The geometry of the space of BPS vortex-antivortex pairs

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Motivation

- Vortices: simplest topological solitons in gauge theory (2D, $U(1)$, $\mathbb{C}$)
- Nice generalization: Higgs field takes values in a kähler mfd X with hamiltonian action of gauge group G
- G action on X can have more than one fixed point: more than one species of vortex
- Different species can coexist in stable equilibrium — but can't coincide
- **Noncompact** vortex moduli spaces (even on compact domains)
- Completeness? Finite volume? Curvature properties?
- Simplest version already interesting: $X = S^2$, $G = U(1)$

$\mathbb{C}P^1$ vortices on a Riemann surface Σ

- Fix $\mathbf{e} \in S^2$ (e.g. $\mathbf{e} = (0,0,1)$)
  $G = U(1)$ acts on S^2 by rotations about $\mathbf{e}$
- $P \to \Sigma$ principal G bundle, degree $n \geq 0$, connexion A
- $\mathbf{n}$ section of $P \times_G S^2$
- Canonical sections $\mathbf{n}_\infty(x) = \mathbf{e}$, $\mathbf{n}_0(x) = -\mathbf{e}$
- **Two** integer topological invariants of a section $\mathbf{n}$:

  $$n_+ = \#(\mathbf{n}(\Sigma), \mathbf{n}_\infty(\Sigma)), \qquad n_- = \#(\mathbf{n}(\Sigma), \mathbf{n}_0(\Sigma))$$

  Constraint: $n = n_+ - n_-$ (so we're assuming $n_+ \geq n_-$)
- Energy

  $$E = \frac{1}{2}\int_\Sigma \left(|d_A\mathbf{n}|^2 + |F_A|^2 + (\mathbf{e}\cdot\mathbf{n})^2\right)$$

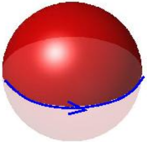
  where, in a local trivialization

  $$d_A\mathbf{n} = d\mathbf{n} - A\mathbf{e}\times\mathbf{n}, \qquad F_A = dA$$

  Aside: $\mu(\mathbf{n}) = -\mathbf{e}\cdot\mathbf{n}$ is moment map for gauge action

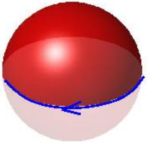
(Anti)vortices

"north" vortex



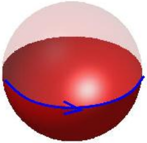
$n_+=1,\ n_-=0$

"north" antivortex



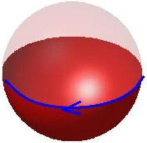
$n_+=-1,\ n_-=0$

"south" vortex



$n_+=0,\ n_-=-1$

"south" antivortex



$n_+=0,\ n_-=1$

“Bogomol’nyi” bound (Schroers)

- Given $(\mathbf{n}, A)$ define a two-form on Σ

$$\Omega(X, Y) = (\mathbf{n} \times d_A\mathbf{n}(X)) \cdot d_A\mathbf{n}(Y)$$

- Let $e_1, e_2 = Je_1$ be a local orthonormal frame on Σ. Then

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}(|d_A\mathbf{n}(e_1)|^2 + |d_A\mathbf{n}(e_2)|^2) + \frac{1}{2}|F_A|^2 + \frac{1}{2}(\mathbf{e}\cdot\mathbf{n})^2 \\
&= \frac{1}{2}|d_A\mathbf{n}(e_1) + \mathbf{n} \times d_A\mathbf{n}(e_2)|^2 + \frac{1}{2}|F_A - *\mathbf{e}\cdot\mathbf{n}|^2 \\
&\qquad + *(\Omega + \mathbf{e}\cdot\mathbf{n}F_A) \\
\implies \quad E &\geq \int_\Sigma (\Omega + \mathbf{e}\cdot\mathbf{n}F_A)
\end{aligned}$$

- Claim: last integral is a homotopy invariant of $(\mathbf{n}, A)$

“Bogomol’nyi” bound

- Suffices to show this in case $D = \mathbf{n}^{-1}(\{\mathbf{e}, -\mathbf{e}\}) \subset \Sigma$ finite
- On $\Sigma \backslash D$ have global one-form $\xi = \mathbf{e} \cdot \mathbf{n}(A - \mathbf{n}^* d\varphi)$ s.t.

$$\Omega + \mathbf{e} \cdot \mathbf{n} F_A = d\xi$$

- Hence

$$\begin{aligned} \int_\Sigma (\Omega + \mathbf{e} \cdot \mathbf{n} F_A) &= \int_{\Sigma \backslash D} (\Omega + \mathbf{e} \cdot \mathbf{n} F_A) \\ &= \lim_{\varepsilon \to 0} \sum_{p \in D} - \oint_{C_\varepsilon(p)} \xi \\ &= 2\pi (n_+ + n_-) \end{aligned}$$

“Bogomol’nyi” bound

- Hence $E \geq 2\pi(n_+ + n_-)$ with equality iff

$$\begin{aligned} \bar{\partial}_A \mathbf{n} &= 0 \qquad (V1) \\ *F_A &= \mathbf{e} \cdot \mathbf{n} \qquad (V2) \end{aligned}$$

- Note solutions of $(V1)$ certainly have D finite (and $n_\pm \geq 0$)

