The geometry of the moduli space of $\mathbb{P}^1$ vortex-antivortex pairs

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joint with
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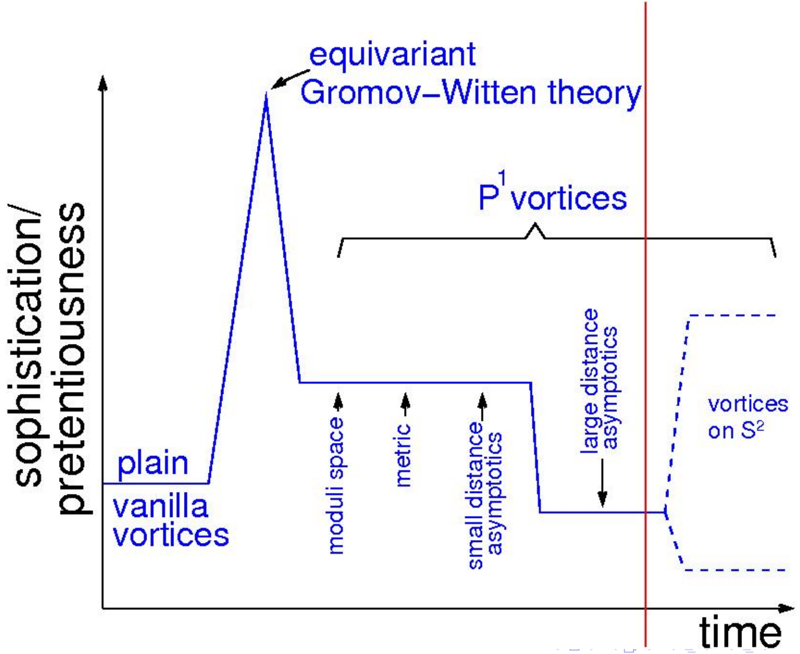
Plan



equivariant
Gromov–Witten theory
P¹ vortices
sophistication/
pretentiousness
plain
vanilla
vortices
moduli space
metric
small distance
asymptotics
large distance
asymptotics
vortices
on S²
time

Plain vanilla vortices

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$$\int_{\mathbb{R}^2} B = \oint_{S^1_\infty} A = \oint_{S^1_\infty} d\chi = 2\pi n$$

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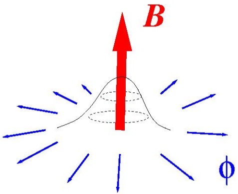

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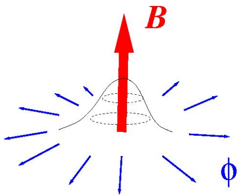

- $\lambda > 1$ vortices repel, $\lambda < 1$ vortices attract.

The Bogomol'nyi argument $\lambda = 1$

$$0 \leq \frac{1}{2}\int_{\mathbb{R}^2} |D_1\varphi + iD_2\varphi|^2 + (*B - \frac{1}{2}(1 - |\varphi|^2))^2$$

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&= E + \frac{1}{2}\int_{\mathbb{R}^2} i(\partial_1(\overline{\varphi}D_2\varphi) - \partial_2(\overline{\varphi}D_1\varphi)) - *B
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So $E \geq \pi n$ with equality iff

$$\begin{aligned} D_1\varphi + iD_2\varphi &= 0 && (BOG1) \\ *B &= \frac{1}{2}(1-|\varphi|^2) && (BOG2) \end{aligned}$$

First order system for (φ, A)

Taubes's Theorem

For each unordered collection of points $z_1, z_2, \dots, z_n \in \mathbb{C} \equiv \mathbb{R}^2$ (with repeats allowed) there exists a unique (up to gauge) solution of (BOG1,2) with $\varphi = 0$ at precisely the points $z_1, \dots, z_n$.

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&\leftrightarrow (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n
\end{aligned}
$$

- $M_n = \mathbb{C}^n$

Huge generalization

- Principal G bundle $P \to \Sigma^2$, connexion A

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- Vanilla version: $G = U(1)$, $X = \mathbb{C}$, $\mu(z) = \frac{1}{2}(1 - |z|^2)$
- A bit *too* simple...

