Vortices and the L^2 volume of spaces of holomorphic maps

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2nd October 2012

Semilocal vortices

- Degree n hermitian line bundle L over compact Riemann surface Σ. Unitary connexion A.
- $k+1$ sections $\varphi = (\varphi_0, \ldots, \varphi_k)$
- Energy $E = \frac{1}{2e^2}\|iF_A\|^2 + \|d_A\varphi\|^2 + \frac{e^2}{2}\|1-|\varphi|^2\|^2$
  $e > 0$ a parameter
- Neat fact $\langle iF_A, |\varphi|^2\omega\rangle_{L^2} = \|\partial_A\varphi\|^2 - \|\bar\partial_A\varphi\|^2$
- Bogomol'nyi bound

$$E \quad = \quad 2\|\bar\partial_A\varphi\|^2 + \frac{1}{2e^2}\|iF_A - e^2(1-|\varphi|^2)\|^2 + \int_\Sigma iF_A \geq 2\pi n$$

equality iff

$$\begin{aligned} \bar\partial_A\varphi &= 0 \qquad (BOG1) \\ *iF_A - e^2(1-|\varphi|^2) &= 0 \qquad (BOG2) \end{aligned}$$

Semilocal vortices

- Solutions called "semilocal vortices". No solutions if $\mathrm{Vol}(\Sigma) < 2\pi n/e^2$
- Moduli space

$$\mathcal{M} = \{\text{Solutions of BOG}\}/\text{gauge equivalence}$$

- Nice description if $2g-2 < n < e^2\mathrm{Vol}(\Sigma)/2\pi$ ($g = \mathrm{genus}(\Sigma)$): there's a rank $r = (k+1)(n-g+1)$ complex vector bundle V over J_Σ such that $\mathcal{M} = \mathbb{P}(V)$
- In particular, it's just a projective space if $g = 0$
- Compact complex mfd of dimension $m = r - 1 + g = n(k+1) - k(g+1)$

Geometry of $\mathcal{M}$

- Natural Riemannian metric

$$\gamma_{\mathcal{M}}((\dot{A},\dot{\varphi}),(\dot{A},\dot{\varphi})) = \frac{1}{4e^2}\|\dot{A}\|^2_{L^2} + \|\dot{\varphi}\|^2_{L^2}$$

where we insist that $(\dot{A},\dot{\varphi})$, solution of LINBOG, is L^2 orthogonal to ∞mal gauge transforms

$$\frac{1}{4e^2}\delta\dot{A} + \langle i\varphi,\dot{\varphi}\rangle = 0 \qquad (G\perp)$$

- Kähler w.r.t. complex structure $i(\dot{A},\dot{\varphi}) = (*\dot{A}, i\dot{\varphi})$
- Baptista has a nice formula for the kähler class

$$[\omega_{\mathcal{M}}] = \pi(\mathrm{Vol}(\Sigma) - \frac{2\pi}{e^2}n)\eta + \frac{2\pi^2}{e^2}\theta$$

where

$$\begin{aligned} \eta &= c_1(S'), \quad S' = \text{antitautological bundle over } \mathbb{P}(V) \\ \theta &= \text{Poincaré dual of } \theta \text{ divisor on } J_\Sigma \end{aligned}$$

- He deduces enough info about $H^*(\mathcal{M},\mathbb{Z})$ to be able to compute $\int_{\mathcal{M}}\eta^{m-i}\wedge\theta^i$, whence

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}) &= \frac{1}{m!}\int_{\mathcal{M}}[\omega_{\mathcal{M}}]^m \\ &= \pi^m\sum_{i=0}^{g}\frac{g!(k+1)^{g-i}}{i!(m-i)!(g-i)!}\left(\frac{2\pi}{e^2}\right)^i\left(\mathrm{Vol}(\Sigma)-\frac{2\pi}{e^2}n\right)^{m-i}\end{aligned}$$

