Geometry of vortex moduli spaces

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What are vortices?

- Kähler mfd X, hamiltonian G action, $\mu : X \to \mathfrak{g}$
- Riemann surface Σ, principal G bundle $P \to \Sigma$
- A **vortex** is a pair (A, φ), connexion on P, section of $P \times_G X$, s.t.

$$\begin{aligned} \overline{\partial_A}\varphi &= 0 \\ *F_A &= -\mu(\varphi) \end{aligned}$$

- Amazing fact: a vortex (if it exists) minimizes

$$E(A, \varphi) = \frac{1}{2}\left(\|F_A\|^2_{L^2} + \|d_A\varphi\|^2_{L^2} + \|\mu(\varphi)\|^2_{L^2}\right)$$

in its "homology class" [Cieliebak-Gaio-Salamon, Mundet i Riera].
- Moduli space of vortices is an object of some interest:
  - **Equivariant** Gromov-Witten theory
  - "Physics" (actually, physics)
  - Has a natural kähler geometry
- We'll stick to $(X, G) = (\mathbb{C}, U(1)), (S^2, U(1)), (\mathbb{C}^2, T^2)$.

Plain vanilla vortices: $X = \mathbb{C}$, $G = U(1)$

- $\mu(z) = \frac{1}{2}(|z|^2 - \tau)$ (can always shift μ by $\tau \in$ centre of $\mathfrak{g}$)
- (Σ, ω) compact
- $P \to \Sigma$ of degree $n \geq 0$.
- φ a section of $L = P \times_{U(1)} \mathbb{C}$.

$$E = \frac{1}{2}\int_\Sigma |d_A\varphi|^2 + |F_A|^2 + \frac{1}{4}(\tau - |\varphi|^2)^2$$

- Nice identity $\langle F_A, |\varphi|^2\omega\rangle = |\partial_A\varphi|^2 - |\overline{\partial_A}\varphi|^2$
- "Bogomol'nyi" argument:

$$\begin{aligned} E &= \frac{1}{2}\|F_A - \frac{1}{2}(\tau - |\varphi|^2)\omega\|^2_{L^2} + \|\overline{\partial_A}\varphi\|^2_{L^2} + \frac{\tau}{2}\langle F_A, \omega\rangle_{L^2} \\ &\geq \tau\pi n \end{aligned}$$

with equality iff

$$\begin{aligned} \overline{\partial_A}\varphi &= 0 \\ *F_A &= \frac{1}{2}(\tau - |\varphi|^2). \end{aligned}$$

Bradlow's obstruction

- Integrate 2nd vortex equation over Σ:

$$\begin{aligned} \int_\Sigma F_A &= \frac{1}{2}\int_\Sigma (\tau - |\varphi|^2)\omega \\ 2\pi n &= \frac{1}{2}\tau \mathrm{Vol}(\Sigma) - \frac{1}{2}\|\varphi|^2_{L^2} \leq \frac{1}{2}\tau \mathrm{Vol}(\Sigma) \end{aligned}$$

- Hence, if $\mathrm{Vol}(\Sigma) < 4\pi n/\tau$, no solutions exist
- "Dissolved" limit: $\mathrm{Vol}(\Sigma) = 4\pi n/\tau$. Vortices have $\varphi = 0$. Moduli space of vortices $=$ space of constant curvature connexions on L
- Interesting case: $\mathrm{Vol}(\Sigma) > 4\pi n/\tau$

Existence – Bradlow's approach

- Choose and fix holomorphic structure on L and background hermitian fibre metric h_0
- Any other hermitian metric is $h = e^{2u} h_0$ for some $u \in C^\infty(\Sigma)$
- For each h there exists unique metric connexion A s.t. $\overline{\partial_A} = \overline{\partial}_L$
- Choose and fix φ a holomorphic section of L. $\overline{\partial_A} \varphi = 0$ by defn of A
- 2nd vortex equation:

$$\Delta u + \frac{1}{2} h_0(\varphi, \varphi) e^{2u} + (*F_0 - \frac{\tau}{2}) = 0$$

