Shape modes of planar solitons

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22/4/26

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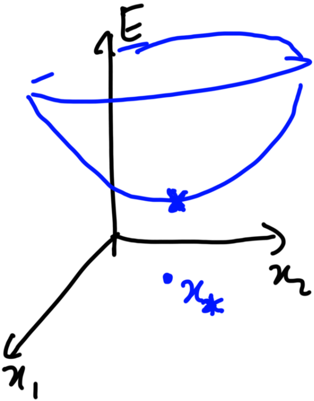
The second variation of an energy functional



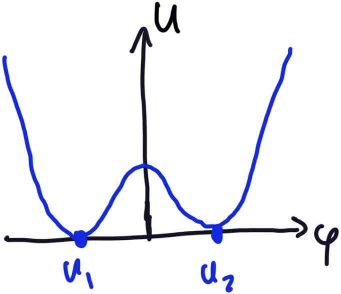
$$H_{ij} = \left.\frac{\partial^2 E}{\partial x_i \partial x_j}\right|_{x_*}$$

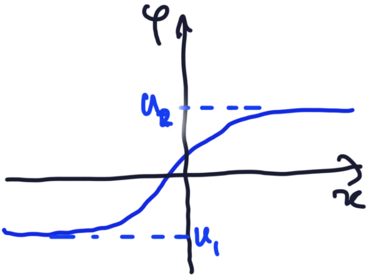
Spectrum $\lambda_1, \lambda_2, \lambda_3, \ldots$

- STABILITY
- VIBRATIONAL DYNAMICS
- PERTURBATIVE QUANTIZATION

Kinks

$$E(\varphi) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + U(\varphi) \right\} dx$$





$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = \int_{-\infty}^{\infty} \varepsilon \left\{ -\frac{d^2\varphi}{dx^2} + U'(\varphi) \right\} dx$$

$$\left(\varepsilon = \partial_t \varphi_t \big|_{t=0} \right)$$

Kinks: second variation

$$E(\varphi) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + U(\varphi) \right\} dx$$

$$\frac{\partial^2 E}{\partial s \partial t}(\varphi_{s,t})\Big|_{s=t=0} = \int_{-\infty}^{\infty} \hat{\varepsilon}\left(-\frac{d^2\varepsilon}{dx^2} + U''(\varphi)\varepsilon\right) dx$$

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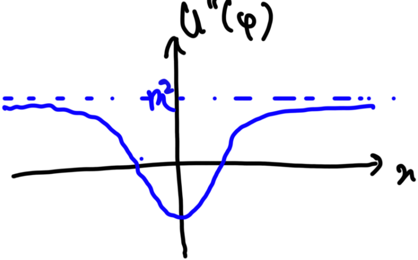

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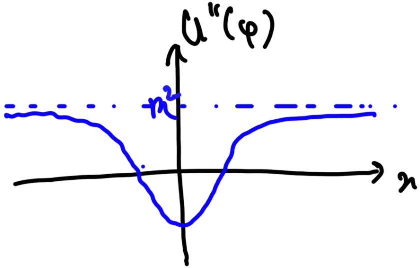
$$\hat{H}\varepsilon = \lambda\varepsilon$$



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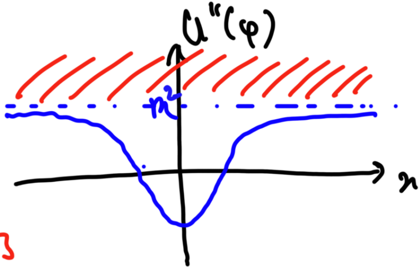
$\lambda \geq m^2$ SCATTERING STATES



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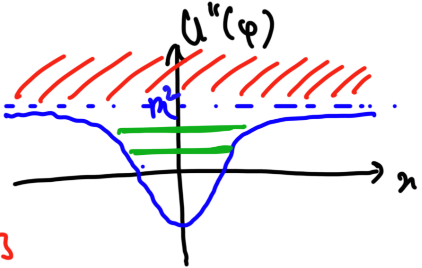
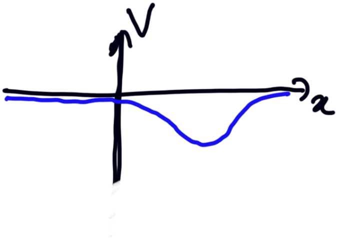
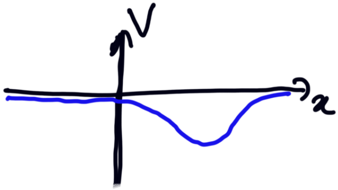
$\lambda < m^2$ BOUND STATES ?