Existence: Yang ($\mathbb{C}$)/Sibner-Sibner-Yang (compact Σ)

- If Σ compact, there's a "Bradlow" obstruction

$$2\pi(n_+ - n_-) = \int_\Sigma F_A = \int_\Sigma \mathbf{e}\cdot\mathbf{n} < \mathrm{Vol}(\Sigma)$$

- **Theorem:** Let $n_+ \geq n_- \geq 0$ and $2\pi(n_+ - n_-) < \mathrm{Vol}(\Sigma)$. For each pair of disjoint effective divisors D_+, D_- in Σ of degrees n_+, n_- there exists a unique gauge equivalence class of solutions of (V1), (V2) with $\mathbf{n}^{-1}(\pm\mathbf{e}) = D_\pm$.
- Moduli space of vortices: $\mathsf{M}_{n_+,n_-} \equiv M_{n_+} \times M_{n_-} \backslash \Delta_{n_+,n_-}$
- If $n_- > 0$, M_{n_+,n_-} is noncompact (in an interesting way)

The "Taubes" equation

$$u = \frac{n_1 + in_2}{1 + n_3}, \qquad h = \log|u|^2, \qquad g_\Sigma = \Omega(z) dz d\bar{z}$$

- h finite except at $\pm$ vortices, $h = \mp\infty$.
- $(V1) \Rightarrow A_{\bar{z}} = -i\frac{\partial_{\bar{z}} u}{u}$, eliminate A from $(V2)$

$$\nabla^2 h - 2\Omega \tanh\frac{h}{2} = 0$$

away from vortex positions

- $(+)$ vortices at z_r^+, $r = 1, \ldots, n_+$, $(-)$ vortices at z_r^-, $r = 1, \ldots, n_-$

$$\nabla^2 h - 2\Omega \tanh\frac{h}{2} = 4\pi\left(\sum_r \delta(z - z_r^+) - \sum_r \delta(z - z_r^-)\right)$$

- Consider $(1,1)$ vortex pairs on $\Sigma = \mathbb{C}$

Solving the (1,1) Taubes equation (numerically)

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta(z-\varepsilon) - \delta(z+\varepsilon)\right)$$

- Regularize: $h = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}$

$$\nabla^2\widehat{h} - 2\frac{|z-\varepsilon|^2 e^{\widehat{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\widehat{h}} + |z+\varepsilon|^2} = 0$$

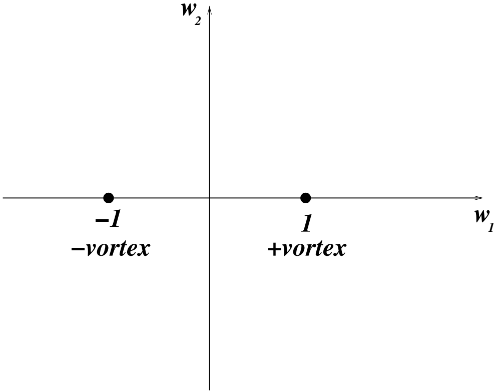
- Rescale: $z =: \varepsilon w$

$$\nabla_w^2\widehat{h} - 2\varepsilon^2\frac{|w-1|^2 e^{\widehat{h}} - |w+1|^2}{|w-1|^2 e^{\widehat{h}} + |w+1|^2} = 0$$

- Solve with b.c. $\widehat{h}(\infty) = 0$

Solving the (1,1) Taubes equation (numerically)

- Symmetry:



Solving the (1,1) Taubes equation (numerically)

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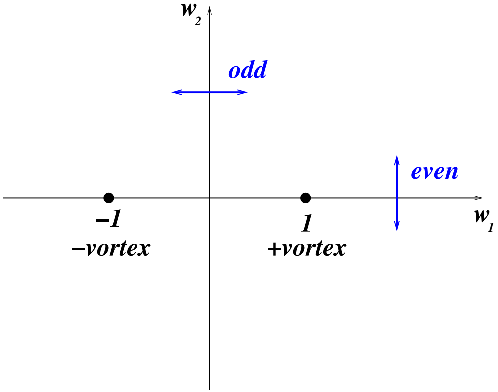

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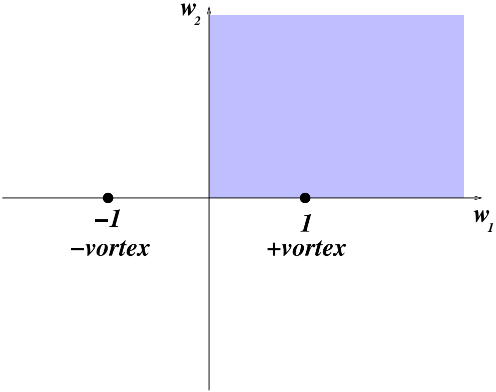

Solving the (1,1) Taubes equation (numerically)

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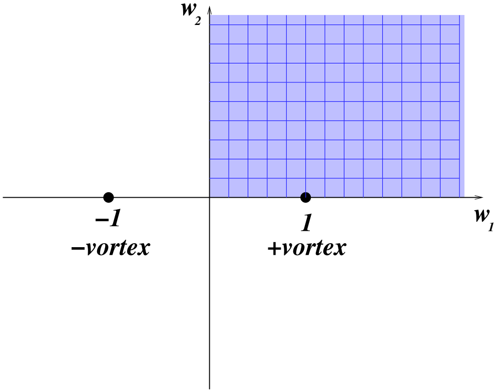

