m\mathbf{k} \times \nabla\,\delta(\mathbf{x}))$$

  Again, (q_1, q_2, m) are shooting parameters, can be found numerically.
- Same trick / leap of faith

$$E_{int} = \frac{1}{2\pi}\left[m^2 K_0(s/\lambda) - q_1^2 K_0(s/\xi_1) - q_2^2 K_0(s/\xi_2)\right]$$

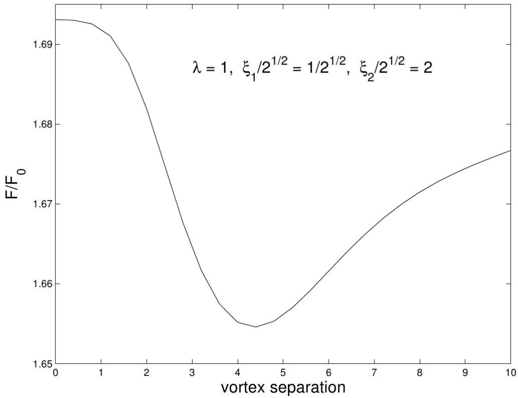
Vortex interactions

$$E_{int} = \frac{1}{2\pi}\left[m^2 K_0(s/\lambda) - q_1^2 K_0(s/\xi_1) - q_2^2 K_0(s/\xi_2)\right]$$

- Now have **three** fundamental length scales in vortex asymptotics: λ, ξ_1, ξ_2
- Interesting (and not uncommon) possibility: $\xi_1 < \lambda < \xi_2$
- Long range intervortex force dominated by scalar attraction, mediated by χ_1
- Naive guess: maybe at short range, magnetic repulsion dominates?
- Answer by computing E_{int} numerically

Non-monotonic vortex interaction: simple model

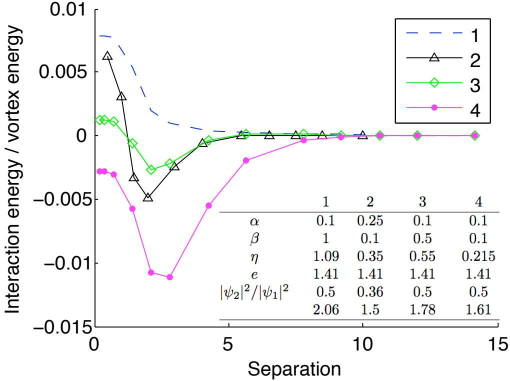
$$V = -\alpha_1|\psi_1|^2 + \frac{\beta_1}{2}|\psi_1|^4 - \alpha_2|\psi_2|^2 + \frac{\beta_2}{2}|\psi_2|^4$$



[Babaev, JMS, PRB72 (2005) 180502]

Non-monotonic vortex interaction: passive band case

$$V = \frac{1}{2}(|\psi_1|^2 - 1)^2 + \alpha|\psi_2|^2 + \frac{\beta}{2}|\psi_2|^4 + \eta Re(\overline{\psi_1}\psi_2)$$

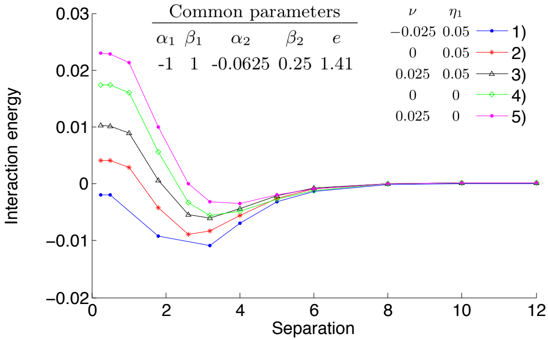


	1	2	3	4
α	0.1	0.25	0.1	0.1
β	1	0.1	0.5	0.1
η	1.09	0.35	0.55	0.215
e	1.41	1.41	1.41	1.41
$\lvert\psi_2\rvert^2/\lvert\psi_1\rvert^2$	0.5	0.36	0.5	0.5
	2.06	1.5	1.78	1.61

[Babaev, Carlström, JMS, PRL105 (2010) 067003]

Non-monotonic vortex interaction: gradient coupling

$$F \to F - \frac{\nu}{2}(\overline{D_i\psi_1}D_i\psi_2 + c.c)$$



[Babaev, Carlström, JMS, PRB83 (2011) 174509]