$\mathbb{P}^1$ vortices

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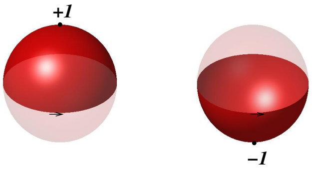

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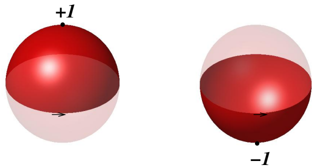

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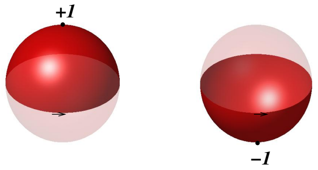

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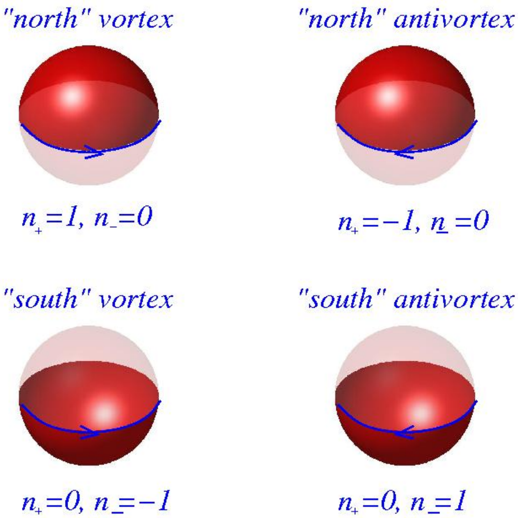
(Anti)vortices



Bogomol'nyi argument

$$E \geq 2\pi(n_+ + n_-)$$

with equality iff

$$\begin{aligned} D_1\mathbf{n} + \mathbf{n} \times D_2\mathbf{n} &= 0, & (BOG1) \\ *B &= \mathbf{e} \cdot \mathbf{n} & (BOG2) \end{aligned}$$

- n_+ = number of vortices (located where $\mathbf{n} = \mathbf{e}$)
- n_- = number of antivortices (located where $\mathbf{n} = -\mathbf{e}$)

The Taubes equation

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- Eliminate A from BOG2

$$\nabla^2 h - 2\tanh\frac{h}{2} = 0$$

away from vortex positions

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$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\sum_{r=1}^{n_+}\delta(z - z_r^+) - \sum_{r=1}^{n_-}\delta(z - z_r^-)\right)$$

- vortices at z_r^+, antivortices at z_r^-

The Taubes equation

- **Theorem** (Yang, 1999): For each pair of disjoint effective divisors $[z_1^+, \ldots, z_{n_+}^+], [z_1^-, \ldots, z_{n_-}^-]$ there exists a unique solution of (TAUBES), and hence a unique (up to gauge) solution of (BOG1), (BOG2).

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- Moduli space of vortices: $\mathsf{M}_{n_+,n_-} \equiv (\mathbb{C}^{n_+} \times \mathbb{C}^{n_-}) \backslash \Delta_{n_+,n_-}$

Solving the (1,1) Taubes equation (numerically)

$$\nabla^2 h - 2\tanh\frac{h}{2} = 4\pi\left(\delta(z-\varepsilon) - \delta(z+\varepsilon)\right)$$

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$$\nabla^2\widehat{h} - 2\frac{|z-\varepsilon|^2 e^{\widehat{h}} - |z+\varepsilon|^2}{|z-\varepsilon|^2 e^{\widehat{h}} + |z+\varepsilon|^2} = 0$$

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- Rescale: $z =: \varepsilon w$

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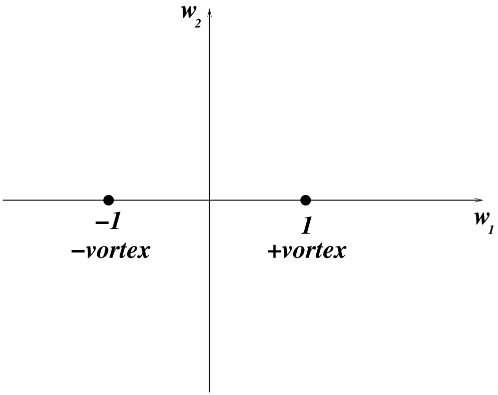
- Solve with b.c. $\widehat{h}(\infty) = 0$

Solving the (1,1) Taubes equation (numerically)

- Symmetry:



