The geometry of the space of vortex-antivortex pairs

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Motivation

- Vortices:
  - simplest topological solitons in gauge theory (2D, $U(1)$, $\mathbb{C}$)
  - topology comes from winding at spatial infinity
  - have a BPS regime
- Sigma model lumps:
  - even simpler topological solitons (2D, no gauge theory, BPS)
  - topology comes from "wrapping" of space around target space $\mathbb{R}^2 \to S^2$
- **Gauged** sigma models: have **both types** of topology
  - Lumps split into vortex antivortex pairs
  - Two species of vortex
  - BPS! Vortices and antivortices ins stable equilibrium
  - Moduli space of vortex-antivortex solutions has interesting geometry.

The gauged $O(3)$ sigma model

- $\mathbf{n} : \Sigma \to S^2$ ($\Sigma = \mathbb{R}^2$, physical space)
- Fix $\mathbf{e} \in S^2$ (e.g. $\mathbf{e} = (0, 0, 1)$)
  $G = U(1)$ acts on S^2 by rotations about $\mathbf{e}$
- Gauge field $A \in \Omega^1(\Sigma)$

$$\mathrm{d}_A \mathbf{n} = \mathrm{d}\mathbf{n} - A\mathbf{e} \times \mathbf{n}$$

  Magnetic field $F_A = \mathrm{d}A \in \Omega^2(\Sigma)$
- Energy

$$E = \frac{1}{2} \int_\Sigma \left(|d_A \mathbf{n}|^2 + |F_A|^2 + (\mathbf{e} \cdot \mathbf{n})^2 \right)$$

  Aside: $\mu(\mathbf{n}) = -\mathbf{e} \cdot \mathbf{n}$ is moment map for gauge action

Magnetic flux quantization

$$E = \frac{1}{2}\int_\Sigma \left(|d_A\mathbf{n}|^2 + |F_A|^2 + (\mathbf{e}\cdot\mathbf{n})^2\right)$$

- As $r \to \infty$, $\mathbf{e}\cdot\mathbf{n} \to 0$: $\mathbf{n} : S^1_\infty \to S^1_{equator}$

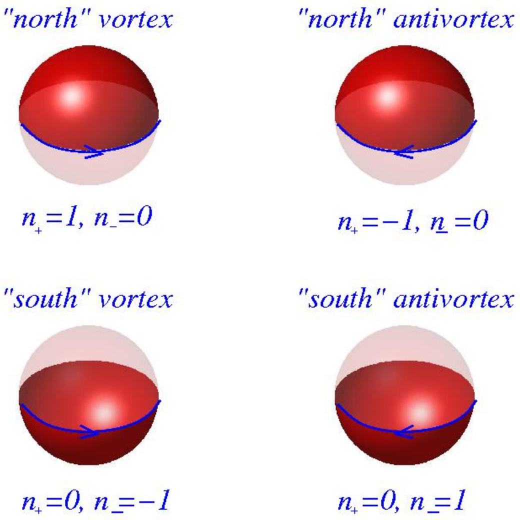
$$\mathbf{n}_\infty(\theta) = (\cos\chi(\theta), \sin\chi(\theta), 0)$$

- $|\mathrm{d}_A\mathbf{n}| \to 0$ also, $A_\infty = \chi'(\theta)\mathrm{d}\theta$
- $\displaystyle\int_\Sigma F_A = \oint_{S^1_\infty} A_\infty = \chi(2\pi) - \chi(0) = 2\pi n$
- **Two** topological charges:

$$n_+ = \sharp\{\mathbf{n}^{-1}(\mathbf{e})\}, \qquad n_- = \sharp\{\mathbf{n}^{-1}(-\mathbf{e})\}.$$

Constraint: $n = n_+ - n_-$

(Anti)vortices



"Bogomol'nyi" bound (Schroers)

- Given $(\mathbf{n}, A)$ define a two-form on Σ

$$\Omega(X, Y) = (\mathbf{n} \times d_A\mathbf{n}(X)) \cdot d_A\mathbf{n}(Y)$$

- Let $e_1, e_2 = Je_1$ be a local orthonormal frame on Σ (e.g. $e_1 = \partial_x$, $e_2 = \partial_y$). Then

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}(|d_A\mathbf{n}(e_1)|^2 + |d_A\mathbf{n}(e_2)|^2) + \frac{1}{2}|B|^2 + \frac{1}{2}(\mathbf{e}\cdot\mathbf{n})^2 \\
&= \frac{1}{2}|d_A\mathbf{n}(e_1) + \mathbf{n} \times d_A\mathbf{n}(e_2)|^2 + \frac{1}{2}|F_A - *\mathbf{e}\cdot\mathbf{n}|^2 \\
&\qquad + *(\Omega + \mathbf{e}\cdot\mathbf{n}F_A) \\
\implies \quad E &\geq \int_\Sigma (\Omega + \mathbf{e}\cdot\mathbf{n}F_A)
\end{aligned}$$