Recall $m=n(k+1)-k(g+1)$

The link with holomorphic maps $\Sigma \to \mathbb{C}P^k$

Bertram-Daskalapoulos-Wentworth

- Consider dense open subset $\mathcal{M}_o = \{[\varphi, A] : \varphi^{-1}(0) = \emptyset\} \subset \mathcal{M}$
- Choose a local section ε of L. Then $\varphi = (f_0\varepsilon, \ldots, f_k\varepsilon)$
- $[f_0, \ldots, f_k]$ is independent of choice of ε, hence globally defined
- Map $\Phi : \Sigma \to \mathbb{C}P^k$ must be holomorphic by BOG1 (choose ε s.t. $\bar{\partial}_A\varepsilon = 0$)
- Hence we have a canonical map $j : \mathcal{M}_o \to \mathcal{H}$, space of degree n holomorphic maps $\Sigma \to \mathbb{C}P^k$

The link with holomorphic maps $\Sigma \to \mathbb{C}P^k$

Bertram-Daskalapoulos-Wentworth

- Conversely, given holo $\Phi : \Sigma \to \mathbb{C}P^k$ can construct associated vortex
- $S =$ tautological bundle over $\mathbb{C}P^k$, $L := \Phi^{-1} S^*$ has degree n
- Given an arbitrary linear map $f : \mathbb{C}^{k+1} \to \mathbb{C}$, $\pi^* f$ is a section of S^*, where $\pi : S \to \mathbb{C}^{k+1}$ is the tautological map
- Choosing $f_i(z_0, \ldots, z_k) = z_i$, get holo sections $\varphi_i = \Phi^* \pi^* f_i$ of L
- Given any hermitian metric on L, $\exists\,!\, A$ s.t. $\bar{\partial}_A = \bar{\partial}^{(L)}$
- Exists unique metric s.t. BOG2 holds (BOG1 automatic)
- Gives canonical map $h : \mathcal{H} \to \mathcal{M}_o$
- $j \circ h = \mathrm{Id}_{\mathcal{H}}$, $h \circ j = \mathrm{Id}_{\mathcal{M}_o}$. Biholomorphism

The link with holomorphic maps $\Sigma \to \mathbb{C}P^k$

- $\mathcal{H}$ also has a natural L^2 metric $\gamma_{\mathcal{H}}$. $\dot{\Phi} \in T_\Phi \mathcal{H} \subset \Gamma(\Phi^{-1} T\mathbb{C}P^k)$

$$\gamma_{\mathcal{H}}(\dot{\Phi}, \dot{\Phi}) = \int_\Sigma |\dot{\Phi}|^2$$

- Convenient to choose FS metric on $\mathbb{C}P^k$ so that Hopf map

$$\mathbb{C}^{k+1} \supset S^{2k+1} \to \mathbb{C}P^k$$

  is a Riemannian submersion (hol. sec. curv. $= 4$)
- Relation between $\gamma_{\mathcal{H}}$ and $\gamma_{\mathcal{M}}$ on $\mathcal{M}_o \equiv \mathcal{H}$?
- Baptista conjectures that $h^* \gamma_{\mathcal{M}} \to \gamma_{\mathcal{H}}$ pointwise on $\mathcal{H}$ as $e \to \infty$

Baptista's conjecture

- Motivated by naive limit of BOG2, $G\perp$ and defn of $\gamma_{\mathcal{M}}$:

$$\begin{aligned} 1-|\varphi|^2 &= 0 & (BOG2)_{e\to\infty} \\ \langle i\varphi,\dot{\varphi}\rangle &= 0 & (G\perp)_{e\to\infty} \\ \gamma_{\mathcal{M}}((\dot{A},\dot{\varphi}),(\dot{A},\dot{\varphi})) &= \int_\Sigma |\dot{\varphi}|^2 \end{aligned}$$

  $\dot{\varphi}$ tangent to unit sphere in L^{k+1}, pointwise orthogonal to gauge orbits, so $|\dot{\varphi}|^2 = |\dot{\Phi}|^2_{FS}$
  ($S^{2k+1}\to\mathbb{C}P^k$ is a Riemannian submersion)
- Of course, not rigorous

Baptista's conjecture

- For each e, can compute

$$\mathrm{Vol}(\mathcal{M}_o) = \int_{\mathcal{M}_o} \frac{\omega_{\mathcal{M}}^m}{m!} = \mathrm{Vol}(\mathcal{M})$$

  using explicit formula
- Natural conjecture:

$$\mathrm{Vol}(\mathcal{H}) = \lim_{e \to \infty} \mathrm{Vol}(\mathcal{M}_o) = \frac{n^g}{m!} (\pi \mathrm{Vol}(\Sigma))^m$$

- Similar conjecture for Einstein-Hilbert action ($\int_{\mathcal{H}} scal$) of $\mathcal{H}$

Why would anyone care about $(\mathcal{H}, \gamma_{\mathcal{H}})$?