  There exists a unique solution u of this PDE by results of Kazdan-Warner
- Gauge equivalence class of solution uniquely determined by **divisor** of φ, i.e. $\varphi^{-1}(0)$ unordered list of n points in Σ with repeats allowed
- Moduli space of n-vortices $M_n = (\Sigma^n)/S_n$. Has canonical desingularization

Vortices on S^2

- n-vortex $\leftrightarrow$ n unordered marked points on S^2 (repeats allowed)
- Roots of
$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$
- Clearly $P(z) \sim \lambda P(z)$
- n-vortex $\leftrightarrow$ $[a_0, a_1, \ldots, a_n] \in \mathbb{C}P^n$
- $M_n = \mathbb{C}P^n$ as a complex mfd (if $\mathrm{Vol}(\Sigma) > 4\pi n/\tau$)
- Shrinks to a point as $\tau \searrow 4\pi n/\mathrm{Vol}(\Sigma)$

The L^2 metric on M_n

- Any curve $(\varphi(t), A(t))$ of solns of vortex eqns represents a tangent vector v to M_n at $[\varphi(0), A(0)]$
- Length of v? Project $(\dot\varphi(0), \dot A(0)) \in \Gamma(L) \oplus \Omega^1(\Sigma)$ L^2 orthogonal to gauge orbit through $(\varphi(0), A(0))$. Then

$$\|v\|^2 := \|(\dot\varphi(0), \dot A(0))_\perp\|^2_{L^2}$$

- Equips M_n with a Riemannian metric γ
- Fairly obvious that γ is hermitian w.r.t.

$$J : (\dot\varphi, \dot A) \mapsto (i\dot\varphi, *\dot A)$$

  Not so obvious that J coincides with J on Σ^n/S_n
- Even less obvious that γ is **kähler**
- Follows from Strachan-Samols **localization formula** for γ on $M_n \backslash \Delta_n$ ($\Delta_n =$ coincidence set)...
- ...or from high-powered general nonsense [Garcia Prada]

Manton's volume calculation

- $H^2(M_n(S^2)) = H^2(\mathbb{C}P^n) = \mathbb{R}$ so $[\omega_{L^2}] = \alpha[\omega_{FS}]$

$$Vol(M_n) = \int_{M_n} \frac{\omega_{L^2}^n}{n!} = \alpha^n \mathrm{Vol}(\mathbb{C}P^n)_{FS}$$

- Just need constant α
- Consider $S^2 = X \subset M_n$, submfd of **coincident** vortices. Can compute $Area(X) = \int_X \omega_{L^2}$ using localization formula
- Corresponding sphere in $\mathbb{C}P^n$:

$$\begin{aligned} P(z) &= (z-t)^n = z^n - ntz^{n-1} + \cdots + (-t)^n \\ X &= \{[1, -nt, \ldots, (-t)^n] \,:\, t \in \mathbb{C} \cup \{\infty\}\}. \end{aligned}$$

- Compute $\int_X \omega_{FS}$, deduce α

$$\mathrm{Vol}(M_n) = \frac{\pi^n(\tau \mathrm{Vol}(\Sigma) - 4\pi n)^n}{n!}$$

Vanishes in dissolving limit $\tau \searrow 4\pi n/\mathrm{Vol}(\Sigma)$

Manton's volume calculation

- Moral: only need kähler **class** of M_n. Idea can be extended to other Σ (S^2 not round, higher genus) [Manton-Nasir]
- Why should we care about $\mathrm{Vol}(M_n)$? Behaviour as $n \to \infty$, with $n/\mathrm{Vol}(\Sigma)$ fixed, tells us about thermodynamics of a gas of vortices.
- Dissolving limit studied in detail for $\Sigma = S^2$ round [Baptista-Manton] and Σ higher genus [Manton-Romão], interesting conjectured asymptotics for $\gamma(\tau)$