- Claim: last integral is a homotopy invariant of $(\mathbf{n}, A)$

"Bogomol'nyi" bound

- Suffices to show this in case $D = \mathbf{n}^{-1}(\{\mathbf{e}, -\mathbf{e}\}) \subset \Sigma$ finite
- On $\Sigma \backslash D$ have global one-form $\xi = \mathbf{e} \cdot \mathbf{n}(A - \mathbf{n}^* d\varphi)$ s.t.

$$\Omega + \mathbf{e} \cdot \mathbf{n} F_A = d\xi$$

- Hence

$$\begin{aligned} \int_\Sigma (\Omega + \mathbf{e} \cdot \mathbf{n} F_A) &= \int_{\Sigma \backslash D} (\Omega + \mathbf{e} \cdot \mathbf{n} F_A) \\ &= \lim_{\varepsilon \to 0} \sum_{p \in D} -\oint_{C_\varepsilon(p)} \xi \\ &= 2\pi(n_+ + n_-) \end{aligned}$$

"Bogomol'nyi" bound

- Hence $E \geq 2\pi(n_+ + n_-)$ with equality iff

$$\begin{aligned} \bar{\partial}_A \mathbf{n} &= 0 \qquad (V1) \\ *F_A &= \mathbf{e}\cdot\mathbf{n} \qquad (V2) \end{aligned}$$

- Note solutions of $(V1)$ certainly have D finite (and $n_\pm \geq 0$)

Existence: Yang

- An **effective divisor** is an unordered, finite list of points in Σ, possibly with repetition,

$$\text{e.g.} \quad [2,0,1+i,2] = [0,2,2,1+i] \neq [2,0,1+i]$$

  The **degree** of the divisor is the length of the list

- **Theorem:** Let $n_+ \geq n_- \geq 0$. For each pair of disjoint effective divisors D_+, D_- in Σ of degrees n_+, n_- there exists a unique gauge equivalence class of solutions of (V1), (V2) with $\mathbf{n}^{-1}(\pm\mathbf{e}) = D_\pm$.
- Moduli space of vortices: $\mathsf{M}_{n_+,n_-} \equiv M_{n_+} \times M_{n_-} \backslash \Delta_{n_+,n_-}$
- Simple case: space of vortex-antivortex pairs

$$M_{1,1} = \{(z_+, z_-) \in \Sigma \times \Sigma : z_+ \neq z_-\} = (\Sigma \times \Sigma) \backslash \Delta$$

The "Taubes" equation

$$u = \frac{n_1 + in_2}{1 + n_3}, \qquad h = \log|u|^2$$

- h finite except at $\pm$ vortices, $h = \mp\infty$.
- $(V1) \Rightarrow A_{\bar{z}} = -i\frac{\partial_{\bar{z}} u}{u}$, eliminate A from $(V2)$

$$\nabla^2 h - 2\tanh\frac{h}{2} = 0$$

  away from vortex positions
- $(+)$ vortices at z_r^+, $r = 1, \ldots, n_+$, $(-)$ vortices at z_r^-, $r = 1, \ldots, n_-$

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\sum_r \delta(z - z_r^+) - \sum_r \delta(z - z_r^-)\right)$$

- Consider $(1, 1)$ vortex pairs

Solving the (1,1) Taubes equation (numerically)

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta(z-\varepsilon) - \delta(z+\varepsilon)\right)$$

- Regularize: $h = \log\left(\frac{|z-\varepsilon|^2}{|z+\varepsilon|^2}\right) + \widehat{h}$

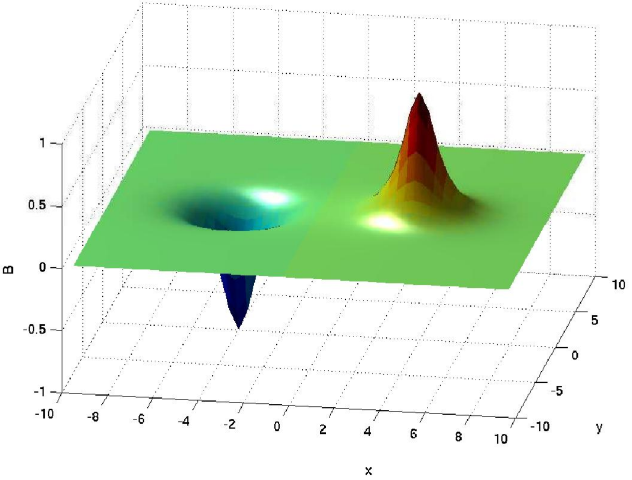
$$\nabla^2\widehat{h} - 2\frac{|z-\varepsilon|^2 e^{\widehat{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\widehat{h}} + |z+\varepsilon|^2} = 0$$