- $\mathcal{H}$ is a soliton moduli space in its own right: sigma model "lumps"
- Holo maps $\Phi : (M, \text{cokähler}) \to (N, \text{kähler})$ globally minimize Dirichlet energy $E_d = \|d\Phi\|^2$ in their htpy class (Lichnerowicz):

$$\begin{aligned} F(\Phi) &:= \|\partial\Phi\|^2 - \|\bar{\partial}\Phi\|^2 = \langle \omega_M, \Phi^*\omega_N \rangle_{L^2} \\ \frac{d}{dt}F(\Phi_t) &= \langle \omega_M, d(\Phi_t^* \iota_{\dot{\Phi}} \omega_N) \rangle_{L^2} = 0 \\ E_d(\Phi, \text{holo}) &= F(\Phi) = F(\Phi' \sim \Phi) \le E_d(\Phi') \end{aligned}$$

- Stable static solutions of ungauged sigma model on $\mathbb{R} \times M$

$$S = \int_{\mathbb{R}} (\|\dot{\Phi}\|^2 - E_d(\Phi))$$

- Geodesic approximation (Ward, after Manton): Given $\Phi(0) \in \mathcal{H}$, $\dot{\Phi}(0) \in T_{\Phi(0)}\mathcal{H}$ small, expect $\Phi(t)$ to be well-approximated by geodesic in $(\mathcal{H}, \gamma_{\mathcal{H}})$.
- Physical interpretation: antiferromagnetic films $(\Sigma \to S^2)$

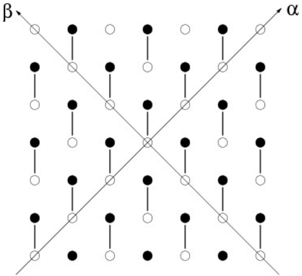
Heisenberg antiferromagnet

- Square lattice of unit spins $\mathbf{n} : \mathbb{Z}^2 \to S^2$, constant $J > 0$

$$\dot{\mathbf{n}}_{ij} = \mathbf{n}_{ij} \times \frac{\partial H}{\partial \mathbf{n}_{ij}}, \qquad H = \sum_{ij} J \mathbf{n}_{ij} \cdot (\mathbf{n}_{i,j+1} + \mathbf{n}_{i+1,j})$$

- Continuum limit? Chessboard, dimerize

$$\Phi_{\alpha,\beta} := \frac{1}{2}(\mathbf{n}_{white} - \mathbf{n}_{black})$$
$$\Psi_{\alpha,\beta} := \frac{1}{2}(\mathbf{n}_{white} + \mathbf{n}_{black}) \approx 0$$



Heisenberg antiferromagnet

- Lattice spacing $\varepsilon \to 0$, eliminate Ψ, rescale time

$$\Phi \times \Box \Phi = 0$$

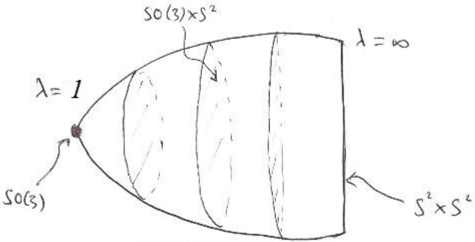
  where $\Box = \partial_t^2 - J^2(\partial_x^2 - \partial_y^2)$.
- (Relativistic) sigma model, target S^2!
- Doping (J position dependent): Sigma model on a curved background $g_\Sigma = (dx^2 + dy^2)/J^2$
- Direct physical interpetation of geodesics in $(\mathcal{H}, \gamma_{\mathcal{H}})$ at least for $k = 1$ - magnetic bubble dynamics

