The L^2 geometry of moduli spaces of $\mathbb{P}^1$ vortices

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The model

- $X = S^2$, $G = U(1)$, $\mu(\mathbf{n}) = \mathbf{e} \cdot \mathbf{n} - \tau$
- P, principal G-bundle over Σ
- Connexion A on P, section $\mathbf{n}$ of P^X

$$E(\mathbf{n}, A) = \frac{1}{2} \int_\Sigma \left(|\mathrm{d}_A \mathbf{n}|^2 + |F_A|^2 + |\mu(\mathbf{n})|^2 \right)$$

- Vortex equations

$$(V1): \quad \bar{\partial}_A \mathbf{n} = 0, \qquad (V2): \quad *F_A = \sharp\mu(\mathbf{n})$$

- General fact: solutions (if they exist) minimize E in their homotopy class

The model

- Canonical sections $\mathbf{n}_+(p) = \mathbf{e}$, $\mathbf{n}_-(p) = -\mathbf{e}$
- Topological invariants: $k_\pm = \sharp(\mathbf{n}(\Sigma), \mathbf{n}_\pm(\Sigma))$
  Assume $k_+ \geq k_- \geq 0$
- $\deg P = k_+ - k_-$
- $E(\mathbf{n}, A) \geq 2\pi(1-\tau)k_+ + 2\pi(1+\tau)k_-$
  with equality iff $(V1), (V2)$
  - $(+)$-vortices located at $\mathbf{n}^{-1}(\mathbf{e})$, mass $2\pi(1-\tau)$
  - $(-)$-vortices located at $\mathbf{n}^{-1}(-\mathbf{e})$, mass $2\pi(1+\tau)$
- "Bradlow" bound

$$
\begin{aligned}
F_A &= *(\mathbf{e}\cdot\mathbf{n} - \tau) \\
2\pi(k_+ - k_-) &= \int_\Sigma (\mathbf{e}\cdot\mathbf{n} - \tau) \leq (1-\tau)|\Sigma|
\end{aligned}
$$

Existence of vortices

Theorem (Sibner, Sibner, Yang 2000; RG 2019):
Assume $|\Sigma| > 2\pi(k_+ - k_-)/(1-\tau)$. Then for each pair D_+, D_- of disjoint effective divisors on Σ there exists a unique (up to gauge) solution of $(V1), (V2)$ with $\mathbf{n}^{-1}(\pm\mathbf{e}) = D_\pm$. This solution is smooth. If all vortex positions are distinct (all elements of $D_\pm$ have multiplicity 1), the solution depends smoothly on the vortex positions also.

- Moduli space of vortices
  $M_{k_+,k_-}(\Sigma) \equiv [Sym^{k_+}(\Sigma) \times Sym^{k_-}(\Sigma)] \backslash \Delta_{fat}$
- **Noncompact** complex mfd, dim $k_+ + k_-$

Existence of vortices

- Key idea in proof: trade $(V1), (V2)$ for "Taubes" equation

$$u = \log\left(\frac{1 - \mathbf{e}\cdot\mathbf{n}}{1 + \mathbf{e}\cdot\mathbf{n}}\right) : \Sigma \to [-\infty, \infty]$$

$$\Delta_\Sigma u + 2\left(\frac{e^u - 1}{e^u + 1} - \tau\right) + 4\pi\left(\sum_{p\in D_+}\delta_p - \sum_{q\in D_-}\delta_q\right) = 0$$

- Regularize
- Green's function: $(\Sigma\times\Sigma)\backslash\Delta \to \mathbb{R}$, $(p, x) \mapsto G_p(x)$
  - smooth
  - $G_p(x) = G_x(p)$
  - $\int_\Sigma G_p = 0$
  - $\Delta_\Sigma G_p = \delta_p - |\Sigma|^{-1}$
  - In a nbhd of p, $G_p(x) = -(2\pi)^{-1}\log d(p, x) + smooth$
- $v := -4\pi(\sum_{p\in D_+} G_p - \sum_{q\in D_-} G_q)$
- $u = v + h$

Existence of vortices

$$\Delta_\Sigma h + F(v+h) - C_0 = 0$$

$C_0 = 4\pi|\Sigma|^{-1}(k_+ - k_-) \geq 0$
$F(t) = 2(\tanh\frac{t}{2} + \tau)$.

