The geometry of the space of vortices

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Dynamics of topological solitons

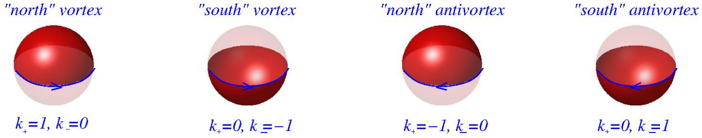
- "Topological solitons" arise quite generically in relativistic classical field theory
- Special situation: theory is "self dual"
  - static solitons exert no net force on each other
  - large moduli space of static n-solitons M_n
  - Examples: instantons, calorons, monopoles, lumps, **vortices**
- Dynamics: **geodesic** motion in M_n w.r.t. a natural Riemannian metric (Manton)
- Goal: understand this metric
- Today: interesting case - model with two different species of vortex

Vortices

- $\mathbf{n} : \Sigma = \mathbb{R}^2 \to S^2$, $A \in \Omega^1(\Sigma)$, $B = \mathrm{d}A$, $\mathrm{d}_A\mathbf{n} = \mathrm{d}\mathbf{n} - A\mathbf{e} \times \mathbf{n}$

$$E = \frac{1}{2}\int_\Sigma \left(|\mathrm{d}_A\mathbf{n}|^2 + |B|^2 + (\mathbf{e}\cdot\mathbf{n})^2\right)$$

- Flux quantization: $\int_\Sigma B = 2\pi k$
- Two species of (anti)vortex:



$k_\pm = \sharp(\mathbf{n}(\Sigma), \pm\mathbf{e})$, $\qquad k = k_+ - k_-$

Bogomol'nyi argument

$$\begin{aligned} 0 &\leq \frac{1}{2}\int_{\Sigma}\left(|\mathrm{d}_A\mathbf{n}(e_1)+\mathbf{n}\times\mathrm{d}_A\mathbf{n}(e_2)|^2+|*B-\mathbf{e}\cdot\mathbf{n}|^2\right) \\ &= E-2\pi(k_+-k_-) \end{aligned}$$

- So $E \geq 2\pi(k_+ - k_-)$ with equality iff

$$\bar{\partial}_A\mathbf{n}=0, \qquad *B=\mathbf{e}\cdot\mathbf{n} \qquad (VE)$$

- First order system $\implies$ Euler-Lagrange eqns
- Cf BPS monopoles on Σ^3, instantons on Σ^4

Existence of vortices

- **Theorem**(Yang 1999 / Sibner-Sibner-Yang 2000)

Gauge equivalence classes of solns of (VE)	$\leftrightarrow$	$D_+ = \{z_1^+, z_2^+, \ldots, z_{k_+}^+\}$, $D_- = \{z_1^-, z_2^-, \ldots, z_{k_-}^-\}$, $D_+ \cap D_- = \emptyset$

where $D_\pm = \mathbf{n}^{-1}(\pm\mathbf{e})$.

- Moduli space of vortices
  $M_{k_+,k_-}(\Sigma) = (Sym^{k_+}\Sigma \times Sym^{k_-}\Sigma)\backslash\Delta_{k_+,k_-}$
- If $k_+k_- \neq 0$, $M_{k_+,k_-}(\Sigma)$ is **noncompact** even if Σ compact
- Simplest example
  $M_{1,1}(\Sigma) = \{(z_+, z_-) \in \Sigma^2 : z_+ \neq z_-\} = (\Sigma \times \Sigma)\backslash\Delta$