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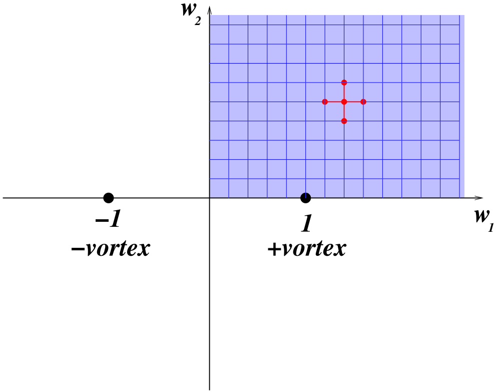
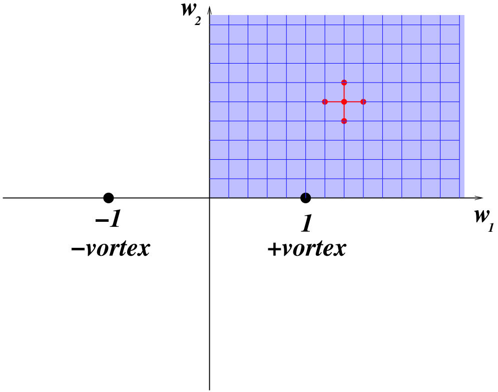


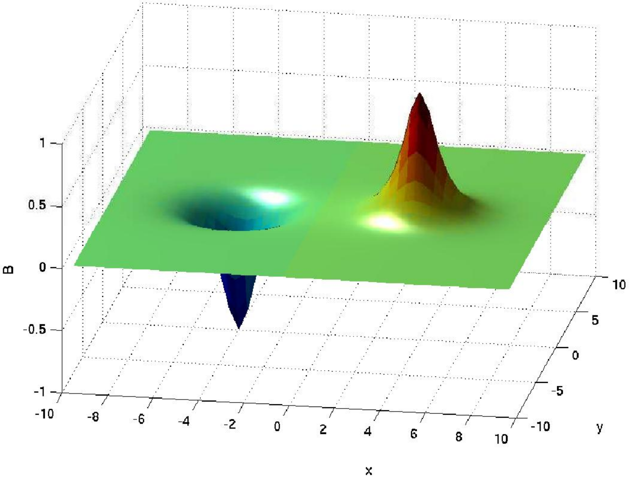
Solving the (1,1) Taubes equation (numerically)

- Symmetry:



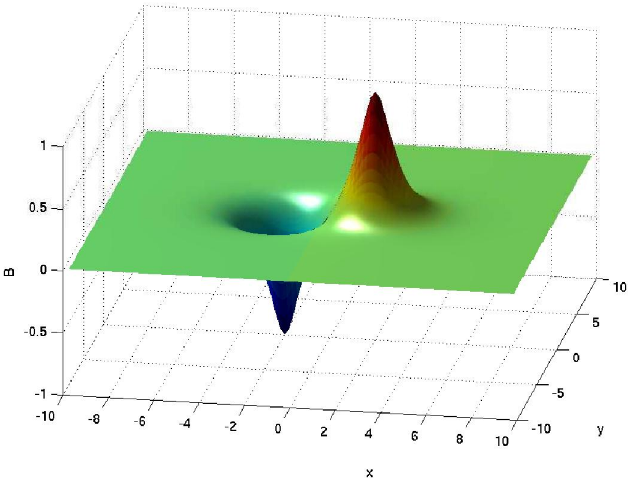
- $F(\widehat{h_{ij}}) = 0$, solve with Newton-Raphson