- "Bradlow" bound implies $F(-\infty) < C_0 < F(\infty)$
- $H^1(\Sigma) = \mathcal{X}(\Sigma) \oplus \mathbb{R}$
  - Given $\widetilde{h} \in \mathcal{X}$, there exists unique $c \in \mathbb{R}$ s.t. $\int_\Sigma (F(v + \widetilde{h} + c) - C_0) = 0$
  - $\mathcal{X} \to \mathbb{R}$, $\widetilde{h} \mapsto c(\widetilde{h})$ is weakly cts
  - Apply Leray-Schauder to $T : \mathcal{X} \to \mathcal{X}$, $\widetilde{h} \mapsto H$,

$$\Delta_\Sigma H + F(v + \widetilde{h} + c(\widetilde{h})) - C_0 = 0$$

- Smoothness: bootstrap
- Uniqueness: max principle/monotonicity of F
- Parametric smoothness: IFT

The L^2 metric on $M_{k_+,k_-}(\Sigma)$

- Curve $c(t) = (\mathbf{n}(t), A(t))$ of solns of $(V1), (V2)$
- Project $(\dot{\mathbf{n}}(0), \dot{A}(0))$ $L^2 \perp$ gauge orbit through $(\mathbf{n}(0), A(0))$ [Physics: $\vec{E} = *\dot{A}$ satisfies Gauss's law]
- $\|\dot{c}(0)\|^2 := \int_\Sigma \left(|\dot{\mathbf{n}}(0)|^2 + |\dot{A}(0)|^2\right)$
- Defines a Riemannian metric g on $M_{k_+,k_-}(\Sigma)$. Very natural:
  - Kähler
  - Geodesic flow on $(M_{k_+,k_-}(\Sigma), g) \leftrightarrow$ low energy dynamics of vortices
  - Quantum dynamics of vortices: $i\partial_t\psi = \frac{1}{2}\Delta_g\psi$
  - Statistical mechanics of vortices: $Vol(M_{k_+,k_-}, g)$ in large k_+, k_- limit

The L^2 metric on $M_{k_+,k_-}(\Sigma)$

- How does one compute g in practice?
  - Cover (almost all) Σ with a coordinate patch
  - Consider collection of time dependent vortex trajectories $(z_1(t), \dots, z_{k_-+k_+}(t))$
  - Construct $(\mathbf{n}(t), A(t))$, project
  - Compute $\int_\Sigma (|\dot{\mathbf{n}}(0)|^2 + |\dot{A}(0)|^2)$.
  - Absolutely hopeless
- Amazing fact: $\int_\Sigma$ localizes around vortex positions

The L^2 metric on $M_{k_+,k_-}(\Sigma)$

- $\pi : \Sigma^{k_+ + k_-} \backslash C \to M_{k_+,k_-}(\Sigma)$

Thm(NR, JMS 2018)

$$\pi^* \omega_{L^2} = 2\pi(1-\tau) \sum_{r=1}^{k_+} pr_r^* \omega_\Sigma + 2\pi(1+\tau) \sum_{r=k_+ +1}^{k_+ + k_-} pr_r^* \omega_\Sigma + i\pi \partial b$$

where $b = \sum_r b_r d\bar{z}_r$ and $b_r(z_1, \ldots, z_{k_- + k_+})$ are defined by

$$\pm u(z) = \log |z - z_r|^2 + a_r + \frac{b_r}{2}(\bar{z} - \bar{z}_r) + \frac{\bar{b}_r}{2}(z - z_r) + \cdots$$

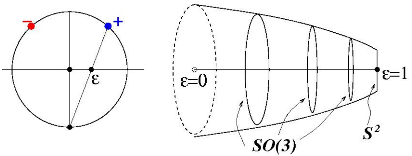
- "Strachan-Samols" localization: gives g almost explicitly on complement of coincidence set
- Moral: to compute g we only need to know how $\mathbf{e} \cdot \mathbf{n}$ (equiv. u, equiv. h) behaves in a nbhd of vortex positions
  In particular: how does $\mathrm{d}h$ at z_r depend on $(z_1, \ldots, z_{k_- + k_+})$?