L^2 geometry of $\mathcal{H}$

- Back to general setting $\mathcal{H} = \{\text{holo maps } \Sigma \to \mathbb{C}P^k \text{ of degree } n\}$
- $\mathcal{H}$ complex manifold if n large compared to g
- Noncompact, incomplete, kähler (w.r.t. $\gamma_{\mathcal{H}}$)
- Explicit formula for $\gamma_{\mathcal{H}}$ in simplest nontrivial case, $\Sigma = S^2$, $k = 1$
    - $\mathcal{H} = \mathrm{Rat}_1 \subset \mathbb{C}P^3$

$$\Phi(z) = \frac{a_0 z + a_1}{a_2 z + a_3}$$

    - $\mathcal{H} = PSL(2,\mathbb{C}) = TSO(3)$, $G = SO(3) \times SO(3)$ acts isometrically, cohomogneity 1



Orbits of λz

L^2 geometry of $\mathcal{H}$

- - Volume $\pi^6/3!$, consistent with Baptista's conjecture
  - Unbounded scalar and holomorphic sectional curvature (no smooth isometric compactification)
  - Set of G invariant kähler metrics on Rat_1 is interesting. Includes some with infinite volume (e.g. complete Ricci flat Stenzel metric), and FS metric on $\mathrm{Rat}_1 \subset \mathbb{C}P^3$
  - They all have hidden isometry $df : TSO(3) \to TSO(3)$, where $f(x) = x^{-1}$. Has strong consequences for spectrum of Laplacian on Rat_1.
- Can we find any other checks on Baptista's volume formula for $\mathcal{H}$?
- Look for cohomogeneity 1 examples: $\Sigma = S^2$, $n = 1$, general k

$\mathcal{H}_{1,k}$

$$\Phi([z_0,z_1]) = [a_0 z_0 + b_0 z_1, \ldots, a_k z_0 + b_k z_1] \leftrightarrow \left[\begin{pmatrix} a_0 & b_0 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix}\right]$$

- $\mathcal{H}_{1,k} \hookrightarrow \mathbb{C}P^{2k+1}$
- γ invariant under action of $G = U(k+1) \times U(2)$, $[M] \mapsto [U_1 M U_2^{-1}]$
- Cohomogeneity one. Each orbit has unique representative

$$\Phi_\mu([z_0,z_1]) = [\mu z_0, z_1, 0, \ldots, 0], \quad \mu \geq 1$$

- Single exceptional orbit $\mu = 1$. All others diffeo G/K where $K = T^3 \times U(k-1)$

$\mathcal{H}_{1,k}$

- γ uniquely determined by one-parameter family of symmetric bilinear forms

$$\gamma_\mu : V_\mu \times V_\mu \to \mathbb{R}$$

where $V_\mu = T_{\Phi_\mu}\mathcal{H}$

- $Ad(G)$ invariant inner product on $\mathfrak{g} = \mathfrak{u}(k+1) \oplus \mathfrak{u}(2)$

$$\langle (A,B),(A',B')\rangle = -\frac{1}{2}(\operatorname{tr} AA' + \operatorname{tr} BB')$$

Define $\mathfrak{p} = \mathfrak{k}^\perp$, identify $T_{gK}(G/K) = \mathfrak{p}$ (well defined up to $Ad(K)$

- action on $\mathfrak{p}$)

$$\begin{array}{rcl} V_\mu & = & \langle \partial/\partial\mu\rangle \oplus \mathfrak{p} \\ & = & \begin{array}{ccccccccc} \langle \partial/\partial\mu\rangle \oplus \mathfrak{p}_0 & \oplus & \mathfrak{p}_\mu & \oplus & \widetilde{\mathfrak{p}}_\mu & \oplus & \hat{\mathfrak{p}} & \oplus & \check{\mathfrak{p}} \\ 1 & & 1 & & 1 & & k-1 & & k-1 \end{array} \end{array}$$

- $Ad(K)$ invariance, hermiticity implies

$$\gamma_\mu = A_0(\mu)(d\mu^2 + 8\mu^2\langle,\rangle_{\mathfrak{p}_0}) + A_1(\mu)\langle,\rangle_{\mathfrak{p}_\mu} + A_2(\mu)\langle,\rangle_{\widetilde{\mathfrak{p}}_\mu} + A_3(\mu)\langle,\rangle_{\hat{\mathfrak{p}}} + A_4(\mu)\langle,\rangle_{\check{\mathfrak{p}}}$$