Not so plain vortices: $(X, G) = (S^2, U(1))$

- Fix $\mathbf{e} \in S^2$ (e.g. $\mathbf{e} = (0, 0, 1)$)
  G acts on S^2 by rotations about $\mathbf{e}$
- Moment map $\mu(\mathbf{n}) = -\mathbf{e} \cdot \mathbf{n}$
- $P \to \Sigma$ principal G bundle, degree $n \geq 0$, connexion A
- $\mathbf{n}$ section of $P \times_G S^2$
- Canonical sections $\mathbf{n}_\infty(x) = \mathbf{e}$, $\mathbf{n}_0(x) = -\mathbf{e}$
- **Two** integer topological invariants of a section $\mathbf{n}$:

$$n_+ = \#(\mathbf{n}(\Sigma), \mathbf{n}_\infty(\Sigma)), \qquad n_- = \#(\mathbf{n}(\Sigma), \mathbf{n}_0(\Sigma))$$

  Constraint: $n = n_+ - n_-$ (so we're assuming $n_+ \geq n_-$)
- Energy

$$E = \frac{1}{2} \int_\Sigma \left(|d_A \mathbf{n}|^2 + |F_A|^2 + (\mathbf{e} \cdot \mathbf{n})^2 \right)$$

  where, in a local trivialization

$$d_A \mathbf{n} = d\mathbf{n} - A\mathbf{e} \times \mathbf{n}$$

“Bogomol'nyi” bound

- Given $(\mathbf{n}, A)$ define a two-form on Σ

$$\Omega(X, Y) = (\mathbf{n} \times d_A\mathbf{n}(X)) \cdot d_A\mathbf{n}(Y)$$

- Let $e_1, e_2 = Je_1$ be a local orthonormal frame on Σ. Then

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}(|d_A\mathbf{n}(e_1)|^2 + |d_A\mathbf{n}(e_2)|^2) + \frac{1}{2}|F_A|^2 + \frac{1}{2}(\mathbf{e}\cdot\mathbf{n})^2 \\
&= \frac{1}{2}|d_A\mathbf{n}(e_1) + \mathbf{n} \times d_A\mathbf{n}(e_2)|^2 + \frac{1}{2}|F_A - *\mathbf{e}\cdot\mathbf{n}|^2 \\
&\qquad + *(\Omega + \mathbf{e}\cdot\mathbf{n}F_A) \\
\implies \quad E &\geq \int_\Sigma (\Omega + \mathbf{e}\cdot\mathbf{n}F_A)
\end{aligned}$$

- Claim: last integral is a homotopy invariant of $(\mathbf{n}, A)$

"Bogomol'nyi" bound

- Suffices to show this in case $D = \mathbf{n}^{-1}(\{\mathbf{e}, -\mathbf{e}\}) \subset \Sigma$ finite
- On $\Sigma \backslash D$ have global one-form

$$\xi = \mathbf{e} \cdot \mathbf{n}(A - \mathbf{n}^* d\varphi)$$

  Furthermore, $\Omega + \mathbf{e} \cdot \mathbf{n} F_A = d\xi$
- Hence

$$\begin{aligned} \int_\Sigma (\Omega + \mathbf{e} \cdot \mathbf{n} F_A) &= \int_{\Sigma \backslash D} (\Omega + \mathbf{e} \cdot \mathbf{n} F_A) \\ &= \lim_{\varepsilon \to 0} \sum_{p \in D} - \oint_{C_\varepsilon(p)} \xi \\ &= 2\pi(n_+ + n_-) \end{aligned}$$

“Bogomol’nyi” bound

- Hence $E \geq 2\pi(n_+ + n_-)$ with equality iff

$$\begin{aligned} \overline{\partial_A}\mathbf{n}(e_1) = d_A\mathbf{n}(e_1) + \mathbf{n} \times d_A\mathbf{n}(Je_1) &= 0 \qquad (V1) \\ *F_A &= \mathbf{e}\cdot\mathbf{n} \qquad (V2) \end{aligned}$$