- Rescale: $z =: \varepsilon w$

$$\nabla_w^2\widehat{h} - 2\varepsilon^2\frac{|w-1|^2 e^{\widehat{h}} - |w+1|^2}{|w-1|^2 e^{\widehat{h}} + |w+1|^2} = 0$$

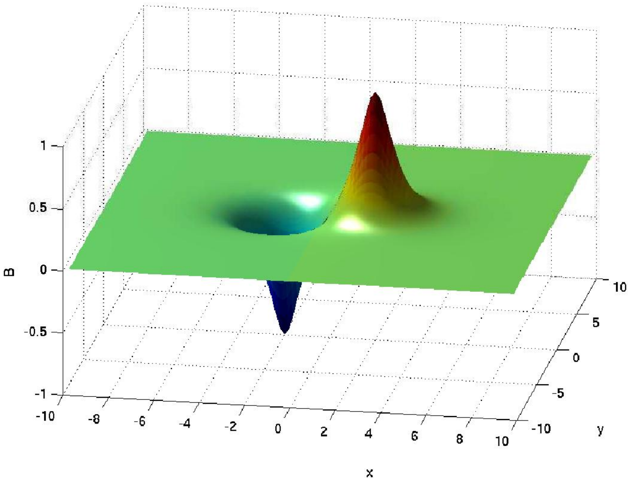
- Solve with b.c. $\widehat{h}(\infty) = 0$