- Kähler implies, for all fixed $X, Y, Z \in \mathfrak{p}$

$$\begin{aligned} \omega([X, Y]_{\mathfrak{p}}, Z) + \text{cyclic perms} &= 0 \\ \frac{d}{d\mu}\omega(X, Y) + \omega(\partial/\partial\mu, [X, Y]_{\mathfrak{p}}) &= 0 \end{aligned}$$

- $\Rightarrow$ there exists positive increasing function $A(\mu)$ and constant $B > 0$ such that

$$A_0 = \frac{A'(\mu)}{4\mu}, \quad A_1 = A_2 = \frac{\mu^2 - 1}{\mu^2 + 1}A, \quad A_3 = B + \frac{A}{2}, \quad A_4 = B - \frac{A}{2}.$$

- Clearly $\lim_{\mu\to\infty} A(\mu)$ exists, $\leq 2B$
- Regularity implies $\lim_{\mu\to 1} A(\mu) = 0$
- **Any** G invariant kähler metric has this structure

$$A_{L^2} = \pi\frac{\mu^4 - 4\mu^2 \log\mu - 1}{(\mu^2 - 1)^2}, \quad B_{L^2} = \frac{\pi}{2}$$

$$A_{FS} = \frac{\mu^2 - 1}{\mu^2 + 1}, \quad B_{FS} = \frac{1}{2}$$

- Straightforward computation

$$\mathrm{vol} = \frac{1}{\sqrt{2}} A^2 (B^2 - A^2/4)^{k-1} \frac{dA}{d\mu} d\mu \wedge \mathrm{vol}_{G/K}$$

- Hence **every** G invariant kähler metric on $\mathcal{H}_{1,k\geq 2}$ has finite volume!

$$\mathrm{Vol}(\mathcal{H}_{1,k}) = 4\sqrt{2}\mathrm{Vol}(G/K) \int_0^{A(\infty)/2B} t^2(1-t^2)^{k-1} dt$$

- Consider the case $A(\infty) = 2B$, as holds for L^2 and FS. Volume depends only on B! Hence L^2 volume = volume of FS metric (of hol sec curv $4/\pi$):

$$\mathrm{Vol}(\mathcal{H}_{1,k}) = \frac{\pi^{4k+2}}{(2k+1)!}$$

which agrees with Baptista's conjecture (we have $\mathrm{Vol}(\Sigma) = \pi$)

$\mathcal{H}_{1,k}$

- Lamia Alqahtani has computed Einstein-Hilbert action of $\mathcal{H}_{1,k}$; also agrees with Baptista's conjecture
- Suggests that $\mathcal{M}$ is the "right" compactification of $\mathcal{H}$ from the viewpoint of L^2 geometry

Dilation cylinders

- Σ, n, k general
- Given degree n meromorphic function W on Σ have dilation cylinder

$$C_W = \{[\mu W, 1, 0, \ldots, 0] : \mu \in \mathbb{C}^\times\} \subset \mathcal{H}_{n,k}$$

- Induced L^2 metric

$$\gamma|_{C_W} = F(\mu) d\mu d\bar{\mu}, \qquad F(\mu) = \int_\Sigma \frac{|W|^2}{(1 + |\mu|^2 |W|^2)^2}$$

- Volume of C_W

$$\begin{aligned}
\mathrm{Vol}(C_W) &= \int_{\mathbb{C}^\times} \left(\int_\Sigma \frac{|W|^2}{(1 + |\mu|^2 |W|^2)^2} \right) \\
&= \int_\Sigma \left(\int_{\mathbb{C}^\times} \frac{|W|^2}{(1 + |\mu|^2 |W|^2)^2} \right) \qquad \text{[Fubini]} \\
&= \int_\Sigma \pi = \pi \mathrm{Vol}(\Sigma) \qquad \text{independent of } W!
\end{aligned}$$