- Note solutions of $(V1)$ certainly have D finite (and $n_\pm \geq 0$)

Existence: Sibner-Sibner-Yang

- Again, there's a "Bradlow" obstruction

$$2\pi(n_+ - n_-) < \mathrm{Vol}(\Sigma)$$

  but derivation is nontrivial
- **Theorem:** Let $n_+ \geq n_- \geq 0$ and $2\pi(n_+ - n_-) < \mathrm{Vol}(\Sigma)$. For each pair of disjoint effective divisors D_+, D_- in Σ of degrees n_+, n_- there exists a unique gauge equivalence class of solutions of (V1), (V2).
- Moduli space of vortices:
  $\mathrm{M}_{n_+,n_-} \equiv (\Sigma^{n_+}/S_{n_+}) \times (\Sigma^{n_-}/S_{n_-}) \backslash \Delta_{n_+,n_-}$
- If $n_- > 0$, M_{n_+,n_-} is noncompact (in an interesting way)
- Again, have kähler L^2 metric. Complete? Finite volume? Isometric compactification?

A formal compactification

- A "linear" model: $G = T^2$, $X = \mathbb{C}^2$, moment map

$$\mu : \mathbb{C}^2 \to \mathfrak{g}^*, \qquad \mu(z_+, z_-) = \frac{1}{2}(|z_+|^2 + |z_-|^2 - 1, |z_+|^2 - \frac{1}{2})$$

- Principal G bundle $P \to \Sigma$ of degree (n_1, n_2)
- Associated X-bundle $P \times_G \mathbb{C}^2 \equiv L_+ \oplus L_-$ where $\deg L_+ = n_+ = n_1 + n_2$, $\deg L_- = n_- = n_2$
- Give $\mathfrak{g}$ a deformed inner product $q^{-2} dt_1^2 + dt_2^2$ [i.e. think of G as $S^1_{1/q} \times S^1$]
- Vortex equations

$$\bar{\partial}_A \varphi_+ = 0 \qquad \bar{\partial}_A \varphi_- = 0$$

$$*F_1 = \frac{1}{2} q^2 (1 - |\varphi_+|^2 + |\varphi_1|^2) \qquad *F_2 = \frac{1}{2}(\frac{1}{2} - |\varphi_+|^2)$$

A formal compactification

- Baptista: vortex solutions $\leftrightarrow$ effective divisors $(\varphi_+^{-1}(0), \varphi_-^{-1}(0))$ of degrees n_+, n_- (if q, $\mathrm{Vol}(\Sigma)$, n_1, n_2 satisfy "Bradlow" bounds)
- $M^q_{n_+,n_-}$ is **compact**
- Obvious dense open embedding $\iota : M_{n_+,n_-} \hookrightarrow M^q_{n_+,n_-}$ [where M_{n_+,n_-} is moduli space of S^2 vortices]
- Have L^2 metrics γ and γ^q on M_{n_+,n_-}, $M^q_{n_+,n_-}$
- **Conjecture** (Romão, JMS): $\iota^*\gamma^q \to \gamma$ uniformly as $q \to \infty$ in the case $n_+ = n_-$.
- Similar conjecture for case $n_+ \neq n_-$: start with a different linear model
- Motivation?

A formal compactification

- Define, in a local triv,

$$T : ((A_1, A_2), (\varphi_+, \varphi_-)) \mapsto (A_2, [\varphi_+ : \varphi_-])$$

Globalizes: $T : \mathscr{A}(P) \times \Gamma(L_+ \oplus L_-) \to \mathscr{A}(P') \times \Gamma(F')$