Volume of $M_{1,1}(S^2_R)$

- $M_{1,1}(\Sigma) = \Sigma \times \Sigma \backslash \Delta$

$$\omega_{L^2} = 2\pi(1-\tau)\omega^+_\Sigma + 2\pi(1+\tau)\omega^-_\Sigma + i\pi\partial b$$

$$b = b_+ d\bar{z}_+ + b_- d\bar{z}_-$$



Thm(NR, JMS 2018, RG, JMS 2019)

$$Vol(M_{1,1}(S^2_R)) = 2\pi(1-\tau)|S^2_R| \times 2\pi(1+\tau)|S^2_R|$$

Proof: symmetry,

$$\omega_{L^2} = A'(\varepsilon)d\varepsilon\wedge\sigma_3 + A(\varepsilon)\sigma_1\wedge\sigma_2 + \frac{c}{1+\varepsilon^2}\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}d\varepsilon\wedge\sigma_1 - \varepsilon\sigma_2\wedge\sigma_3\right)$$

- $Vol = 4\pi^2 \lim_{\varepsilon\to 0} A(\varepsilon) - c^2\pi^2$

- localization formula $\Rightarrow A(\varepsilon)$ in terms of $b_+(\varepsilon, -\varepsilon)$
- it's enough to show $|\mathrm{d}h_\varepsilon| \leq C$ as $\varepsilon \to 0$.
- So we need to understand solution h_ε of

$$\Delta_\Sigma h_\varepsilon + F(v_\varepsilon + h_\varepsilon) = 0$$

where $v_\varepsilon = -2\pi(G_\varepsilon - G_{-\varepsilon})$ in coalescing limit, $\varepsilon \to 0$.

- Naively: $v_\varepsilon \to 0$ pointwise, suggests $h_\varepsilon \to h_0$ where

$$\Delta_\Sigma h_0 + F(h_0) = 0$$

Max principle, monotonicity $\Rightarrow h_0 = c_* = F^{-1}(0)$

Coalescing vortices

Thm(RG, JMS 2019) On any Σ, $\|h_\varepsilon - c_*\|_{C^1} \to 0$ as $\varepsilon \to 0$.
Proof: $h_\varepsilon = \widetilde{h}_\varepsilon + c_\varepsilon$

- $\Delta_\Sigma \widetilde{h}_\varepsilon = -F(v_\varepsilon + \widetilde{h}_\varepsilon + c_\varepsilon) \Rightarrow \|\Delta_\Sigma \widetilde{h}_\varepsilon\|_{L^2} \le C$
- SEE $\Rightarrow \|\widetilde{h}_\varepsilon\|_{H^2} \le C$
- Sobolev $\Rightarrow \|\widetilde{h}_\varepsilon\|_{C^0} \le C$
- $\int_\Sigma F(v_\varepsilon + \widetilde{h}_\varepsilon + c_\varepsilon) = 0 \Rightarrow |c_\varepsilon| \le C$
- Alaoglu/Rellich-Kondrachov/Bolzano-Weierstrass:
  $\widetilde{h}_\varepsilon \rightharpoonup h'$ in H^2, $\widetilde{h}_\varepsilon \to h'$ in H^1, $c_\varepsilon \to c'$
- $v_\varepsilon \to 0$ in L^2, MVT $\Rightarrow F(v_\varepsilon + \widetilde{h}_\varepsilon + c_\varepsilon) \rightharpoonup F(h' + c')$
- h' weak soln of $\Delta_\Sigma h' + F(h' + c') = 0$
- Max principle $\Rightarrow h' = 0$, $c' = c_*$

Thm(RG, JMS 2019) On any Σ, $\|h_\varepsilon - c_*\|_{C^1} \to 0$ as $\varepsilon \to 0$.
Proof cont:

- $\widetilde{h}_\varepsilon \to 0$ in H^1, $c_\varepsilon \to c_*$. MVT

$$|F(v_\varepsilon + \widetilde{h}_\varepsilon + c_\varepsilon) - F(c_*)| \leq 2|v_\varepsilon + \widetilde{h}_\varepsilon + c_\varepsilon - c_*|$$

$$\Rightarrow \|\Delta_\Sigma \widetilde{h}_\varepsilon\|_{L^2} \leq 2(\|v_\varepsilon\|_{L^2} + \|\widetilde{h}_\varepsilon\|_{L^2} + |c - \varepsilon - c_*|) \to 0$$