(1,1) vortices



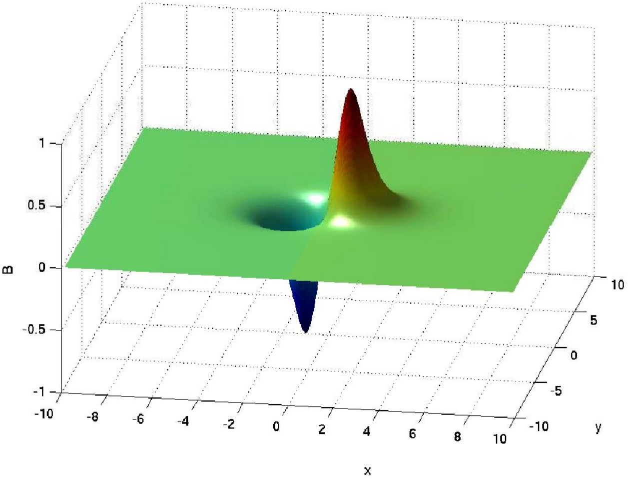
$\varepsilon = 4$

(1,1) vortices



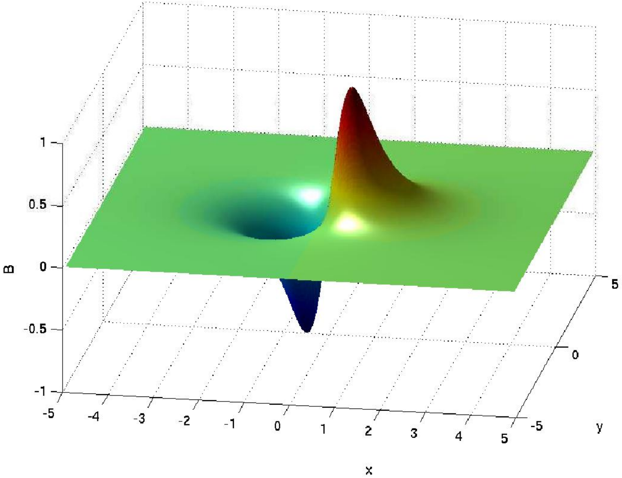
$\varepsilon = 2$

(1,1) vortices



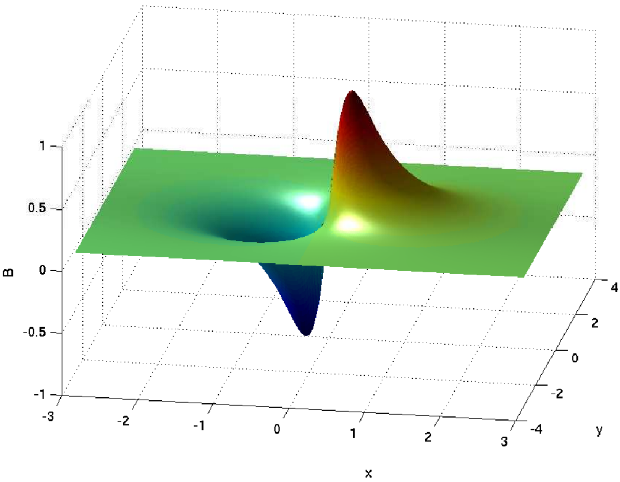
$\varepsilon = 1$

(1,1) vortices



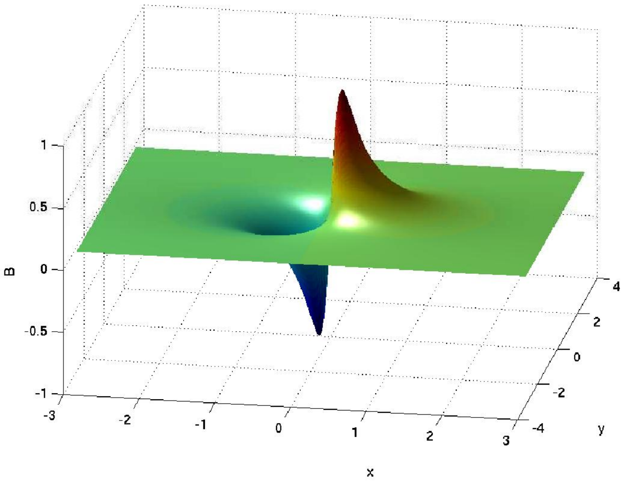
$\varepsilon = 0.5$

(1,1) vortices



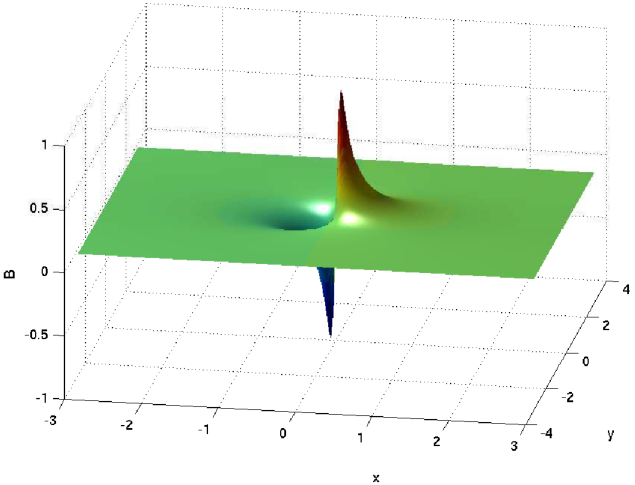
$\varepsilon = 0.3$

(1,1) vortices



$\varepsilon = 0.15$

(1,1) vortices



$\varepsilon = 0.06$

The metric on M_{n_+,n_-}

$$S = \frac{1}{2}\int_{\Sigma\times\mathbb{R}} \left(D_\mu \mathbf{n}\cdot D^\mu \mathbf{n}) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \mathbf{e}\cdot\mathbf{n}^2 \right) = \int_{\mathbb{R}} (T-E)dt$$

- Restriction of kinetic energy

$$T = \frac{1}{2}\int_\Sigma (|\dot{\mathbf{n}}|^2 + |\dot{A}|^2)$$

  to M_{n_+,n_-} equips it with a Riemannian metric g
- Low energy (anti)vortex dynamics: **geodesic motion** on (M_{n_+,n_-}, g)
- Expand solution h of Taubes eqn about $\pm$ vortex position z_s:

$$\pm h = \log|z - z_s|^2 + a_s + \frac{1}{2}\bar{b}_s(z - z_s) + \frac{1}{2}b_s(\bar{z} - \bar{z}_s) + \cdots$$

- $b_r(z_1, \ldots, z_{n_+ + n_-})$ (unknown) complex functions

The metric on M_{n_+,n_-}

- **Proposition** (Romão-JMS, following Strachan-Samols):

$$g = 2\pi \left\{ \sum_r |dz_r|^2 + \sum_{r,s} \frac{\partial b_s}{\partial z_r} dz_r d\bar{z}_s \right\}$$

- Corollary: g is **kähler**

The metric on $M_{1,1}$

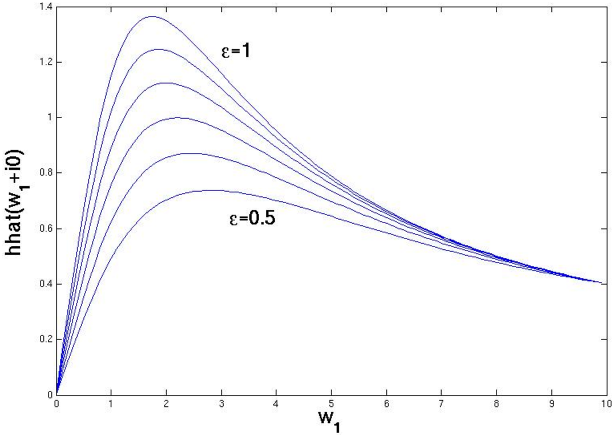
- $M_{1,1} = (\mathbb{C} \times \mathbb{C}) \backslash \Delta = \mathbb{C}_{com} \times \mathbb{C}^\times$
- $M^0_{1,1} = \mathbb{C}^\times$

$$g^0 = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d}{d\varepsilon} (\varepsilon b(\varepsilon)) \right) (d\varepsilon^2 + \varepsilon^2 d\psi^2)$$