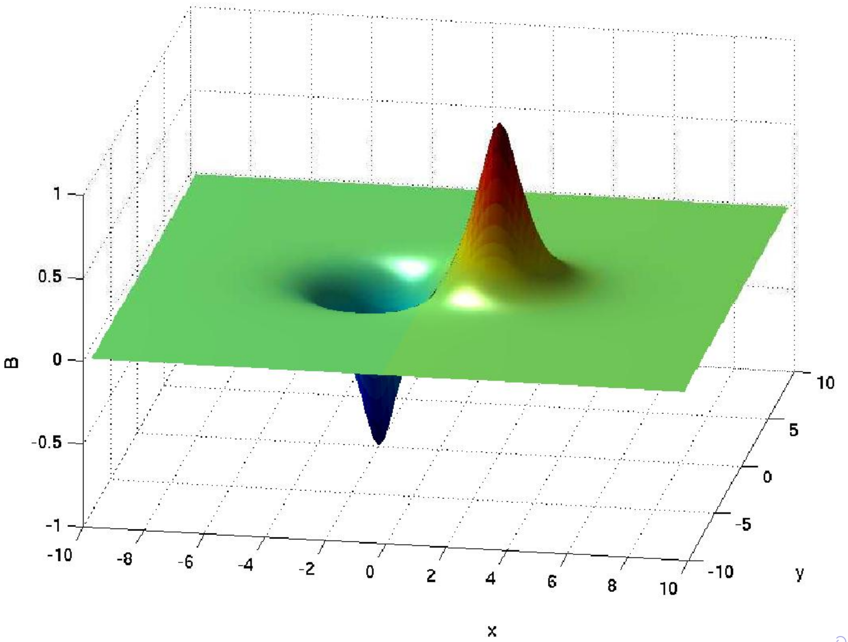
Taubes equation

$$\bar{\partial}_A \mathbf{n} = 0, \qquad *B = \mathbf{e} \cdot \mathbf{n}$$

- $h := \log\left(\frac{1-\mathbf{e}\cdot\mathbf{n}}{1+\mathbf{e}\cdot\mathbf{n}}\right)$
- h has logarithmic singularities at (anti)vortex positions $z_i^{\pm}$

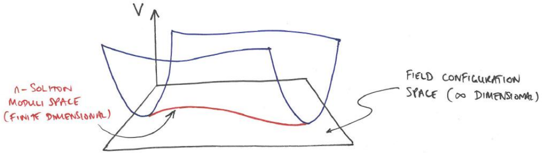
$$\Delta h - 2\tanh\frac{h}{2} = 4\pi\left(\sum \delta(z - z_i^{+}) - \sum \delta(z - z_i^{-})\right)$$

Taubes equation



The L^2 metric on M_{k_+,k_-}: dynamics

$$S = \frac{1}{2}\int_{\mathbb{R}\times\Sigma}\left(D_\mu\mathbf{n}\cdot D^\mu\mathbf{n} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - (\mathbf{e}\cdot\mathbf{n})^2\right) = \int_{\mathbb{R}}(T-E)dt$$



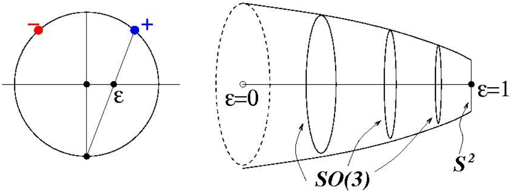
- Restrict S to curve $(\mathbf{n}(t), A(t))$ of solutions of (VE)
- **Geodesic** motion on M_{k_+,k_-} w.r.t. metric γ with

$$\|(\dot{\mathbf{n}}, \dot{A})\| = \int_\Sigma (|\dot{\mathbf{n}}|^2 + |\dot{A}|^2)$$

- **Fact**: γ is Kähler and *localizes* (a la Strachan-Samols)... Have "explicit" formula for γ in terms of behaviour of $\mathbf{n}\cdot\mathbf{e}$ at (anti)vortex positions

What do we know about γ?

$$M_{1,1}(S^2_R) = (S^2 \times S^2)\backslash\Delta = [SO(3) \times (0,1)] \cup [S^2 \times \{1\}]$$



$$\gamma = A(\varepsilon)\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\sigma_1^2 + \frac{1+\varepsilon^2}{1-\varepsilon^2}\sigma_2^2\right) - \frac{A'(\varepsilon)}{\varepsilon}\left(\mathrm{d}\varepsilon^2 + \varepsilon^2\sigma_3^2\right)$$