A surprising theorem



Q: WHEN DOES A POTENTIAL WELL HAVE A BOUND STATE?



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A: ALWAYS! (IF DIM = 1, 2)

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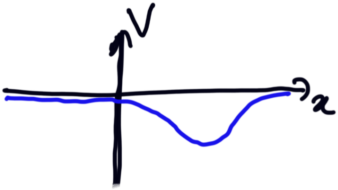

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$$\langle \varepsilon, \hat{H}\varepsilon \rangle_{L^2} / \|\varepsilon\|_{L^2}^2 < 0 .$$

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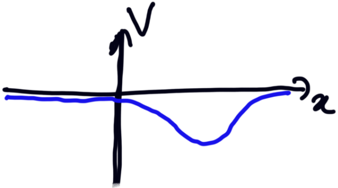

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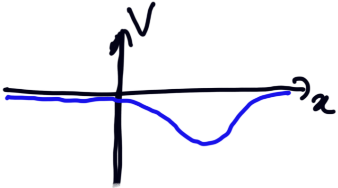

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$\varepsilon = e^{-x^2/R^2}$ will do, $R \gg 0$ DIM = 1

$\varepsilon = e^{-(|x|+1)^\alpha}$ will do, $0 < \alpha \ll 1$ DIM = 2. $\square$

Kinks: so what?

$$\hat{H}\varepsilon = -\frac{d^2\varepsilon}{dx^2} + U''(\varphi(x))\,\varepsilon \qquad \boxed{-\frac{d^2\varphi}{dx^2} + U'(\varphi) = 0}$$

Kinks: so what?

$$\hat{H}\varepsilon = -\frac{d^2\varepsilon}{dx^2} + U''(\varphi(x))\varepsilon$$

$$\boxed{-\frac{d^2\varphi}{dx^2} + U'(\varphi) = 0}$$

$\Rightarrow \hat{H}\varphi' = 0$ $\qquad \varepsilon = \varphi'$ TRANSLATION MODE.

ANY OTHER BOUND STATES?

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ANY OTHER BOUND STATES? DEPENDS....

$U(\varphi) = (1-\varphi^2)^2$ YES!

$U(\varphi) = 1-\cos\varphi$ NO!

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THE SURPRISING THEOREM HAS MORE IMPACT IN DIM 2

Vortices

$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2$$

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$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2$$

$\varphi : \mathbb{R}^2 \to \mathbb{C}$ $\quad (\varphi \in \Gamma(L))$

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$$A = A_1 dx_1 + A_2 dx_2 \in \Omega^1(\mathbb{R}^2)$$

$$(A \in \mathcal{A}(L))$$

Vortices

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$$D\varphi = d\varphi - iA\varphi \quad \left(\begin{array}{c}\text{COVARIANT}\\ \text{DERIVATIVE}\end{array}\right)$$

Vortices

$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2$$

$F_A = dA \in \Omega^2(\mathbb{R}^2)$, MAGNETIC FIELD

(F_A = CURVATURE OF A)

Vortices

$$E(\varphi, A) = \underbrace{\frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2}$$

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^2} |f|^2 d^2x$$

$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2$$

GAUGE INVARIANCE

$$(\varphi, A) \mapsto (e^{i\chi}\varphi, A + d\chi)$$

$$\chi \in C^\infty(\mathbb{R}^2)$$

Vortices

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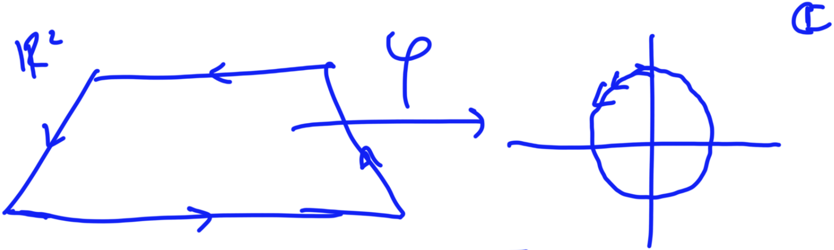

WINDING # $n \in \mathbb{Z}$.