where $b(\varepsilon) = b_+(\varepsilon, -\varepsilon)$

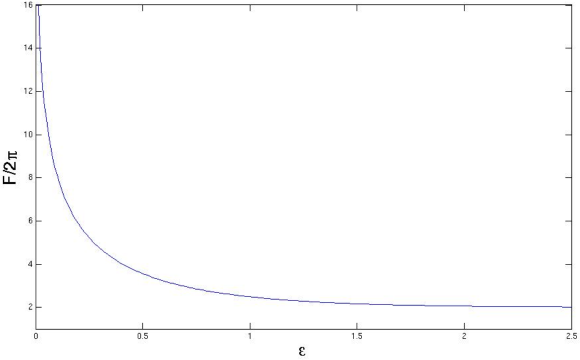
- $\varepsilon b(\varepsilon) = \frac{\partial \widehat{h}}{\partial w_1}\Big|_{w=1} - 1$
- Can easily extract this from our numerics

The metric on $M_{1,1}(\mathbb{C})$



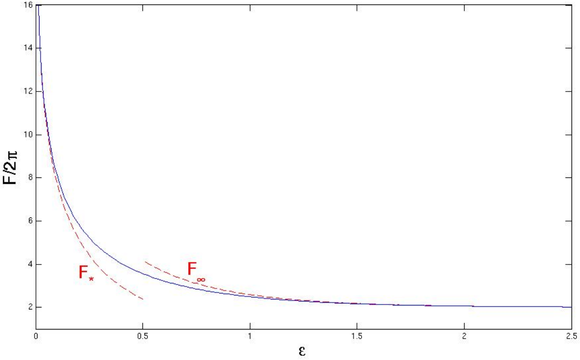
$$\varepsilon b(\varepsilon) = \left.\frac{\partial \widehat{h}}{\partial w_1}\right|_{w=1} - 1$$

The metric on $M_{1,1}(\mathbb{C})$



$$F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d(\varepsilon b(\varepsilon))}{d\varepsilon} \right)$$

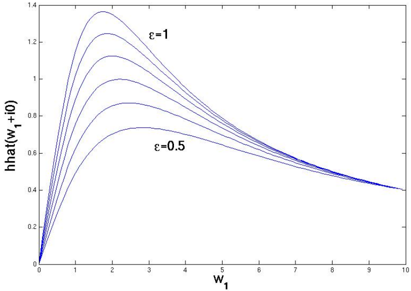
The metric on $M_{1,1}(\mathbb{C})$: conjectured asymptotics



$$F_*(\varepsilon) = 2\pi\left(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)\right) \sim -8\pi \log \varepsilon$$

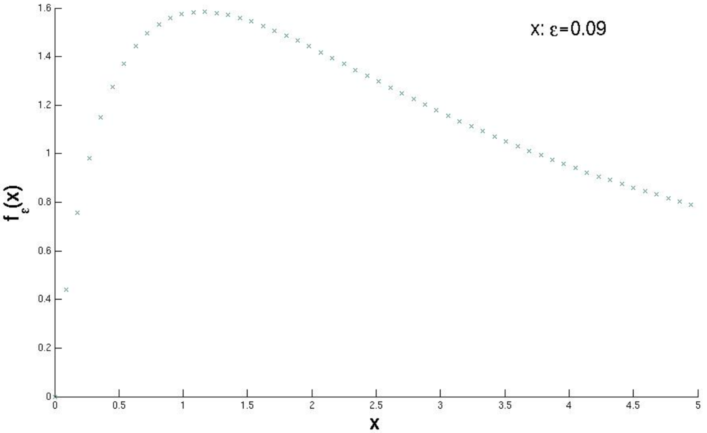
$$F_\infty(\varepsilon) = 2\pi\left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon)\right)$$

Self similarity as $\varepsilon \to 0$

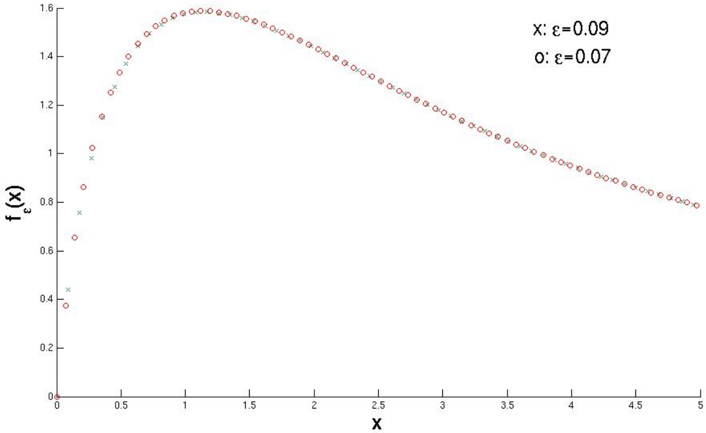


- Suggests $\widehat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w)$ for small ε, where f_* is fixed?
- Define $f_\varepsilon(z) := \varepsilon^{-1}\widehat{h}_\varepsilon(\varepsilon^{-1}z)$