- SEE $\|\widetilde{h}_\varepsilon\|_{H^2} \to 0$, Sobolev $\|h_\varepsilon\|_{C^0} \to 0$
- Calderon-Zygmund: $\|f\|_{L^p_2} \leq C(\|\Delta_\Sigma f\|_{L^p} + \|f\|_{L^p})$
- LDCT $\Rightarrow$ for all $p > 2$

$$\|h_\varepsilon\|_{L^p_2} \to 0$$

- Sobolev $\Rightarrow \|\widetilde{h}_\varepsilon\|_{C^1} \to 0$. $\square$

Coalescing vortices

- $p_\varepsilon := \partial_\varepsilon h_\varepsilon$

$$\Delta_\Sigma p_\varepsilon + F'(v_\varepsilon + h_\varepsilon)p_\varepsilon = -F'(v_\varepsilon + h_\varepsilon)\partial_\varepsilon v_\varepsilon$$

  Lax-Milgram gives estimate for $\|p_\varepsilon\|_{H^1}$ ...
- **Thm**(RG, JMS 2019) $\|p_\varepsilon\|_{H^3} \leq C/\varepsilon$
- **Cor**(NR, JMS 2018, RG, JMS 2019)
  $(M_{1,1}(\Sigma), g)$ is geodesically incomplete

Compactification of $M_{k_+,k_-}(\Sigma)$

$$[Sym^{k^+}(\Sigma) \times Sym^{k_-}(\Sigma)]\backslash\Delta_{fat} \hookrightarrow Sym^{k^+}(\Sigma) \times Sym^{k_-}(\Sigma)$$

- Can we extend g smoothly to R.H. mfd? No!
- Identify R.H. mfd as moduli space of vortices in a **linear** gauged sigma model
- $X = \mathbb{C}^2$, $G = T^2$, $g_{T^2} = d\theta_1^2/e^2 + d\theta_2^2$

$$(\lambda_1, \lambda_2) : (X_+, X_-) \mapsto (\lambda_1\lambda_2 X_+, \lambda_1 X_-)$$

- Moment map

$$\begin{aligned} \mu_1(X_+, X_-) &= \frac{1}{2}(4 - |X_+|^2 - |X_-|^2) \\ \mu_2(X_+, X_-) &= \frac{1}{2}(2 - 2\tau - |X_+|^2) \end{aligned}$$

Compactification of $M_{k_+,k_-}(\Sigma)$

- Vortex equations

$$\begin{aligned} \overline{\partial^A}\varphi_\pm &= 0 && (1)\\ *F_{A_1} &= \frac{e^2}{2}(4-|\varphi_+|^2-|\varphi_-|^2) && (2)\\ *F_{A_2} &= 1-\tau-\frac{1}{2}|\varphi_+|^2 && (3) \end{aligned}$$

- Provided $k_+ \geq k_- > max\{0, 2genus(\Sigma)-2\}$ and a Bradlow bound is satisfied

$$M^{C^2,e}_{k_+,k_-}(\Sigma) \equiv Sym^{k^+}(\Sigma) \times Sym^{k_-}(\Sigma)$$

- One-parameter family of metrics g_e on $Sym^{k^+}(\Sigma) \times Sym^{k_-}(\Sigma)$
- $\iota^* g_e$: one-parameter family of metrics on $M_{k_+,k_-}(\Sigma)$

Compactification of $M_{k_+,k_-}(\Sigma)$

Conjecture(NR, JMS 2018) $\iota^* g_e$ converges uniformly to g (the L^2 metric on $M_{k_+,k_-}(\Sigma)$) as $e \to \infty$.