Vortices

$$E(\varphi,A)=\frac{1}{2}\|D\varphi\|_{L^2}^2+\frac{1}{2}\|F_A\|_{L^2}^2+\frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2$$

$E(\varphi,A)\geq \pi n$ WITH EQUALITY $\iff$

$$D_1\varphi+iD_2\varphi=0, \qquad *F_A=\frac{1}{2}(1-|\varphi|^2)$$

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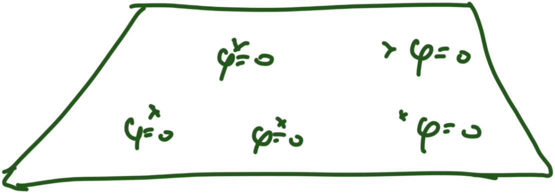
$E(\varphi,A)\geq \pi n$ WITH EQUALITY $\Longleftrightarrow$

$$D_1\varphi+iD_2\varphi=0,\qquad *F_A=\frac{1}{2}(1-|\varphi|^2)$$

$M_n=\mathbb{C}^n$

$\mathbb{R}^2=\mathbb{C}$



Vortices: second variation of the energy

$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2$$

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$\varphi_{s,t}$, $A_{s,t}$

$$\left.\begin{array}{l} \partial_s \varphi_{s,t}|_{0,0} = \hat{\varepsilon} \\ \partial_t \varphi_{s,t}|_{0,0} = \varepsilon \end{array}\right\} \in \Gamma(L) \qquad \left.\begin{array}{l} \partial_s A_{s,t}|_{0,0} = \hat{\alpha} \\ \partial_t A_{s,t}|_{0,0} = \alpha \end{array}\right\} \in \Omega^1(\mathbb{R}^2)$$

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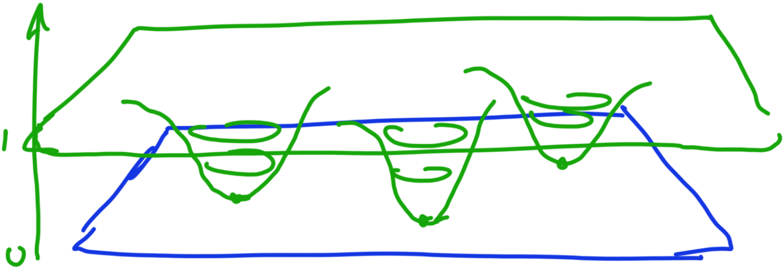
$$\left.\frac{\partial^2}{\partial s\,\partial t} E(\varphi_{s,t}, A_{s,t})\right|_{0,0} = \left\langle (\hat{\varepsilon}, \hat{\alpha}), H(\varepsilon, \alpha) \right\rangle_{L^2}$$

$$H\begin{pmatrix}\varepsilon \\ \alpha\end{pmatrix} = \begin{pmatrix} \Delta_A \varepsilon + \frac{1}{2}(1-|\varphi|^2)\varepsilon + (\varphi,\varepsilon)\varphi + i*(\alpha\wedge *D\varphi + D(*\alpha\varphi)) \\ -*d*d\alpha + |\varphi|^2\alpha + (\varepsilon, iD\varphi) + (\varphi, iD\varepsilon) \end{pmatrix}$$

Another surprising theorem

$$\hat{H} = -\nabla^2 + |\varphi|^2$$



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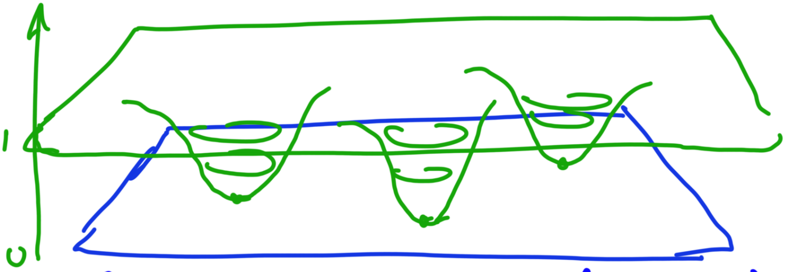

If $\hat{H}\chi = \lambda\chi$, let $(\varepsilon, \alpha) = (i\varphi\chi, d\chi)$.