(1,1) vortices



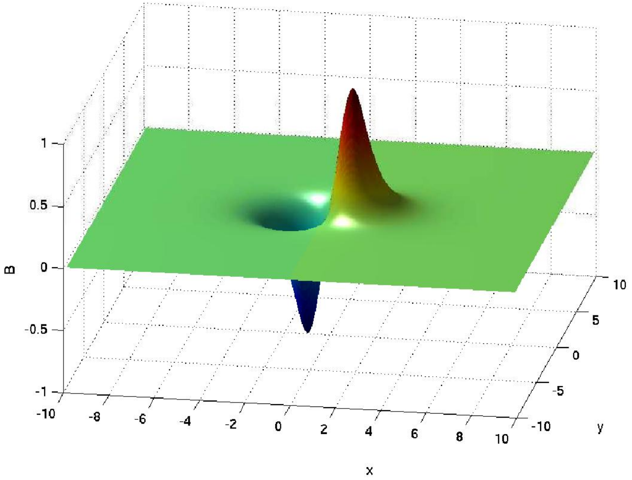
$\varepsilon = 4$

(1,1) vortices



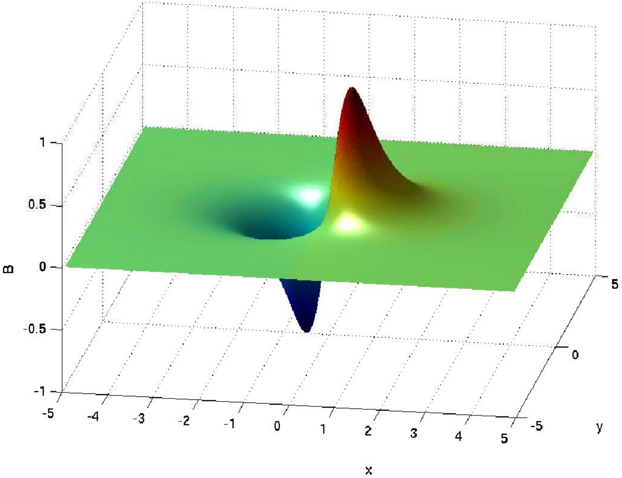
$\varepsilon = 2$

(1,1) vortices



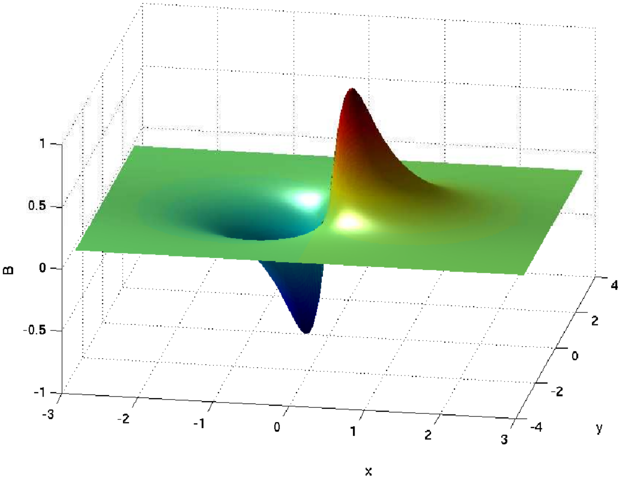
$\varepsilon = 1$

(1,1) vortices



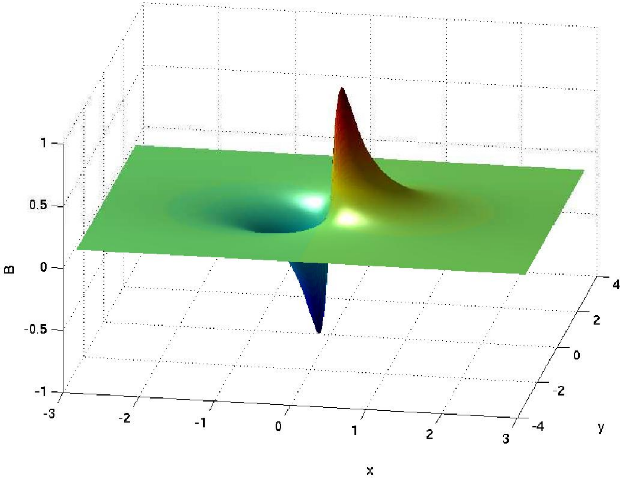
$\varepsilon = 0.5$

(1,1) vortices



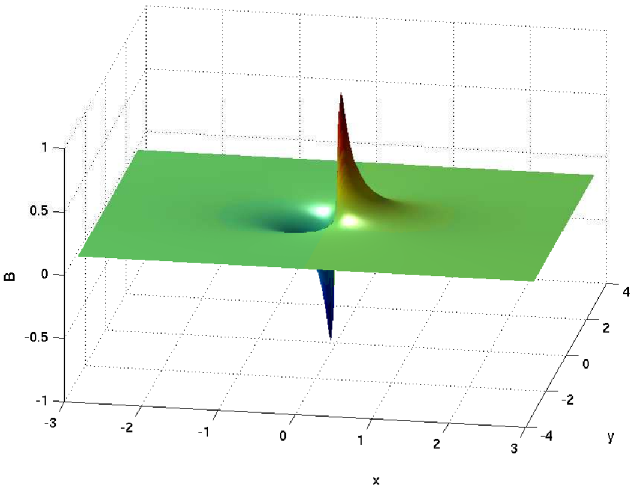
$\varepsilon = 0.3$

(1,1) vortices



$\varepsilon = 0.15$

(1,1) vortices



$\varepsilon = 0.06$

The metric on M_{n_+,n_-}

- Restriction of kinetic energy

$$T = \frac{1}{2}\int_\Sigma |\dot{\mathbf{n}}|^2 + |\dot{A}|^2$$

  to M_{n_+,n_-} equips it with a Riemannian metric
- Expand solution h of Taubes eqn about ± vortex position z_s:

$$\pm h = \log|z - z_s|^2 + a_s + \frac{1}{2}\bar{b}_s(z - z_s) + \frac{1}{2}b_s(\bar{z} - \bar{z}_s) + \cdots$$

- $b_r(z_1, \ldots, z_{n_+ + n_-})$ (unknown) complex functions
- **Proposition** (Romão-JMS, following Strachan-Samols):

$$g = 2\pi\left\{\sum_r \Omega(|z_r|)|dz_r|^2 + \sum_{r,s}\frac{\partial b_s}{\partial z_r}dz_r d\bar{z}_s\right\}$$

  Holds on any Riemann surface (including $\mathbb{C}$)