- Motivation: forgetful map
  $T : (\varphi_+, \varphi_-, A_1, A_2) \mapsto ([\varphi_+ : \varphi_-], A_2)$ globalizes

$$\Gamma(P^{\mathbb{C}^2}) \times \mathscr{A}(P) \to \Gamma(P_2^{\mathbb{P}^1}) \times \mathscr{A}(P_2)$$

  Formally a Riemannian submersion
- For fixed pair of (disjoint) divisors, apply to $(\varphi_+^e, \varphi_-^e, A_1^e, A_2^e)$ solution of vortex equations
  - $\overline{\partial}_{A_2^e}[\varphi_+^e : \varphi_-^e] = 0$ automatically — solves $(V1)$
  - Expect $|\varphi_+^e|^2 + |\varphi_-^e|^2 = 4 + O(e^{-2})$
  - Then $*F_{A_2^e} = \frac{|\varphi_+^e|^2 - |\varphi_-^e|^2}{|\varphi_+^e|^2 + |\varphi_-^e|^2} - \tau + O(e^{-2})$
  - So $([\varphi_+^e : \varphi_-^e], A_2)$ solves $(V2)$ up to an error of order e^{-2}
- Similar conjecture for **ungauged** maps $\Sigma \to \mathbb{P}^{n-1}$ and $U(1)$ vortices with $X = \mathbb{C}^n$ proved by Liu

Testing the conjecture

- Conjecture implies $Vol(M_{k_+,k_-}(\Sigma), g) = \lim_{e\to\infty} Vol(M^{\mathbb{C}^2,e}_{k_+,k_-}(\Sigma, g_e))$
- Using ideas of Baptista, can write down Kähler class of $M^{\mathbb{C}^2}$ exactly
- Computing

$$Vol(M^{\mathbb{C}^2}, g_e) = \int_{M^{\mathbb{C}^2}} \frac{\omega_e^{k_+ + k_-}}{(k_+ + k_-)!}$$

reduces to an exercise in understanding the cohomology ring of $M^{\mathbb{C}^2} = Sym^{k^+}(\Sigma) \times Sym^{k_-}(\Sigma)$

Testing the conjecture

Thm (NR, JMS 2019)

$$Vol\left(\mathrm{M}^{\mathbb{C}^2,e}_{k_+,k_-}(\Sigma)\right) = \sum_{\ell=0}^{g} \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^{g} \frac{(2\pi)^{2\ell} J_\sigma^{k_\sigma - j_\sigma} K_\sigma^{j_\sigma - \ell}}{(j_\sigma - \ell)!(g-j_\sigma)!(k_\sigma - j_\sigma)!}.$$

where

$$\begin{aligned}
J_+ &:= 2\pi(1-\tau)|\Sigma| - 4\pi^2(k_+ - k_-),\\
J_- &:= 2\pi(1+\tau)|\Sigma| - 4\pi^2 e^{-2} k_- + 4\pi^2(k_+ - k_-),\\
K_+ &:= 4\pi^2,\\
K_- &:= 4\pi^2(1+e^{-2}).
\end{aligned}$$

Testing the conjecture

- So the conjecture implies

$$Vol\,(\mathrm{M}_{k_+,k_-}(\Sigma)) = \sum_{\ell=0}^{g} \frac{g!(g-\ell)!}{(-1)^\ell \ell!} \prod_{\sigma=\pm} \sum_{j_\sigma=\ell}^{g} \frac{(2\pi)^{2\ell} J_\sigma^{k_\sigma - j_\sigma} K_\sigma^{j_\sigma - \ell}}{(j_\sigma - \ell)!(g-j_\sigma)!(k_\sigma - j_\sigma)!}.$$

where

$$\begin{aligned} J_+ &:= 2\pi(1-\tau)|\Sigma| - 4\pi^2(k_+ - k_-), \\ J_- &:= 2\pi(1+\tau)|\Sigma| + 4\pi^2(k_+ - k_-), \\ K_\pm &:= 4\pi^2. \end{aligned}$$

- This is consistent with $Vol(M_{1,1}(S^2_R))$.
- Also checked (RG) $M_{k_+,0}(S^2_R)$, $M_{0,k_-}(S^2_R)$
- No cases with $genus(g) > 0$ have been checked. Note that the conjecture implies

$$Vol(M_{1,1}(\Sigma)) \neq 2\pi(1-\tau)|\Sigma| \times 2\pi(1+\tau)|\Sigma|$$

when $g = 1$

Testing the conjecture

- Can also compute Einstein-Hilbert action of $M^{\mathbb{C}^2,e}_{k_+,k_-}(\Sigma)$ explicitly.
- Get conjectures for EH of $M_{k_+,k_-}(\Sigma)$.
- **None** of these have been checked.