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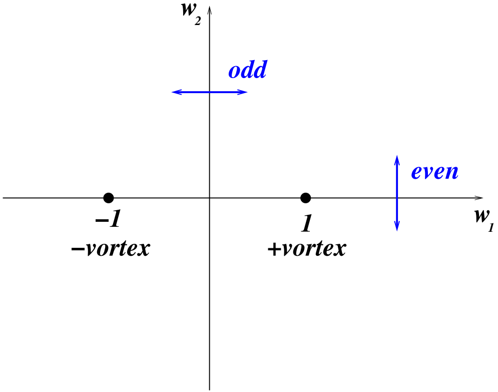

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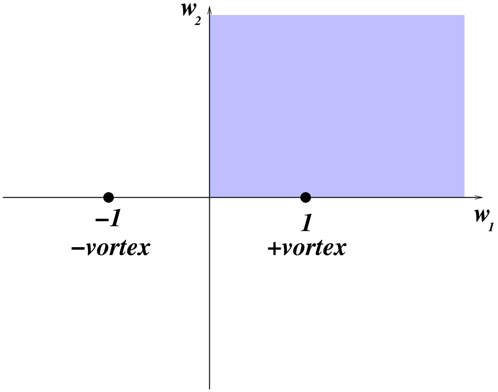

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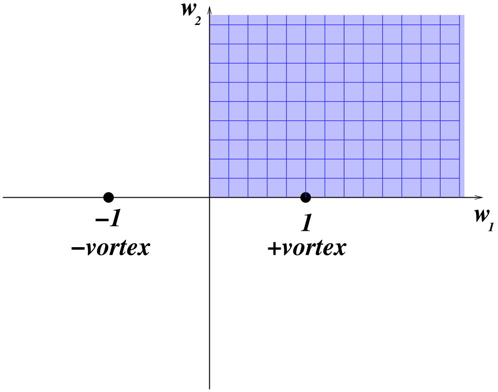

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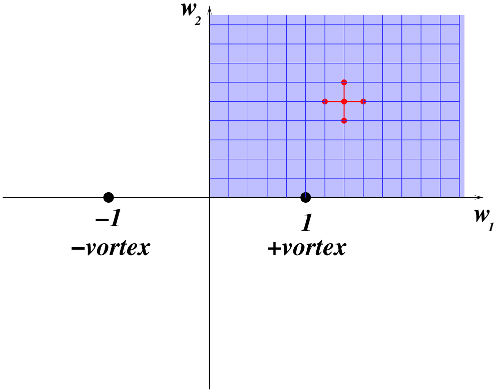

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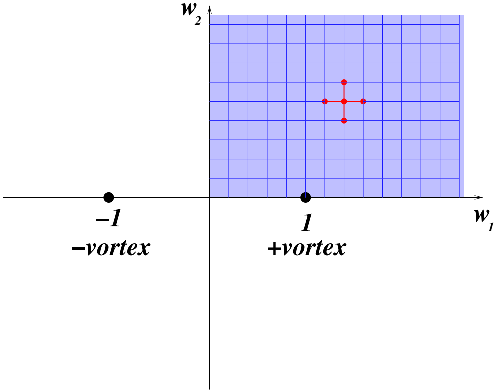
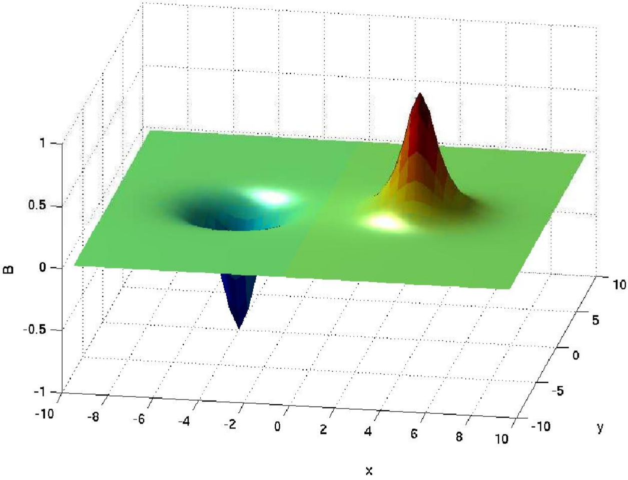