The metric on $M_{1,1}(\mathbb{C})$

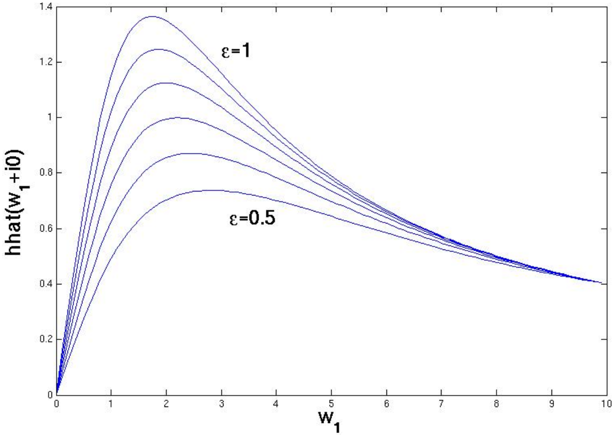
- $M_{1,1} = (\mathbb{C} \times \mathbb{C}) \backslash \Delta = \mathbb{C}_{com} \times \mathbb{C}^{\times}$
- $M^0_{1,1} = \mathbb{C}^{\times}$

$$g^0 = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d}{d\varepsilon} (\varepsilon b(\varepsilon)) \right) (d\varepsilon^2 + \varepsilon^2 d\psi^2)$$

where $b(\varepsilon) = b_+(\varepsilon, -\varepsilon)$

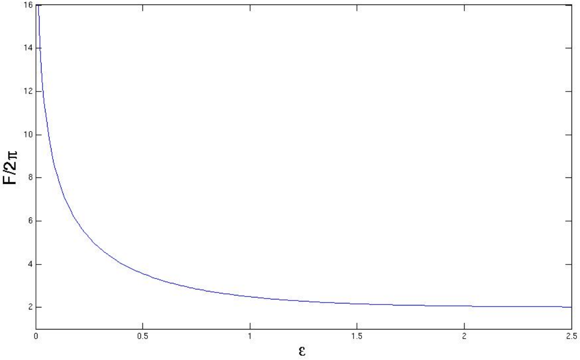
- $\varepsilon b(\varepsilon) = \frac{\partial \widehat{h}}{\partial w_1}\Big|_{w=1} - 1$
- Can easily extract this from our numerics

The metric on $M_{1,1}(\mathbb{C})$



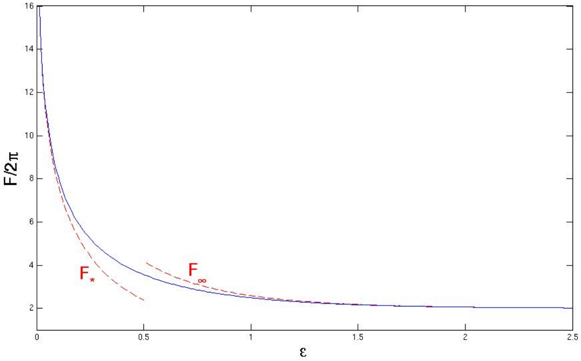
$$\varepsilon b(\varepsilon) = \left.\frac{\partial \widehat{h}}{\partial w_1}\right|_{w=1} - 1$$

The metric on $M_{1,1}(\mathbb{C})$



$$F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d(\varepsilon b(\varepsilon))}{d\varepsilon} \right)$$

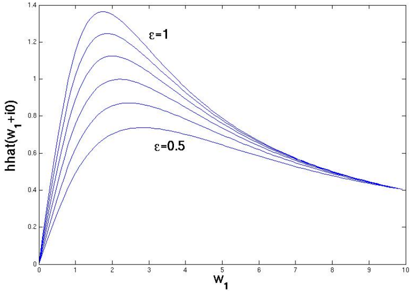
The metric on $M_{1,1}(\mathbb{C})$: conjectured asymptotics



$$F_*(\varepsilon) = 2\pi\left(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)\right) \sim -8\pi \log \varepsilon$$

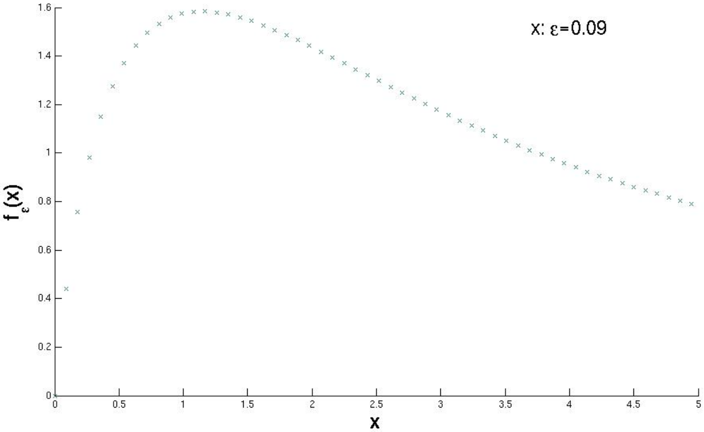
$$F_\infty(\varepsilon) = 2\pi\left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon)\right)$$

Self similarity as $\varepsilon \longrightarrow 0$



- Suggests $\widehat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w)$ for small ε, where f_* is fixed?
- Define $f_\varepsilon(z) := \varepsilon^{-1}\widehat{h}_\varepsilon(\varepsilon^{-1}z)$