- Complete? Finite volume? Need $A(\varepsilon)$ as $\varepsilon \to 0$
- Behaviour of $\mathbf{e}\cdot\mathbf{n}$ at $z = \pm\varepsilon$

The regularized Taubes equation

- Taubes eqn on S^2_R, $h := \log\left(\frac{1-\mathbf{e}\cdot\mathbf{n}}{1+\mathbf{e}\cdot\mathbf{n}}\right)$

$$\Delta_{\mathbb{R}^2} h - \frac{8R^2 \tanh(h/2)}{(1+|z|^2)^2} = 4\pi(\delta(z-\varepsilon) - \delta(z+\varepsilon))$$

- h has logarithmic singularities at (anti)vortex positions
- Regularize: $\tilde{h} := h - \log\left|\frac{z-\varepsilon}{z+\varepsilon}\right|^2$
- Dilate: $\hat{h}(z) = \tilde{h}(z/\varepsilon)$
- $\hat{h}$ satisfies a nice elliptic PDE on S^2

$$\Delta_{S^2}\hat{h} - 8R^2\varepsilon^2\left(\frac{1+|z|^2}{1+\varepsilon^2|z|^2}\right)^2 \frac{|z-1|^2 e^{\hat{h}} - |z+1|^2}{|z-1|^2 e^{\hat{h}} + |z+1|^2} = 0$$

- Control behaviour of $\partial_1 \hat{h}|_{z=1}$ using elliptic estimates...

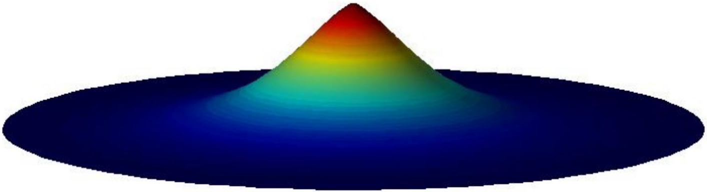
Geometry of $M_{1,1}(S^2)$

- **Theorem** $(M_{1,1}(S^2_R), \gamma)$ has volume $(2\pi)^2(4\pi R^2)^2$
- **Theorem** $(M_{1,1}(S^2_R), \gamma)$ is geodesically incomplete
- **Conjecture** $(M_{k_+,k_-}(S^2), \gamma)$ has volume

$$(2\pi)^{k_+ + k_-}[A - 2\pi(k_+ - k_-)]^{k_+}[A + 2\pi(k_+ - k_-)]^{k_-}$$

  where $A = area(S^2)$.
- Vortex thermodynamics...

Geometry of $M_{1,1}(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^\times$



Interesting generalizations

- Replace target S^2 by a general kähler mfd X with holomorphic, hamiltonian action of G
- Moment map $\mu : X \to \mathfrak{g}^*$
- Principal G bundle $P \to \Sigma$, connexion A

$$E = \frac{1}{2}\int_\Sigma \left(|\mathrm{d}_A\varphi|^2 + |F_A|^2 + |\mu\circ\varphi|^2\right)$$

- Solutions of $\overline{\partial}_A\varphi = 0$, $*F_A = \sharp\mu\circ\varphi$ minimize E in their homotopy class.
- Our case: $X = S^2$, $G = S^1$, $\mu(\mathbf{n}) = \mathbf{e}\cdot\mathbf{n} + \tau$
  breaks vortex-antivortex symmetry
- $G = T^n$, X toric: should still have metric localization
  - e.g. T^2 action on $\mathbb{C}P^2$: now have **three** types of vortex, which can coalesce pairwise, provided all 3 types never coalesce together
- Chern-Simons deformation almost completely unexplored (only $X = \mathbb{C}$, $G = U(1)$)