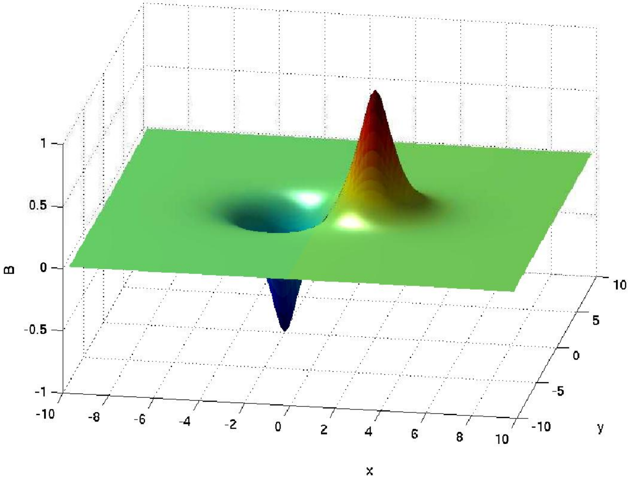
- $F(\widehat{h}_{ij}) = 0$, solve with Newton-Raphson

(1,1) vortices



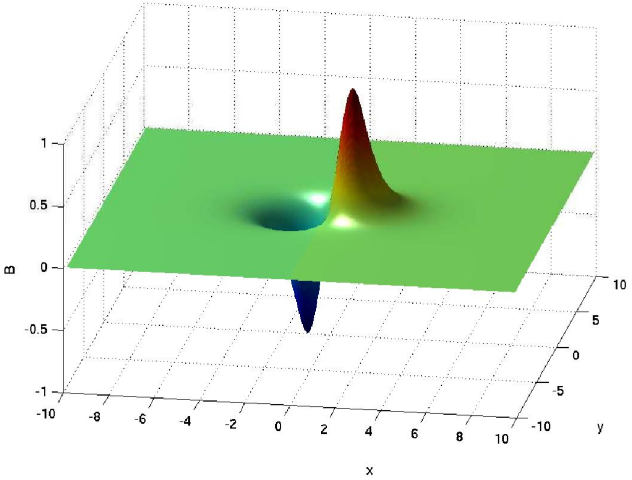
$\varepsilon = 4$

(1,1) vortices



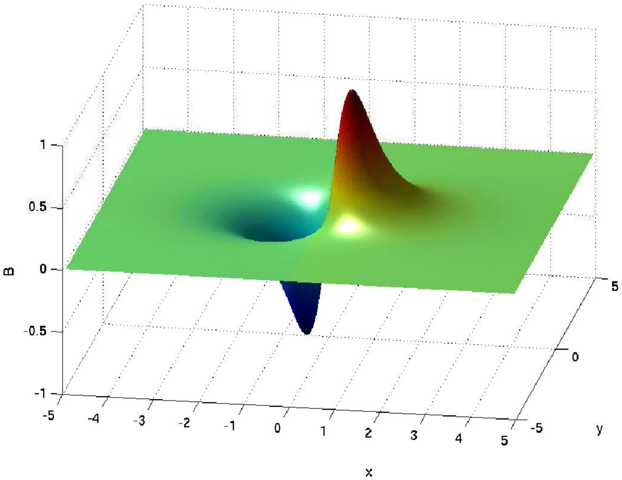
$\varepsilon = 2$

(1,1) vortices



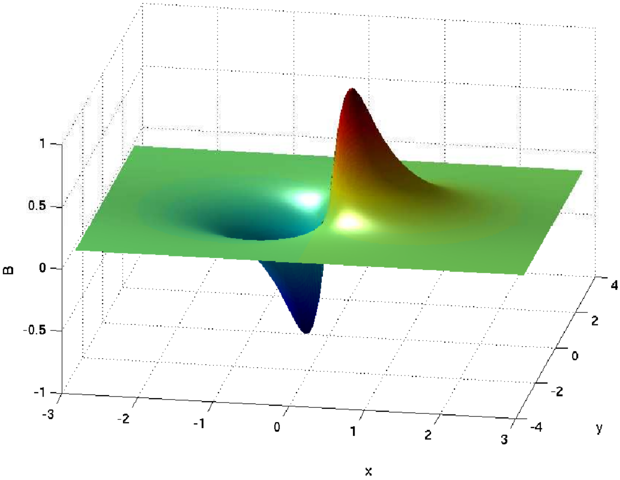
$\varepsilon = 1$

(1,1) vortices



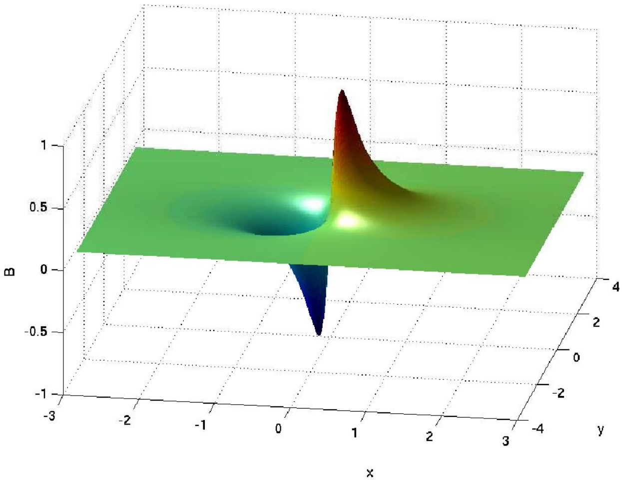
$\varepsilon = 0.5$

(1,1) vortices



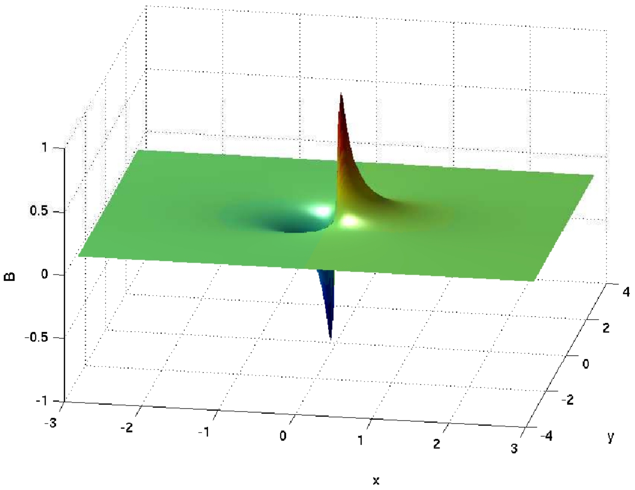
$\varepsilon = 0.3$

(1,1) vortices



$\varepsilon = 0.15$

(1,1) vortices