Self similarity as $\varepsilon \longrightarrow 0$



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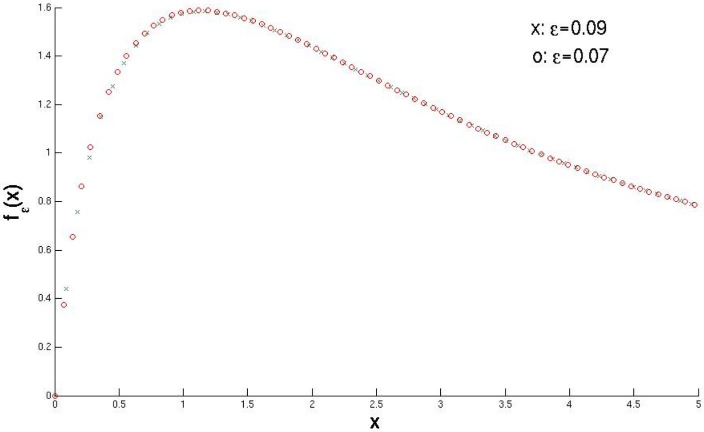

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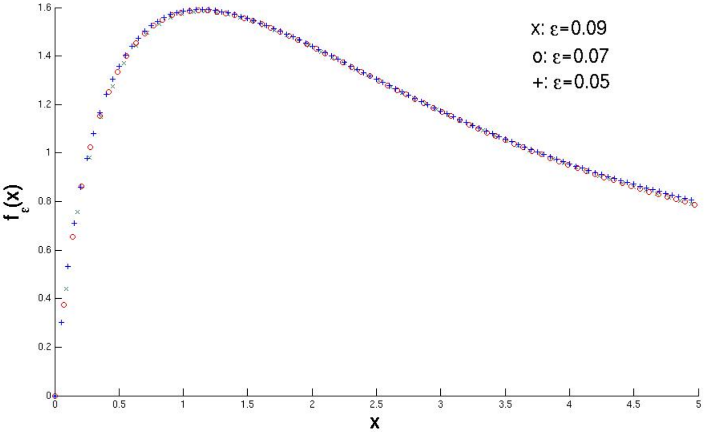

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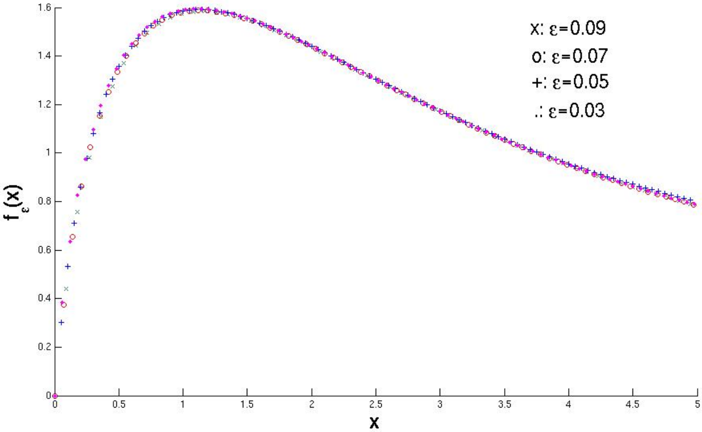

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 \widehat{h})(w) = 2\varepsilon^2 \frac{|w-1|^2 e^{\widehat{h}(w)} - |w+1|^2}{|w-1|^2 e^{\widehat{h}(w)} + |w+1|^2}$$

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 \widehat{h})(w) = 2\varepsilon^2 \frac{|w-1|^2 e^{\widehat{h}(w)} - |w+1|^2}{|w-1|^2 e^{\widehat{h}(w)} + |w+1|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 f_\varepsilon)(z) = \frac{2}{\varepsilon} \frac{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} + |z+\varepsilon|^2}$$

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- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \longrightarrow 0$

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \longrightarrow 0$

Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \longrightarrow 0$
- Screened inhomogeneous Poisson equation, source $-4\cos\theta/r$

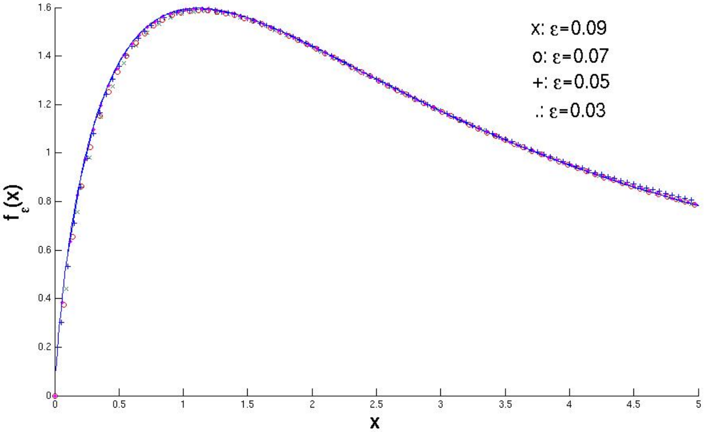
Self similarity as $\varepsilon \longrightarrow 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \longrightarrow 0$
- Screened inhomogeneous Poisson equation, source $-4\cos\theta/r$
- Unique solution (decaying at infinity)