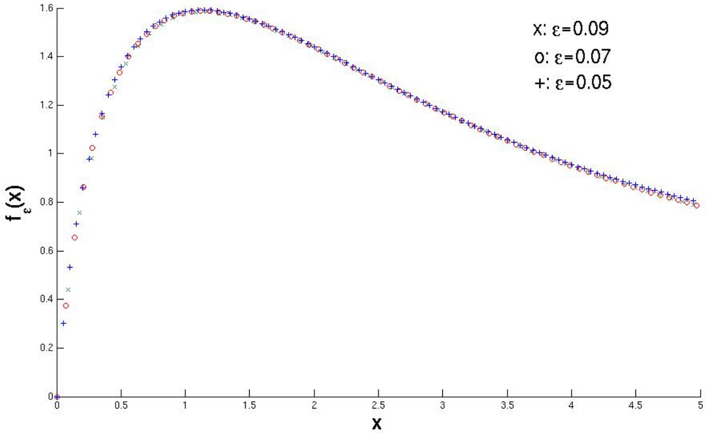
Self similarity as $\varepsilon \to 0$



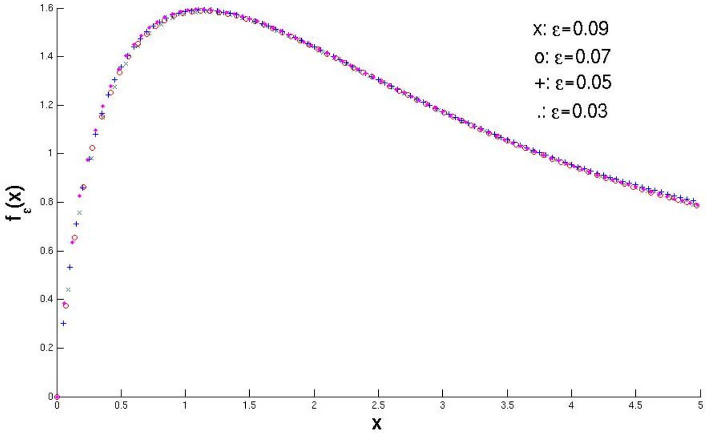
Self similarity as $\varepsilon \to 0$



Self similarity as $\varepsilon \to 0$



Self similarity as $\varepsilon \to 0$



Self similarity as $\varepsilon \to 0$

$$(\nabla^2 \widehat{h})(w) = 2\varepsilon^2 \frac{|w-1|^2 e^{\widehat{h}(w)} - |w+1|^2}{|w-1|^2 e^{\widehat{h}(w)} + |w+1|^2}$$

Self similarity as $\varepsilon \to 0$

$$(\nabla^2 \widehat{h})(w) = 2\varepsilon^2 \frac{|w-1|^2 e^{\widehat{h}(w)} - |w+1|^2}{|w-1|^2 e^{\widehat{h}(w)} + |w+1|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$

Self similarity as $\varepsilon \to 0$

$$(\nabla^2 f_\varepsilon)(z) = \frac{2}{\varepsilon} \frac{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} + |z+\varepsilon|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$

Self similarity as $\varepsilon \to 0$

$$(\nabla^2 f_\varepsilon)(z) = \frac{2}{\varepsilon} \frac{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\varepsilon f_\varepsilon(z)} + |z+\varepsilon|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \to 0$

Self similarity as $\varepsilon \to 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \to 0$

Self similarity as $\varepsilon \to 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \to 0$
- Screened inhomogeneous Poisson equation, source $-4\cos\theta/r$

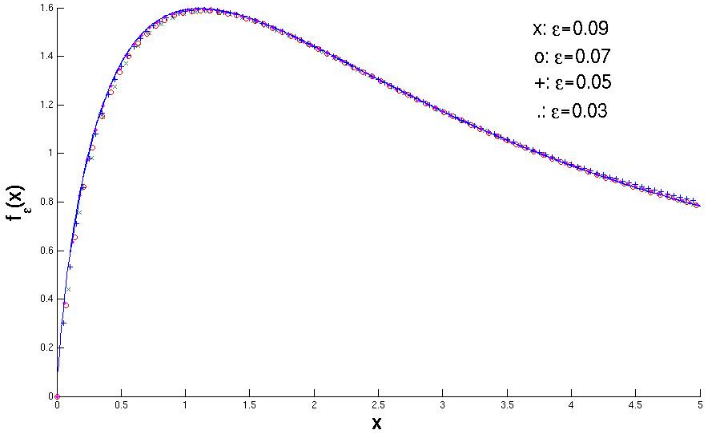
Self similarity as $\varepsilon \to 0$

$$(\nabla^2 f_*)(z) = f_*(z) - \frac{2(z+\bar{z})}{|z|^2}$$

- Subst $\widehat{h}(w) = \varepsilon f_\varepsilon(\varepsilon w)$
- Take formal limit $\varepsilon \to 0$
- Screened inhomogeneous Poisson equation, source $-4\cos\theta/r$
- Unique solution (decaying at infinity)