$\varepsilon = 0.06$

Vortex dynamics

$$S = \int (T - E)dt, \qquad T = \frac{1}{2}\int_{\mathbb{R}^2} |\dot{\mathbf{n}}|^2 + |\dot{A}|^2$$

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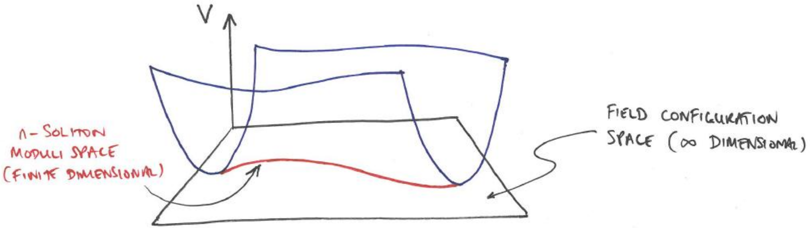

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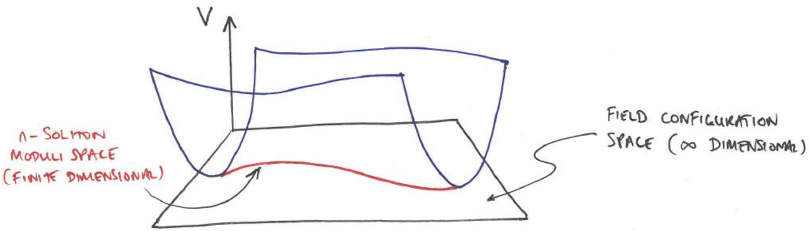

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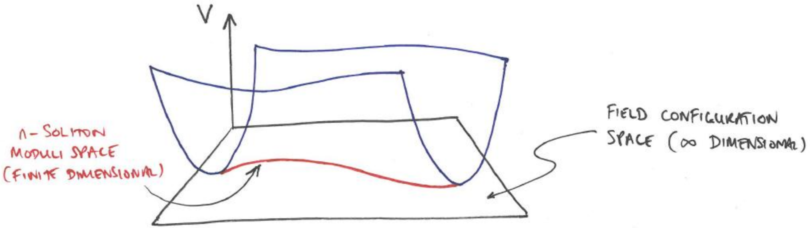

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- **Geodesic** motion in M_{n_+,n_-} w.r.t. the L^2 metric.