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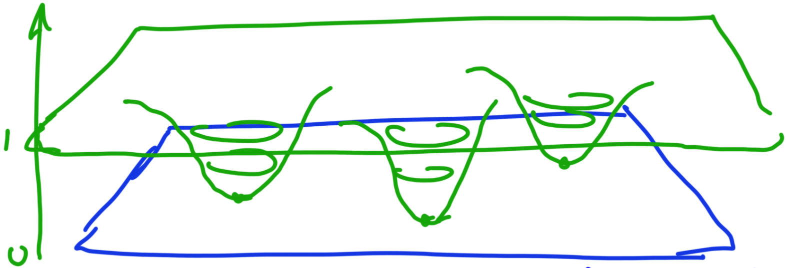

IF $\hat{H}\chi = \lambda\chi$, let $(\varepsilon, \alpha) = (i\varphi\chi, d\chi)$.

THEN $H\begin{pmatrix} i\varepsilon \\ *\alpha \end{pmatrix} = \lambda \begin{pmatrix} i\varepsilon \\ *\alpha \end{pmatrix}$!!!

$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{8}\|1-|\varphi|^2\|_{L^2}^2$$

$$= \frac{1}{2}\left\|\frac{1}{\sqrt{2}}(* + i)D\varphi\right\|_{L^2}^2 + \frac{1}{2}\left\|*F_A - \frac{1}{2}(1-|\varphi|^2)\right\|_{L^2}^2 + \pi n$$

$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|^2_{L^2} + \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{8}\|1-|\varphi|^2\|^2_{L^2}$$

$$= \frac{1}{2}\left\|\frac{1}{\sqrt{2}}(* + i)D\varphi\right\|^2_{L^2} + \frac{1}{2}\left\|*F_A - \frac{1}{2}(1-|\varphi|^2)\right\|^2_{L^2} + \pi n$$

$$= \frac{1}{2}\|\mathrm{Bog}(\varphi, A)\|^2_{L^2} + \pi n$$

$$\mathrm{Bog} : \Gamma(L) \times \mathcal{A}(L) \to \Omega^1(L) \times C^\infty(\mathbb{R}^2)$$

$$\mathrm{Bog}(\varphi, A) = \left(\frac{1}{\sqrt{2}}(* + i)D\varphi,\ *F_A - \frac{1}{2}(1-|\varphi|^2)\right)$$

Vortices: second variation of the energy (take 2)

$$E(\varphi, A) = \frac{1}{2}\|\mathrm{Bog}(\varphi, A)\|_{L^2}^2$$

Vortices: second variation of the energy (take 2)

$$E(\varphi_{s,t}, A_{s,t}) = \frac{1}{2}\|\mathrm{Bog}(\varphi_{s,t}, A_{s,t})\|_{L^2}^2$$

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$$= \left\langle(\hat{\varepsilon},\hat{\alpha}), \mathcal{B}^\dagger\mathcal{B}(\varepsilon,\alpha)\right\rangle_{L^2}$$

$$\mathcal{B} : \Gamma(L)\oplus\Omega^1(\mathbb{R}^2) \to \Omega^1(L)\oplus C^\infty(\mathbb{R}^2)$$

$$(\varepsilon,\alpha) \longmapsto \left(\frac{1}{\sqrt{2}}(*+i)(D\varepsilon - i\alpha\varphi),\ *d\alpha + (\varphi,\varepsilon)\right)$$

$$E(\varphi_{s,t}, A_{s,t}) = \frac{1}{2}\|\mathrm{Bog}(\varphi_{s,t}, A_{s,t})\|_{L^2}^2$$

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$$H = \mathcal{B}^\dagger\mathcal{B} \;!$$

∞'mal G.T.'s: $\qquad \mathcal{G} : C^\infty(\mathbb{R}^2) \longrightarrow \Gamma(L) \oplus \Omega^1(\mathbb{R}^2)$

$$\chi \longmapsto (i\varphi\chi, d\chi)$$

Extending $\mathscr{B}$

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GAUGE INVARIANCE $\Rightarrow$ $H\mathcal{G} = 0$.