$$f_*(re^{i\theta}) = \frac{4}{r}(1 - rK_1(r))\cos\theta$$

Self similarity as $\varepsilon \to 0$



The metric on $M^0_{1,1}$

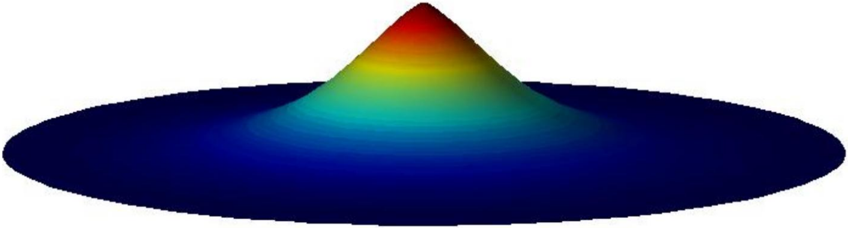
- Predict, for small ε,

$$\widehat{h}(w_1 + i0) \approx \varepsilon f_*(\varepsilon w_1) = \frac{4}{w_1}(1 - \varepsilon w_1 K_1(\varepsilon w_1))$$

whence we extract predictions for $\varepsilon b(\varepsilon)$, $F(\varepsilon)$

$$g^0 = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2)$$

- Conjecture: $F(\varepsilon) \sim -8\pi \log \varepsilon$ as $\varepsilon \to 0$
- $M_{1,1}$ is **incomplete**, with unbounded curvature



Vortices on compact Σ

- Why?
  - Thermodynamics of vortex gas: technical trick to get finite vortex density without losing BPS structure
  - Regularized Taubes equation is now an elliptic PDE on a **compact** domain: can do rigorous analysis
- There are extra technicalities...
- ... and an extra condition for existence of vortices

The Bradlow bound

$$\bar{\partial}_A \mathbf{n} = 0, \qquad *F_A = \mathbf{e} \cdot \mathbf{n}$$

- Integrate 2nd eqn over Σ:

$$2\pi(n_+ - n_-) = \int_\Sigma \mathbf{e} \cdot \mathbf{n} \in [-Vol(\Sigma), Vol(\Sigma)]$$

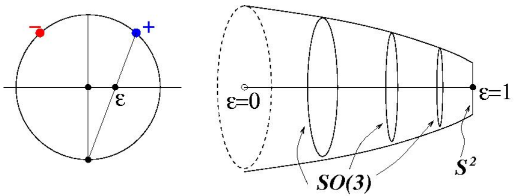
- Hence, if vortex equations have a solution,

$$2\pi|n_+ - n_-| \leq Vol(\Sigma).$$

  Upper bound on **excess** of vortices over antivortices, and vice versa.

- **Theorem**(Sibner,Sibner,Yang): Let $n_+ \geq n_- \geq 0$ and $2\pi(N_+ - n_-) < Vol(\Sigma)$. For each pair of disjoint effective divisors D_+, D_- in Σ of degrees n_+, n_- there exists a unique gauge equivalence class of solutions of (V1), (V2) with $\mathbf{n}^{-1}(\pm\mathbf{e}) = D_\pm$.

Vortices on S^2: $M_{1,1}(S^2)$



- $M_{1,1} = S^2 \times S^2 \backslash \Delta = (0,1) \times SO(3) \sqcup \{1\} \times S^2$
- g is $SO(3)$-invariant, kähler, and invariant under $(z_+, z_-) \mapsto (z_-, z_+)$
- Every such metric takes the form

$$g = -\frac{Q'(\varepsilon)}{\varepsilon}(d\varepsilon^2 + \varepsilon^2\sigma_3^2) + Q(\varepsilon)\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\sigma_1^2 + \frac{1+\varepsilon^2}{1-\varepsilon^2}\sigma_2^2\right),$$

for $Q : (0,1] \to [0,\infty)$ decreasing with $Q(1) = 0$.

Vortices on S^2: $M_{1,1}(S^2)$

- $Vol(M_{1,1}(S^2)$ is finite iff $Q : (0,1] \to [0,\infty)$ is bounded

$$Vol(M_{1,1}(S^2)) = \left[\lim_{\varepsilon\to 0} 2\pi Q(\varepsilon)\right]^2$$

- By means of elliptic estimates on Taubes eqn we can prove:
  - $M_{1,1}(S^2_R)$ has volume $[2\pi \times 4\pi R^2]^2$
  - The "radial" geodesic $0 < \varepsilon \leq 1$ in $M_{1,1}$ has finite length, and hence
  - $M_{1,1}(S^2_R)$ is geodesically incomplete