$$f_*(re^{i\theta}) = \frac{4}{r}(1 - rK_1(r))\cos\theta$$

Self similarity as $\varepsilon \longrightarrow 0$



The metric on $M^0_{1,1}$

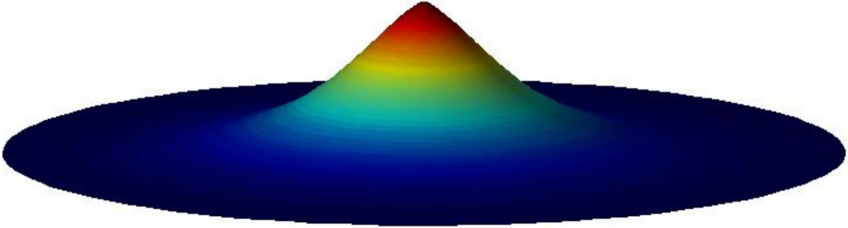
- Predict, for small ε,

  $$\widehat{h}(w_1 + i0) \approx \varepsilon f_*(\varepsilon w_1) = \frac{4}{w_1}(1 - \varepsilon w_1 K_1(\varepsilon w_1))$$

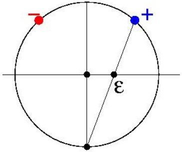
  whence we extract predictions for $\varepsilon b(\varepsilon)$, $F(\varepsilon)$

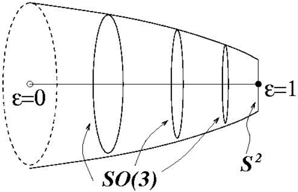
  $$g^0 = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2)$$

- Conjecture: $F(\varepsilon) \sim -8\pi \log \varepsilon$ as $\varepsilon \to 0$
- $M_{1,1}$ is **incomplete**, with unbounded curvature



Vortices on S^2: $M_{1,1}(S^2)$





- $M_{1,1} = S^2 \times S^2 \backslash \Delta = (0,1) \times SO(3) \sqcup \{1\} \times S^2$
- g is $SO(3)$-invariant, kähler, and invariant under $(z_+, z_-) \mapsto (z_-, z_+)$
- Every such metric takes the form

$$g = -\frac{Q'(\varepsilon)}{\varepsilon}(d\varepsilon^2 + \varepsilon^2\sigma_3^2) + Q(\varepsilon)\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\sigma_1^2 + \frac{1+\varepsilon^2}{1-\varepsilon^2}\sigma_2^2\right),$$

  for $Q : (0,1] \to \mathbb{R}$ decreasing with $Q(1) = 0$.
- Once again, can deduce $Q(\varepsilon)$ from $\partial\widehat{h}/\partial w_1$ at $w = 1 + i0$

Solving the (1,1) Taubes equation on S^2 (numerically)

$$\nabla^2 h - \frac{8R^2}{(1+|z|^2)^2}\tanh\frac{h}{2} = 4\pi\left(\delta(z-\varepsilon) - \delta(z+\varepsilon)\right)$$

Solving the (1,1) Taubes equation on S^2 (numerically)

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- Regularize: $h = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}$

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- Regularize: $h = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}$
- Rescale: $z =: \varepsilon w$

Solving the (1,1) Taubes equation on S^2 (numerically)

$$\nabla_w^2 \widehat{h} - \frac{8R^2\varepsilon^2}{(1+\varepsilon^2|w|^2)} \frac{|w-1|^2 e^{\widehat{h}} - |w+1|^2}{|w-1|^2 e^{\widehat{h}} + |w+1|^2} = 0$$

- Regularize: $h = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}$
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Solving the (1,1) Taubes equation on S^2 (numerically)

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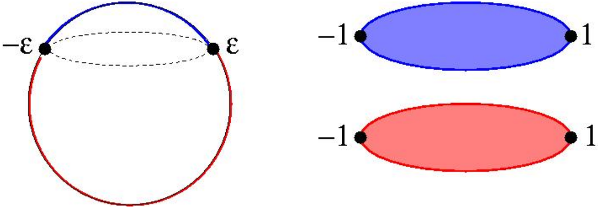
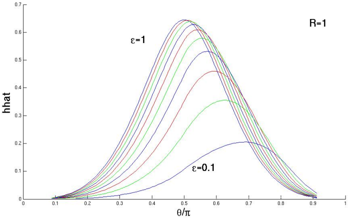
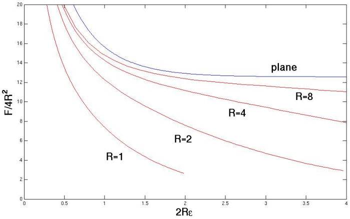
- Regularize: $h = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}$
- Rescale: $z =: \varepsilon w$
- Split S^2 into 2 caps. On lower cap $\varepsilon \mapsto 1/\varepsilon$



Solving the (1,1) Taubes equation on S^2 (numerically)





- $\varepsilon b(\varepsilon) = \widehat{h}_x(1,0) - 1$
- $Q(\varepsilon) = -2\pi\left(1 + 2R^2 + \varepsilon b(\varepsilon) - \dfrac{4R^2}{1+\varepsilon^2}\right)$