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GAUGE INVARIANCE $\Rightarrow \quad H\mathcal{G} = 0.$

$$\Rightarrow \mathcal{G}^\dagger H = 0$$

So $\quad H(\varepsilon,\alpha) = \lambda(\varepsilon,\alpha) \neq 0 \Rightarrow \mathcal{G}^\dagger(\varepsilon,\alpha) = 0$

($L^2 \perp$ GAUGE ORBIT)

Extending $\mathscr{B}$

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$$\tilde{\mathscr{B}} : \Gamma(L) \oplus \Omega^1(\mathbb{R}^2) \to \Omega^1(L) \oplus C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2)$$

$$\tilde{\mathscr{B}} = \mathscr{B} \oplus \mathcal{G}^\dagger$$

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$$\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathcal{G}^\dagger$$

$$\tilde{\mathcal{H}} = \tilde{\mathcal{B}}^\dagger \tilde{\mathcal{B}} = \mathcal{H} + \mathcal{G}\mathcal{G}^\dagger : \mathbb{P} \to \mathbb{P}$$

Extending $\mathscr{B}$

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$$\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathcal{G}^\dagger$$

$$\tilde{\mathcal{H}} = \tilde{\mathcal{B}}^\dagger \tilde{\mathcal{B}} = \mathcal{H} + \mathcal{G}\mathcal{G}^\dagger : \mathcal{P} \to \mathcal{P}$$

$$\hat{\mathcal{H}} = -\nabla^2 + |\varphi|^2 = \mathcal{G}^\dagger \mathcal{G}$$

A symmetry of $\widetilde{H}$

$$\mathfrak{B} : \mathbb{P} \longrightarrow \mathbb{O}$$

$$\Gamma(L) \oplus \Omega^1(\mathbb{R}^2) \qquad\qquad \Omega^1(L) \oplus C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2)$$

$$\mathfrak{B} : \mathbb{P} \longrightarrow \mathbb{O}$$

$$S_1 : \mathbb{P} \longrightarrow \mathbb{P}$$
$$(\varepsilon, \alpha) \longmapsto (i\varepsilon, *\alpha)$$

$$S_2 : \mathbb{O} \longrightarrow \mathbb{O}$$
$$(\xi, f_1, f_2) \longmapsto (*\xi, -f_2, f_1)$$

$$\widetilde{\mathcal{B}} : \mathbb{P} \longrightarrow \mathbb{O}$$

$$S_1 : \mathbb{P} \longrightarrow \mathbb{P}$$

$$(\varepsilon, \alpha) \longmapsto (i\varepsilon, *\alpha)$$

$$S_1^2 = -\mathrm{Id}_{\mathbb{P}}$$

$$\widetilde{\mathcal{B}} S_1 = S_2 \widetilde{\mathcal{B}}$$

$$S_2 : \mathbb{O} \longrightarrow \mathbb{O}$$

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$$S_2^2 = -\mathrm{Id}_{\mathbb{O}}$$

$$\widetilde{\mathcal{B}}^\dagger S_2 = S_1 \widetilde{\mathcal{B}}^\dagger$$

A symmetry of $\widetilde{H}$

$$\widetilde{\mathcal{B}} : \mathbb{P} \longrightarrow \mathbb{O}$$

$$S_1 : \mathbb{P} \to \mathbb{P} \qquad\qquad S_2 : \mathbb{O} \longrightarrow \mathbb{O}$$

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$$S_1^2 = -\mathrm{Id}_{\mathbb{P}} \qquad\qquad S_2^2 = -\mathrm{Id}_{\mathbb{O}}$$

$$\widetilde{\mathcal{B}} S_1 = S_2 \widetilde{\mathcal{B}} \qquad\qquad \widetilde{\mathcal{B}}^{\dagger} S_2 = S_1 \widetilde{\mathcal{B}}^{\dagger}$$

$$\widetilde{H} S_1 = \widetilde{\mathcal{B}}^{\dagger} \widetilde{\mathcal{B}} S_1 = S_1 \widetilde{H}$$

Proof of the (second) surprising theorem

$$\mathbb{P} = \underbrace{\mathrm{Im}\,\mathcal{G}}_{G_{\infty}} \oplus \underbrace{\ker \mathcal{G}^{\dagger}}_{G_{\infty}^{\perp}}$$