- Formally, T is an L^2 Riemannian submersion
- Fix a disjoint pair of divisors and let $((A_1, A_2), (\varphi_+, \varphi_-))_q$ be the corresponding q-vortex, $q > 0$ large
- Then $T((A_1, A_2), (\varphi_+, \varphi_-))_q$ satisfes the first F' vortex equation by construction:

$$\overline{\partial}_{A_2}[\varphi_+ : \varphi_-] = 0$$

- For all q,

$$*F_1 = \frac{1}{2} q^2 (1 - |\varphi_+|^2 - |\varphi_-|^2)$$

Suggests $|\varphi_+|^2 + |\varphi_-|^2 = 1 + O(q^{-2})$

A formal compactification

- Then last q-vortex equation is

$$\begin{aligned} *F_2 &= \frac{1}{2}(\frac{1}{2} - |\varphi_+|^2) = \frac{1}{2}\frac{|\varphi_-|^2 - |\varphi_+|^2}{|\varphi_-|^2 + |\varphi_+|^2} + O(q^{-2}) \\ &= \frac{1}{2}\mathbf{e}\cdot\mathbf{n} + O(q^{-2}) \end{aligned}$$

  suggesting $T((A_1, A_2), (\varphi_+, \varphi_-))_q$ converges to a F' vortex as $q \to \infty$

- Similar statement for $(G, X) = (S^1, \mathbb{C}^2)$, $\iota : Hol_k(\Sigma, \mathbb{C}P^1) \hookrightarrow M_k^q$, proved by Chih-Chung Liu

Testing the conjecture

- Since $M^q_{n_+,n_-}$ is compact, we can compute its volume if we know the kähler class. We do for $\Sigma = S^2$ [Baptista]. Take limit $q \to \infty$, get conjectural formula for $\mathrm{Vol}(M_{n_+,n_+})$,

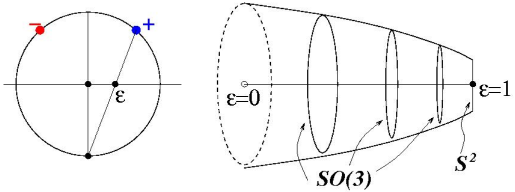
$$\mathrm{Vol}(M_{n,n}(S^2)) = \frac{(2\pi)^{2n}}{(n!)^2}\mathrm{Vol}(S^2)^{2n}$$

  for any S^2
- Can prove this for $n = 1$, on any **round** S^2
- **Theorem** (Romão, JMS) Let Σ be a round two-sphere. Then

$$\mathrm{Vol}(M_{1,1}(\Sigma)) = (2\pi\mathrm{Vol}(\Sigma))^2.$$

- Proof has 3 ingredients:
  - Symmetry
  - Taubes equation
  - Localization formula

Ingredient 1: symmetry



- $M_{1,1} = S^2 \times S^2 \backslash \Delta = (0,1) \times SO(3) \sqcup \{1\} \times S^2$
- γ is $SO(3)$-invariant, kähler, and invariant under $(z_+, z_-) \mapsto (z_-, z_+)$
- Every such metric takes the form

$$\gamma = -\frac{Q'(\varepsilon)}{\varepsilon}(d\varepsilon^2 + \varepsilon^2\sigma_3^2) + Q(\varepsilon)\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\sigma_1^2 + \frac{1+\varepsilon^2}{1-\varepsilon^2}\sigma_2^2\right),$$

  for $Q : (0,1] \to \mathbb{R}$ decreasing with $Q(1) = 0$.
- Has finite total volume iff Q is bounded

$$Vol(M_{1,1}) = \frac{1}{4}(4\pi)^2 \lim_{\varepsilon \to 0} Q(\varepsilon)^2$$

Ingredient 2: Taubes equation

$$\overline{\partial_A}\mathbf{n} = 0, \qquad *F_A = \frac{1}{2}\mathbf{e}\cdot\mathbf{n}$$

- Stereographic coords z, u on S^2_R, S^2_{target}

$$g = \Omega(|z|)dzd\overline{z} = \frac{4R^2}{(1+|z|^2)^2}dzd\overline{z}.$$

- $h : \Sigma \to \mathbb{R} \cup \{\pm\infty\}$, $h = \log|u|^2$

$$\nabla^2 h - 2\Omega \tanh\frac{h}{2} = 4\pi(\sum \delta(z - z_+) - \sum \delta(z - z_-))$$