Type 1.5: magnetic response

$$G = F - H \int_{\mathbb{R}^2} B$$

- Vortices have binding energy: $\frac{F_n}{n}$ decreasing function of n
- $H_{c1} \neq F_1/2\pi$
- $G_{n\text{-vortex}} = F_n - 2n\pi H$, $G_{\text{Meissner}} = 0$,

$$H_{c1} = \lim_{n \to \infty} \frac{(F_n/n)}{2\pi}$$

- As $H \nearrow H_{c1}$ a huge clump of flux penetration becomes energetically favourable
  1st order phase transition, like type I
- But favoured penetrating state is a **lattice** of vortices (like type II) with a fixed preferred vortex separation (unlike type II)
- As H increases further, vortex lattice gets squeezed, vortex density grows continuously, so magnetization declines continuously (like type II)

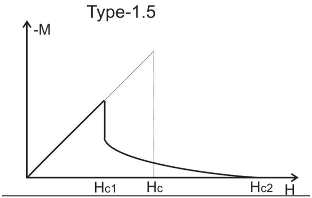
Type 1.5: magnetic response

- Eventually reach H_{c2}

$$H_{c2} = \left|\,\min\{\text{eigenvalues of } \mathscr{H}|_{\psi_1=\psi_2=0}\}\right|$$

after which normal state is linearly stable.

$M = \langle B - H \rangle$



Signature: semi-Meissner state - macroscopic clumps of vortex lattice, with fixed lattice spacing, in a sea of Meissner state.

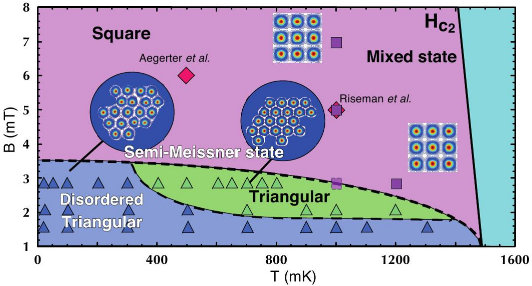
Type 1.5: experimental observations

MgB_2 at $H = 5$ Oe and $T = 4.2$ K, imaged by Bitter decoration

[Moshchalkov et al, PRL102 (2009) 117001]

Type 1.5: experimental observations

Sr_2RuO_4, muon spin rotation measurements



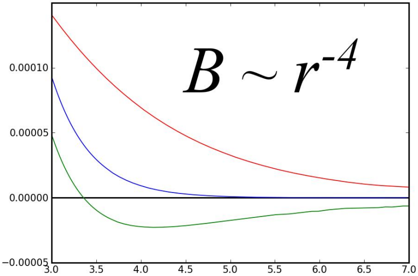
[Ray et al, PRB89 (2014) 094504]

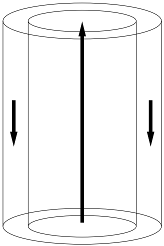
A menagerie of topological solitons

- Fractional flux vortices: $\psi_1 = \sigma_1(r)e^{i\theta}$, $\psi_2 = \sigma_2(r)$

$$\int_{\mathbb{R}^2} B = \frac{2\pi u_1^2}{u_1^2 + u_2^2}$$

- $\sigma_a(r)$, $B(r)$ power law localized, magnetic flux reversal. (N.B. these have divergent energy)





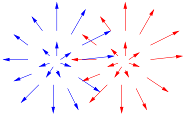
A menagerie of topological solitons

- Skyrmions: penalize coincident vanishing of ψ_1, ψ_2

$$\varphi = [\psi_1 : \psi_2] : \mathbb{R}^2 \to \mathbb{C}P^1 \cong S^2$$

Topological charge $n = (4\pi)^{-1} \int_{\mathbb{R}^2} \varphi \cdot (\partial_1 \varphi \times \partial_2 \varphi)$, $\Phi = 2\pi n$

- These excite relative phase mode $\theta_1 - \theta_2$ (where $\psi_a = |\psi_a| e^{i\theta_a}$)



- scalar **dipole** moment, extra "coherence" length. Can get long range orientation dependent attraction deep in naively type II regime

A menagerie of topological solitons

- $\mathbb{C}P^{k-1}$ skyrmions: penalize coincident vanishing of $\psi_1, \psi_2, \ldots, \psi_k$

$$\varphi = [\psi_1 : \psi_2 : \cdots : \psi_k] : \mathbb{R}^2 \to \mathbb{C}P^{k-1}, \quad \rho = |\Psi|, \quad J$$

- Topological charge $n = (4\pi)^{-1} \int_{\mathbb{R}^2} \varphi^* \omega_{\mathbb{C}P^{k-1}}$, $\Phi = 2\pi n$
- Domain walls: arise when V has (at least) two gauge inequivalent vacua
- Bound states of domain walls and vortices etc. etc.

Do any of these exist in real superconductors?