Strachan-Samols localization

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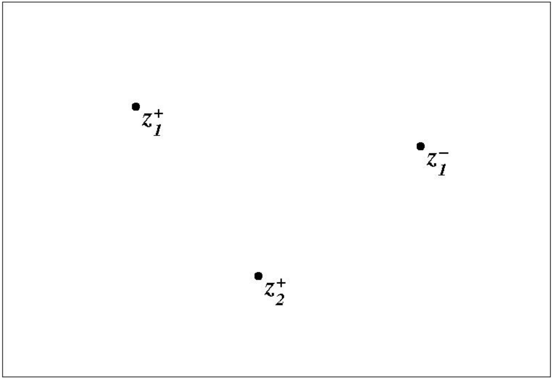
$$\eta = \sum_r \dot{z}_r^+\frac{\partial h}{\partial z_r^+} + \sum_r \dot{z}_r^-\frac{\partial h}{\partial z_r^-}$$

- η is a very good way to characterize $(\dot{\mathbf{n}}, \dot{A})$. Why?

Strachan-Samols localization



$$T = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\dot{A}|^2 + \frac{4|\dot{u}|^2}{(1+|u|^2)^2}\right)$$

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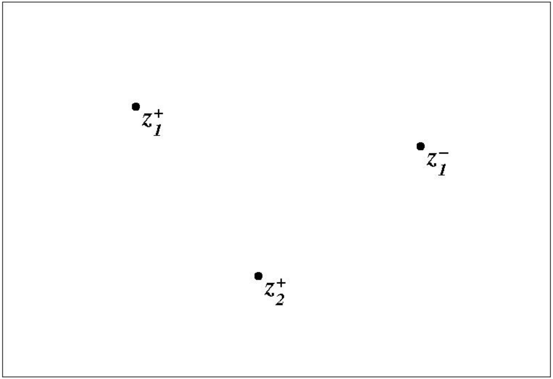

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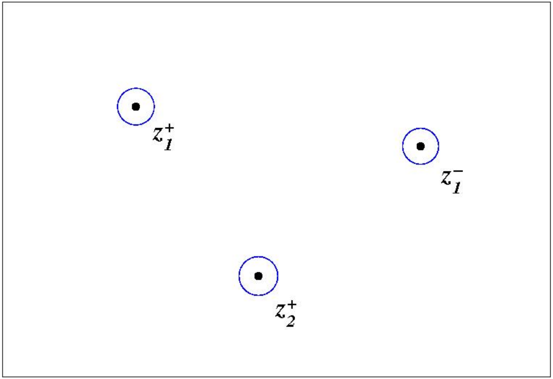

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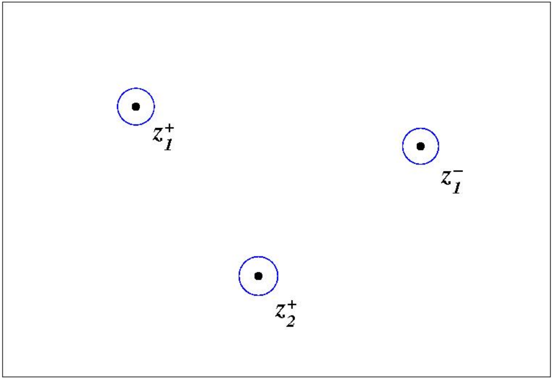

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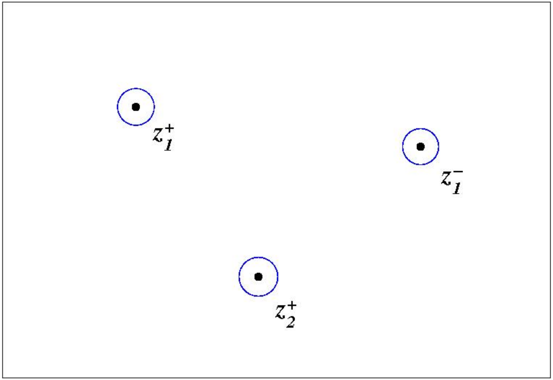