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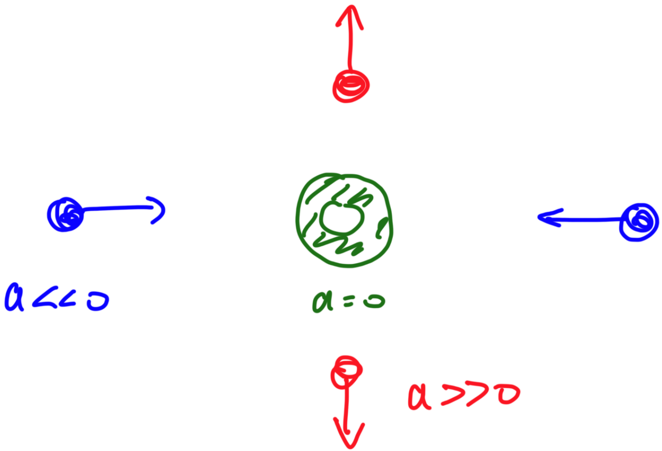
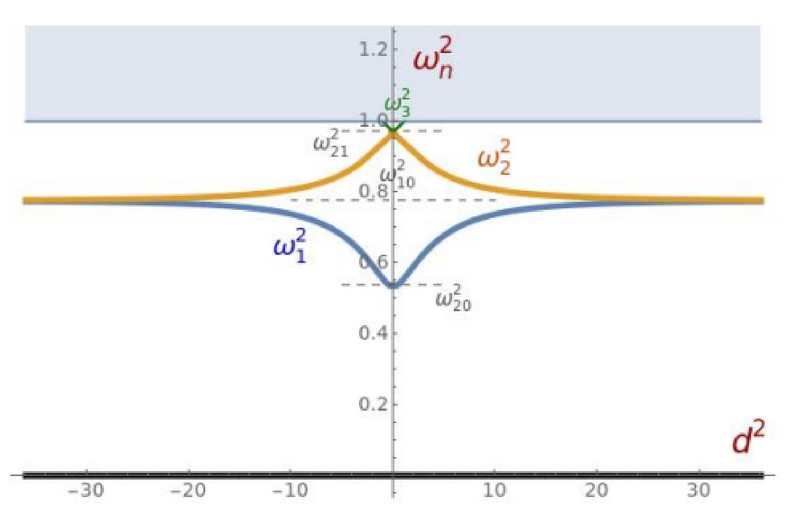
$$\mathcal{G}\mathcal{G}^\dagger x = \lambda x \quad \Rightarrow \quad \tilde{H}\mathcal{G}(x) = \mathcal{G}\mathcal{G}^\dagger \mathcal{G} x = \lambda \mathcal{G}(x), \quad \mathcal{G}(x) \in G_\infty$$

$$p(z) = (z - z_1)(z - z_2) = z^2 + a$$



Two-vortex scattering



$\mathbb{C}$

S^2

Let us "generalize": $\mathbb{C}P^1$ vortices

$\mathbb{C}$

$\mu = \frac{1}{2}(1-|\varphi|^2)$

S^2

$\mu = \underline{n}\cdot\varphi$

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$\mathbb{C}$

$\mu = \frac{1}{2}(1-|\varphi|^2)$

$$D\varphi = d\varphi - iA\varphi$$

$$\begin{cases} *D\varphi + iD\varphi = 0 \\ *F_A - \mu(\varphi) = 0 \end{cases}$$

$$E(\varphi, A) = \frac{1}{2}\|D\varphi\|^2 + \frac{1}{2}\|F_A\|^2 + \frac{1}{2}\|\mu(\varphi)\|^2$$

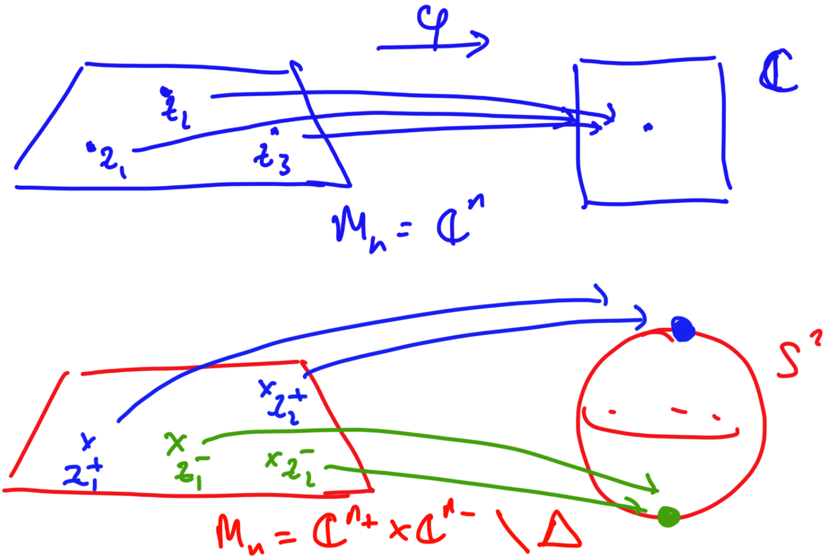
S^2

$\mu = \underline{n}\cdot\underline{\varphi}$

$$D\underline{\varphi} = d\underline{\varphi} - A\,\underline{n}\times\underline{\varphi}$$