Dilation cylinders

- Gives heuristic support for Baptista's conjecture in some more cases
- $\Sigma = S^2$ (**any** metric)

$$\Phi = [\sum_{j=0}^{n} a_j z_0^j z_1^{n-j}], \qquad a_0, \ldots, a_n \in \mathbb{C}^{k+1}$$

Open dense inclusion $\mathcal{H}_{n,k} \hookrightarrow \mathbb{C}P^{nk+n+k}$

- Assume (pretend) that $\gamma_{\mathcal{H}}$ extends smoothly to $\mathbb{C}P^m$. Then

$$\mathrm{Vol}(\mathcal{H}) = \int_{\mathbb{C}P^m} \frac{\omega_{\mathcal{H}}^m}{m!} = \frac{1}{m!}\left(\int_X \omega_{\mathcal{H}}\right)^m$$

where X is a generator of $H_2(\mathbb{C}P^m, \mathbb{Z})$

- Choose $X = C_W \cup \{0, \infty\}$ where $W = (z_0/z_1)^n$:

$$\int_X \omega_{\mathcal{H}} = \mathrm{Vol}(X) = \pi \mathrm{Vol}(\Sigma)$$

Dilation cylinders

- "Hence"
$$\mathrm{Vol}(\mathcal{H}_{n,k}) = \frac{\pi \mathrm{Vol}(\Sigma)}{(nk+n+k)!}$$
as claimed by Baptista
- Similar argument for $\Sigma = T^2$ (any metric), $n = 2$, $k = 1$, $C_{\wp}$,
$$\mathrm{Rat}_1 \times \Sigma \xrightarrow{4:1} \mathcal{H}_{2,1} \qquad \Phi(z) = \frac{a_0 \wp(z-s) + a_1}{a_2 \wp(z-s) + a_3}$$
also consistent with Baptista's conjecture:
$$\mathrm{Vol}(\mathcal{H}_{2,1}) = \frac{1}{4}(2\pi)\mathrm{Vol}(\Sigma)\left(\frac{1}{3!}(\pi \mathrm{Vol}(\Sigma))^3\right)$$

Concluding remarks

- Why would we care about $\mathrm{Vol}(\mathcal{H})$ (or $\mathrm{Vol}(\mathcal{M})$)?
  - Statistical mechanics of geodesic motion on $\mathcal{H}$ (or $\mathcal{M}$) at large n
  - Controlled by growth of $\mathrm{Vol}(\mathcal{H})$ with n and $\mathrm{Vol}(\Sigma)$
  - Can extract equation of state of a soliton gas (Manton)
- This all assumes that soliton dynamics is well-approximated by geodesic motion in $\mathcal{M}$. Is it?
  - Aim to prove that real dynamics stays (uniformly) ε^2 close to geodesic in $\mathcal{M}$ for times of order ε^{-1} if initial velocities are of order ε
  - Proved for basic vortices on $\mathbb{R}^2$ and monopoles on $\mathbb{R}^3$ (Stuart)
  - Proved for S^2 sigma model on compact Σ (JMS)
  - It's a long and complicated story...

Concluding Remarks

Precise statement of theorem for wave map flow $\mathbb{R} \times \Sigma \to S^2$:

- Let Σ be a compact Riemann surface of genus g and $n \geq g$.
- For fixed $\Phi_0 \in \mathcal{H}$ and $\Phi_1 \in T_{\Phi_0}\mathcal{H}$ consider the one parameter family of wave-map IVPs

$$\Phi(0) = \Phi_0, \qquad \Phi_t(0) = \varepsilon\Phi_1,$$

  parametrized by $\varepsilon > 0$.
- There exist constants $\tau_* > 0$ and $\varepsilon_* > 0$ such that for all $\varepsilon \in (0, \varepsilon_*]$, the problem has a unique solution for $t \in [0, \tau_*/\varepsilon]$.
- Furthermore, the time re-scaled solution

$$\Phi_\varepsilon : [0, \tau_*] \times \Sigma \to S^2, \qquad \Phi_\varepsilon(\tau, x) = \Phi(\tau/\varepsilon, x)$$

  converges uniformly in C^1 to $\Psi : [0, \tau_*] \times \Sigma \to S^2$, the geodesic in $\mathcal{H}$ with the same initial data, as $\varepsilon \to 0$.