The volume of $M_{1,1}(S^2)$

- Formula for g implies finite total volume iff Q is bounded:

$$Vol(M_{1,1}) = \frac{1}{4}(4\pi)^2 \lim_{\varepsilon\to 0} Q(\varepsilon)^2$$

- We can **prove** that $|\widehat{h}_x(1,0)| \leq C\varepsilon$, whence
- **Theorem** (Romão, JMS) Let Σ be a round two-sphere. Then

$$Vol(M_{1,1}(\Sigma)) = (2\pi Vol(\Sigma))^2.$$

The volume of $M_{n,n}(S^2)$

- $M_{n,n}(S^2) = \{\text{disjoint pairs of } n\text{-divisors on } S^2\} = (\mathbb{P}^n \times \mathbb{P}^n) \backslash \Delta$
- Consider gauged **linear** sigma model:
  - fibre $\mathbb{C}^2$
  - gauge group $\widetilde{U}(1) \times U(1) : (\varphi_1, \varphi_2) \mapsto (e^{i(\widetilde{\theta}+\theta)}\varphi_1, e^{i\widetilde{\theta}}\varphi_2)$

$$\begin{aligned} E_{\widetilde{e}} &= \frac{1}{2}\int_\Sigma \left\{ \frac{|\widetilde{F}|^2}{\widetilde{e}^2} + |F|^2 + |\mathrm{d}_{\widetilde{A}}\varphi|^2 + |\mathrm{d}_A\varphi|^2 \right. \\ &\qquad \left. + \frac{\widetilde{e}^2}{4}(4 - |\varphi_1|^2 - |\varphi_2|^2)^2 + \frac{1}{4}(2 - |\varphi_1|^2)^2 \right\} \end{aligned}$$

- For any $\widetilde{e} > 0$, has compact moduli space of (n, n)-vortices

$$M^{lin}_{n,n} = \mathbb{P}^n \times \mathbb{P}^n$$

- Baptista found a formula for $[\omega_{L^2}]$ of $M^{lin}_{n_1,n_2}(\Sigma)$
- Can compute $Vol(M^{lin}_{n,n}(S^2))$ by evaluating $[\omega_{L^2}]$ on $\mathbb{P}^1 \times \{p\}$, $\{p\} \times \mathbb{P}^1$

The volume of $M_{n,n}(S^2)$

$$
\begin{aligned}
E_{\widetilde{e}} &= \frac{1}{2}\int_{\Sigma}\left\{\frac{|\widetilde{F}|^2}{\widetilde{e}^2} + |F|^2 + |\mathrm{d}_{\widetilde{A}}\varphi|^2 + |\mathrm{d}_A\varphi|^2\right. \\
&\qquad \left. + \frac{\widetilde{e}^2}{4}(4 - |\varphi_1|^2 - |\varphi_2|^2)^2 + \frac{1}{4}(2 - |\varphi_1|^2)^2\right\}
\end{aligned}
$$

- Take formal limit $\widetilde{e} \to 0$:
    - $|\varphi_1|^2 + |\varphi_2|^2 = 4$ pointwise
    - $\widetilde{A}$ frozen out, fibre $\mathbb{C}^2$ collapses to $S^3/\widetilde{U}(1) = \mathbb{P}^1$
    - E-L eqn for $\widetilde{A}$ is algebraic: eliminate $\widetilde{A}$ from E_∞

      $$E_\infty = \frac{1}{2}\int_\Sigma |F|^2 + 4\frac{|\mathrm{d}u - iAu|^2}{(1+|u|^2)^2} + \left(\frac{1-|u|^2}{1+|u|^2}\right)^2$$

      where $u = \varphi_1/\varphi_2$
    - Exactly our $\mathbb{P}^1$ sigma model!

The volume of $M_{n,n}(S^2)$

- Leads us to conjecture that

$$Vol(M_{n,n}(S^2)) = \lim_{\tilde{e}\to\infty} Vol(M^{lin}_{n,n}(S^2)) = \frac{(2\pi Vol(S^2))^{2n}}{(n!)^2}$$

  Agrees with $M_{1,1}(S^2_R)$.
- More elaborate choice of linear model gives more general conjecture:

$$Vol(M_{n,m}(S^2)) = \frac{(2\pi)^{n+m}}{n!m!}(Vol(S^2) - \pi(n-m))^n(Vol(S^2) + \pi(n-m))^m$$

- Can generalize to other Σ, Einstein-Hilbert action...
- Similar limit ($\mathbb{C}^k$ fibre, $U(1)$ gauge → *ungauged* $\mathbb{P}^{k-1}$ model) studied rigorously by Chih-Chung Liu.

Summary / What next?

- Case $\Sigma = \mathbb{C}$ is most interesting
- $M_{1,1}(\mathbb{C}) = \mathbb{C} \times \mathbb{C} \backslash \Delta = \mathbb{C}_{com} \times \mathbb{C}^\times$
- Numerics: metric on SoR $\mathbb{C}^\times$, $g^0 = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2)$
- Conjectured asymptotics in small ε region

$$F(\varepsilon) \quad \sim \quad -8\pi \log \varepsilon$$

- Would imply $M_{1,1}(\mathbb{C})$ is incomplete with unbounded scalar curvature
- Can we prove it?
- We can shift the vacuum manifold:

$$\mu(\mathbf{n}) = \tau - \mathbf{e} \cdot \mathbf{n}$$

Case $0 < \tau < 1$ very sparsely explored