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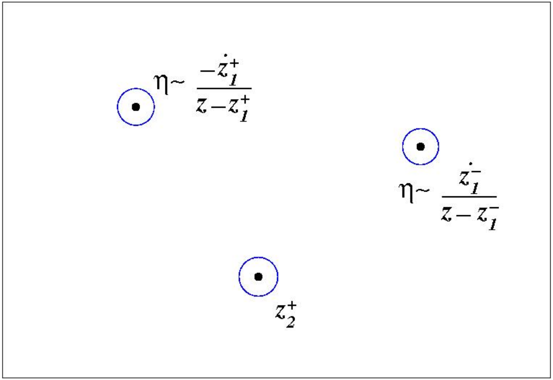

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where sums over all (anti)vortex positions and, in a nbhd of $z_s^{\pm}$,

$$h = \pm \left\{ \log |z - z_s^{\pm}|^2 + a_s + \frac{1}{2}\bar{b}_s(z - z_s^{\pm}) + \frac{1}{2} b_s(\bar{z} - \bar{z}_s^{\pm}) + \cdots \right\}$$

Strachan-Samols localization

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- Can compute g if we know $b_r(z_1^+, \ldots, z_{n_-}^-)$

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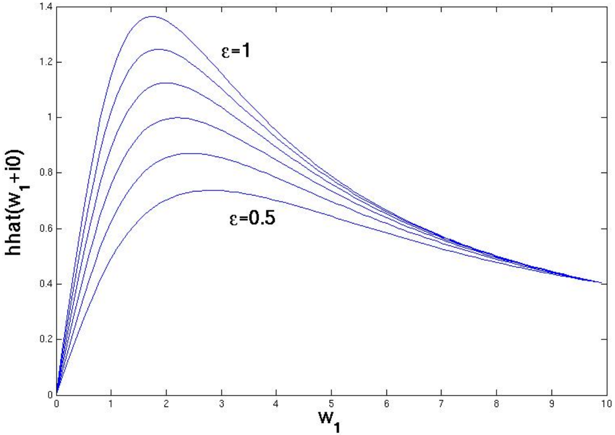

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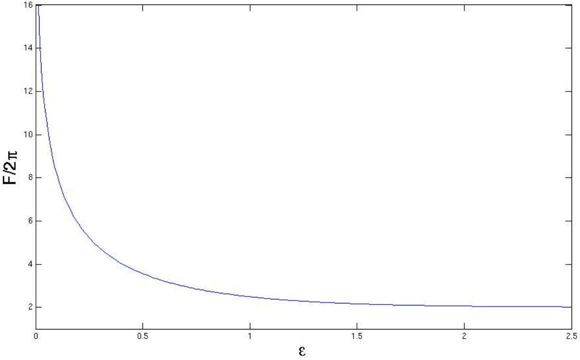
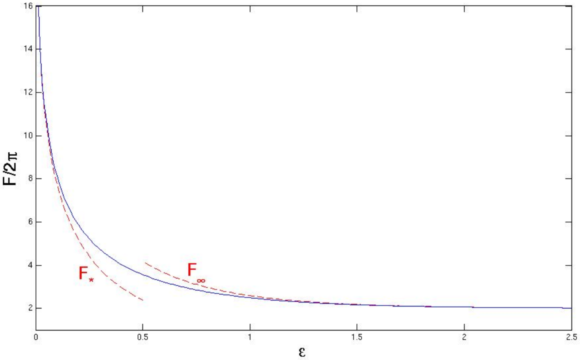


$$F(\varepsilon) = 2\pi \left(2 + \frac{1}{\varepsilon} \frac{d(\varepsilon b(\varepsilon))}{d\varepsilon} \right)$$

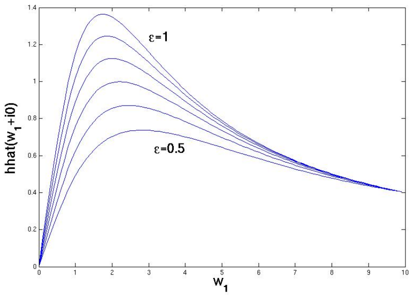
The metric on $M_{1,1}$: conjectured asymptotics



$$F_*(\varepsilon) = 2\pi\left(2 + 4K_0(\varepsilon) - 2\varepsilon K_1(\varepsilon)\right) \sim -8\pi \log \varepsilon$$

$$F_\infty(\varepsilon) = 2\pi\left(2 + \frac{q^2}{\pi^2} K_0(2\varepsilon)\right)$$