$$\begin{cases} *D\underline{\varphi} + \underline{\varphi}\times D\underline{\varphi} = \underline{0} \\ *F_A - \mu(\underline{\varphi}) = 0 \end{cases}$$

Two winding numbers (n_+, n_-)



Let us "generalize": $\mathbb{C}P^1$ vortices

ALL THE ABOVE NATURALLY GENERALIZE...

$\mathcal{B} = d\mathcal{B}og_{(\phi,A)} : \mathbb{P} \to \mathbb{O}$ $\qquad \mathbb{P} = \ldots, \ \mathbb{O} = \ldots$

ALL THE ABOVE NATURALLY GENERALIZE....

$\mathcal{B} = d\mathcal{B}o_{g(\varphi,A)} : \mathbb{P} \to \mathbb{D}$ $\qquad \mathbb{P} = \ldots, \; \mathbb{D} = \ldots$

$\tilde{\mathcal{B}} = \mathcal{B} \oplus G^\dagger, \quad G(\chi) = (\chi \underline{n} \times \underline{\varphi}, d\chi)$

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$$\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathcal{G}^\dagger, \quad \mathcal{G}(x) = (x_1 x_2 x_3, dx)$$

$$\mathcal{H} = \mathcal{B}^\dagger \mathcal{B}, \quad \tilde{\mathcal{H}} = \tilde{\mathcal{B}}^\dagger \tilde{\mathcal{B}}$$

Let us “generalize”: $\mathbb{C}P^1$ vortices

ALL THE ABOVE NATURALLY GENERALIZE....

$$B = d\mathrm{Bog}_{(\varphi,A)} : \mathbb{P} \to \mathbb{O} \qquad \mathbb{P} = \ldots, \quad \mathbb{O} = \ldots$$

$$\tilde{B} = B \oplus G^\dagger, \quad G(\chi) = (\chi \underline{1} \times \varphi, d\chi)$$

$$H = B^\dagger B, \quad \tilde{H} = \tilde{B}^\dagger \tilde{B}$$

$$S_1 : \mathbb{R} \to \mathbb{P} \qquad S_1(\varepsilon, \alpha) = (\varphi \times \varepsilon, \alpha)$$

Let us "generalize": $\mathbb{C}P^1$ vortices

ALL THE ABOVE NATURALLY GENERALIZE....

$$\mathcal{B} = d\mathcal{B}og_{(\varphi,A)} : \mathbb{P} \to \mathbb{D} \qquad \mathbb{P} = \ldots, \quad \mathbb{D} = \ldots$$

$$\widetilde{\mathcal{B}} = \mathcal{B} \oplus \mathcal{G}^\dagger, \quad \mathcal{G}(x) = (x\underline{n} \times \underline{\varphi}, dx)$$

$$H = \mathcal{B}^\dagger \mathcal{B}, \quad \widetilde{H} = \widetilde{\mathcal{B}}^\dagger \widetilde{\mathcal{B}}$$

$$S_1 : \mathbb{R} \to \mathbb{P} \qquad S_1(\underline{\varepsilon}, \alpha) = (\underline{\varphi} \times \underline{\varepsilon}, \star\alpha)$$

$$\hat{H} = \mathcal{G}^\dagger \mathcal{G} = -\nabla^2 + |\underline{n} \times \underline{\varphi}|^2 \leftarrow$$ ALWAYS HAS AT LEAST 1 BOUND STATE

Let us "generalize": $\mathbb{C}P^1$ vortices

ALL THE ABOVE NATURALLY GENERALIZE....