The volume of $M_{n_+,n_-}(S^2)$

- $M_{n_+,n_-}(S^2) = \{\text{disjoint pairs of } n_\pm\text{-divisors on } S^2\} = (\mathbb{P}^{n_+} \times \mathbb{P}^{n_-})\backslash\Delta$
- Consider gauged **linear** sigma model:
  - fibre $\mathbb{C}^2$
  - gauge group $\widetilde{U}(1) \times U(1) : (\varphi_1, \varphi_2) \mapsto (e^{i(\widetilde{\theta}+\theta)}\varphi_1, e^{i\widetilde{\theta}}\varphi_2)$

$$\begin{aligned} E_{\widetilde{e}} &= \frac{1}{2}\int_\Sigma \left\{ \frac{|\widetilde{F}|^2}{\widetilde{e}^2} + |F|^2 + |\mathrm{d}_{\widetilde{A}}\varphi|^2 + |\mathrm{d}_A\varphi|^2 \right. \\ &\quad \left. + \frac{\widetilde{e}^2}{4}(4 - |\varphi_1|^2 - |\varphi_2|^2)^2 + \frac{1}{4}(2 - |\varphi_1|^2)^2 \right\} \end{aligned}$$

- For any $\widetilde{e} > 0$, has compact moduli space of (n_+, n_-)-vortices

$$M^{lin}_{n_+,n_-} = \mathbb{P}^{n_+} \times \mathbb{P}^{n_-}$$

- Baptista found a formula for $[\omega_{L^2}]$ of $M^{lin}_{n_+,n_-}(\Sigma)$
- Can compute $Vol(M^{lin}_{n_+,n_-}(S^2))$ by evaluating $[\omega_{L^2}]$ on $\mathbb{P}^1 \times \{p\}$, $\{p\} \times \mathbb{P}^1$

The volume of $M_{n_+,n_-}(S^2)$

$$\begin{aligned} E_{\widetilde{e}} &= \frac{1}{2}\int_\Sigma \left\{ \frac{|\widetilde{F}|^2}{\widetilde{e}^2} + |F|^2 + |\mathrm{d}_{\widetilde{A}}\varphi|^2 + |\mathrm{d}_A\varphi|^2 \right. \\ &\quad \left. + \frac{\widetilde{e}^2}{4}(4 - |\varphi_1|^2 - |\varphi_2|^2)^2 + \frac{1}{4}(2 - |\varphi_1|^2)^2 \right\} \end{aligned}$$

- Take formal limit $\widetilde{e} \to \infty$:
  - $|\varphi_1|^2 + |\varphi_2|^2 = 4$ pointwise
  - $\widetilde{A}$ frozen out, fibre $\mathbb{C}^2$ collapses to $S^3/\widetilde{U}(1) = \mathbb{P}^1$
  - E-L eqn for $\widetilde{A}$ is algebraic: eliminate $\widetilde{A}$ from E_∞

    $$E_\infty = \frac{1}{2}\int_\Sigma |F|^2 + 4\frac{|\mathrm{d}u - iAu|^2}{(1+|u|^2)^2} + \left(\frac{1-|u|^2}{1+|u|^2}\right)^2$$

    where $u = \varphi_1/\varphi_2$
  - Exactly our $\mathbb{P}^1$ sigma model!

The volume of $M_{n,n}(S^2)$

- Leads us to conjecture that

$$Vol(M_{n_+,n_-}(S^2)) = \lim_{\tilde{e}\to\infty} Vol(M^{lin}_{n_+,n_-}(S^2)$$
$$= \frac{(2\pi)^{n_+ + n_-}}{n_+! n_-!}(V - \pi(n_+ - n_-))^{n_+}(V + \pi(n_+ - n_-))^{n_-}$$

  where $V = Vol(S^2)$
- Agrees with rigorous formula for $n_+ = n_- = 1$, $S^2 = S^2_R$
- Can generalize to $genus(\Sigma) > 0$ (it's complicated), and Einstein-Hilbert action
- Thermodynamics of vortex gas **mixture**

Summary / What next?

- Case $\Sigma = \mathbb{C}$ is most interesting
- $M_{1,1}(\mathbb{C}) = \mathbb{C} \times \mathbb{C} \backslash \Delta = \mathbb{C}_{com} \times \mathbb{C}^\times$
- Numerics: metric on SoR $\mathbb{C}^\times$, $g^0 = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2)$
- Conjectured asymptotics in small ε region

$$F(\varepsilon) \quad \sim \quad -8\pi \log \varepsilon$$

- Would imply $M_{1,1}(\mathbb{C})$ is incomplete with unbounded scalar curvature
- **Can we prove it?**
- We can shift the vacuum manifold:

$$V(\mathbf{n}) = \frac{1}{2}(\tau - \mathbf{e} \cdot \mathbf{n})^2$$

  Case $0 < \tau < 1$ very sparsely explored
- Other kähler targets ($\mathbb{P}^n, \mathbb{C}^k \times \mathbb{P}^n, \ldots$), other gauge groups, Chern-Simons term...