- Suffices to consider $z_+ = \varepsilon > 0$, $z_- = -\varepsilon$. Can regularize

$$h(z) = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}(z/\varepsilon)$$

Ingredient 2: Taubes equation

- Then

$$\nabla^2 \widehat{h} - \frac{8R^2\varepsilon^2}{(1+\varepsilon^2|z|^2)^2} \frac{|z-1|^2 e^{\widehat{h}} - |z+1|^2}{|z-1|^2 e^{\widehat{h}} + |z+1|^2} = 0$$

- Nice semilinear elliptic PDE

Ingredient 3: localization formula

- The solution h of Taubes equation with sources at $z_1, z_2, \ldots, z_{n_+}, z_{n_++1}, \ldots, z_{n_++n_-}$ has expansion

$$\pm h = \log |z - z_s|^2 + a_s + \frac{1}{2}\bar{b}_s(z - z_s) + \frac{1}{2}b_s(\bar{z} - \bar{z}_s) + \cdots$$

  about $\pm$ vortex position z_s
- Think of (z_s) as local coords on $M_{n_+,n_-}(\Sigma)\backslash\Delta$
- $b_r(z_1, \ldots, z_{n_++n_-})$ are (unknown) complex functions of vortex positions
- **Proposition** (Romão-JMS, following Strachan-Samols):

$$\gamma = 2\pi \left\{ \sum_r \Omega(|z_r|)|dz_r|^2 + \sum_{r,s} \frac{\partial b_s}{\partial z_r} dz_r d\bar{z}_s \right\}$$

  Holds on any Riemann surface (including $\mathbb{C}$)
- Expect such a formula whenever target X is toric w.r.t. $G^{\mathbb{C}}$

Proof of volume formula for $M_{1,1}(S^2)$

- Localization formula $\Longrightarrow$

$$Q(\varepsilon) = -2\pi\left(\varepsilon b_1(\varepsilon,-\varepsilon) - \frac{4R^2}{1+\varepsilon^2} + 1 + 2R^2\right)$$

  So symmetry $\Longrightarrow$ volume formula holds if

$$\lim_{\varepsilon\to 0} \varepsilon b_1(\varepsilon,-\varepsilon) = -1$$

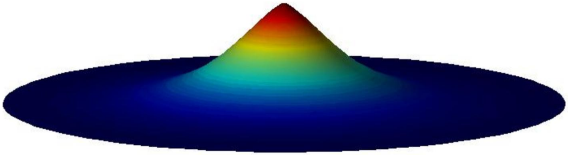
- But

$$\varepsilon b_1(\varepsilon,-\varepsilon) = \left.\frac{\partial \widehat{h}(x+iy)}{\partial x}\right|_{(1,0)} - 1$$

  so it remains to show $\widehat{h}_x(1+i0) \to 0$ as $\varepsilon \to 0$.
- Go do elliptic estimates on the PDE for $\widehat{h}$

What else?

- Case $\Sigma = \mathbb{C}$ is interesting (more interesting from "physics" viewpoint)
- $M_{1,1}(\mathbb{C}) = \mathbb{C} \times \mathbb{C} \backslash \Delta = \mathbb{C}_{com} \times \mathbb{C}^{\times}$
- Numerics: metric on SoR $\mathbb{C}^{\times}$, $\gamma^0 = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2)$



- Conjectured asymptotics in small, large ε regions

$$F(\varepsilon) \sim -8\pi \log \varepsilon \qquad \text{as } \varepsilon \to 0$$
$$F(\varepsilon) \sim 2\pi \left(2 + \frac{m^2}{\pi^2} K_0(2\varepsilon)\right) \qquad \text{as } \varepsilon \to \infty$$

- Would imply $M_{1,1}(\mathbb{C})$ is incomplete with unbounded scalar curvature
- Model with $\mu(\mathbf{n}) = \tau - \mathbf{e} \cdot \mathbf{n}$ completely unexplored