$$B = dB \circ g_{(\varphi,\lambda)} : \mathbb{P} \to \mathbb{D} \qquad \mathbb{P} = \dots, \quad \mathbb{D} = \dots$$

$$\tilde{B} = B \oplus G^\dagger, \qquad G(\chi) = (\chi\, \underline{n} \times \underline{\varphi}, d\chi)$$

$$H = B^\dagger B, \qquad \tilde{H} = \tilde{B}^\dagger \tilde{B}$$

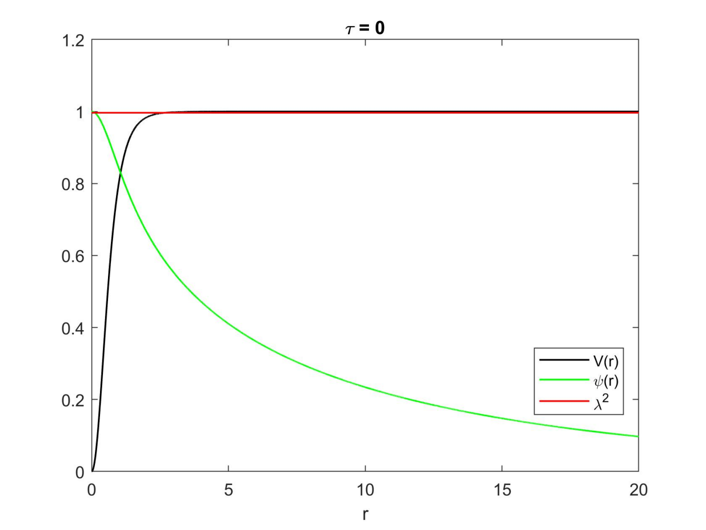
$$S_1 : \mathbb{R} \to \mathbb{P} \qquad S_1(\varepsilon, \alpha) = (\varphi \times \varepsilon, *\alpha)$$

$$\hat{H} = G^\dagger G = -\nabla^2 + |\underline{n} \times \underline{\varphi}|^2$$ ← ALWAYS HAS AT LEAST 1 BOUND STATE

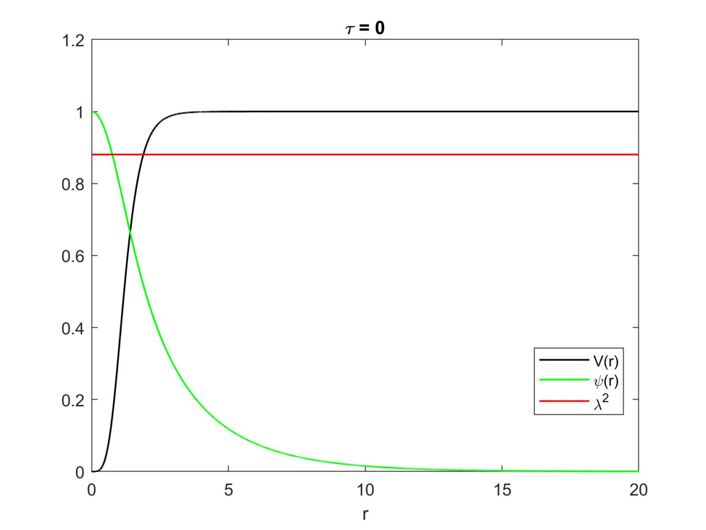
$$\hat{H}\chi = \lambda\chi \Rightarrow H(S_1 G(\chi)) = \lambda S_1 G(\chi)$$

EVERY $(1, n-1)$ VORTEX HAS A SHAPE MODE !!

Shape mode is very lightly bound



$(n_+, n_-) = (2, 0)$: spectral wall?



Credits/bibliography

- The key fact that (for linear vortices) eigenstates of $-\nabla^2 + |\varphi|^2$ generate eigenmodes of H is contained (somewhat hidden) in A. Alonso-Izquierdo, W. Garcia Fuertes, and J. Mateos Guilarte, Phys. Lett. **B 753** (2016), 29–33.
- The explanation given here is based on N. Gavrea, D. Harland and M. Speight, arXiv:2603.21343
- The graph of spectral data for $n = 2$ linear vortices is taken from A. Alonso-Izquierdo, W. Garcia Fuertes, N.S. Manton, and J. Mateos Guilarte, JHEP **2024** (2024), 20.
- The effect of excited shape modes on linear two-vortex scattering was understood in S. Krusch, M. Rees, and T. Winyard, Phys. Rev. D **110** (2024), 056050.
- Completing the generalization to arbitrary kähler targets is Nora's next Ph.D. problem.