Self similarity as $\varepsilon \longrightarrow 0$



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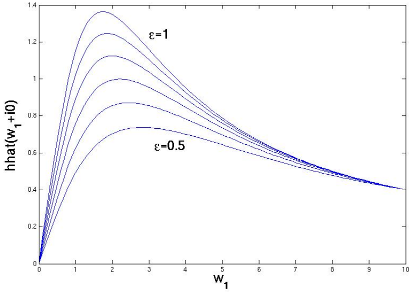

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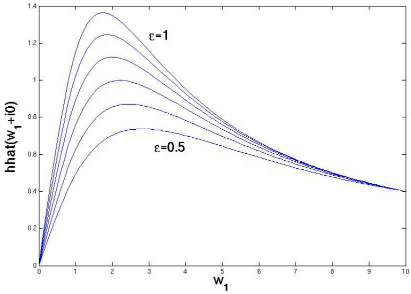

- Suggests $\widehat{h}_\varepsilon(w) \approx \varepsilon f_*(\varepsilon w)$ for small ε, where f_* is fixed?
- Define $f_\varepsilon(z) := \varepsilon^{-1}\widehat{h}_\varepsilon(\varepsilon^{-1} z)$

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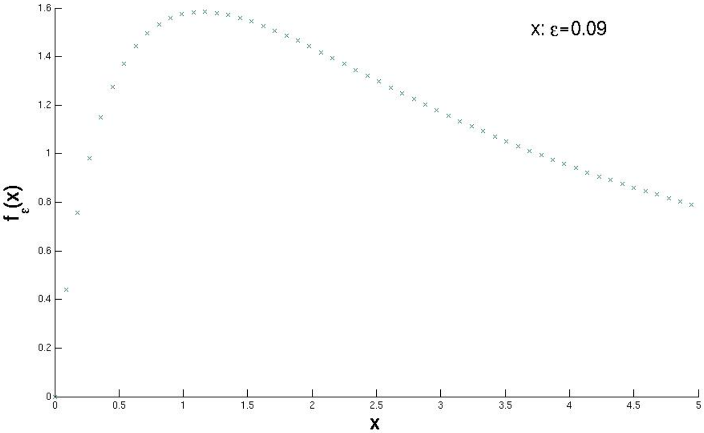

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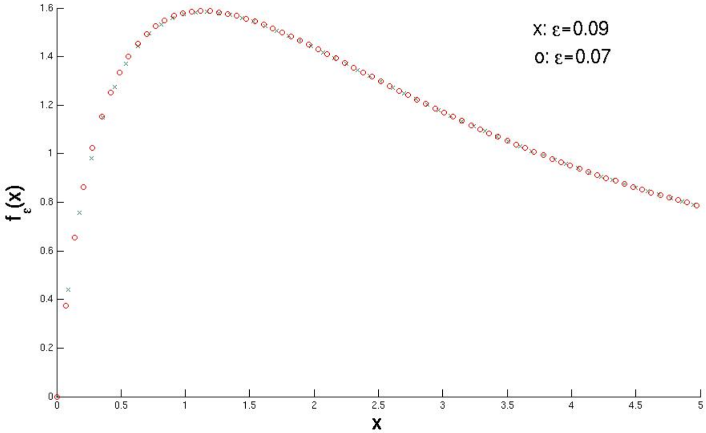

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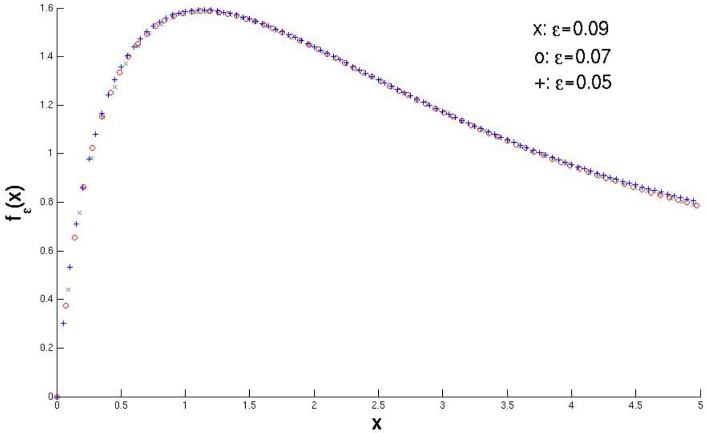

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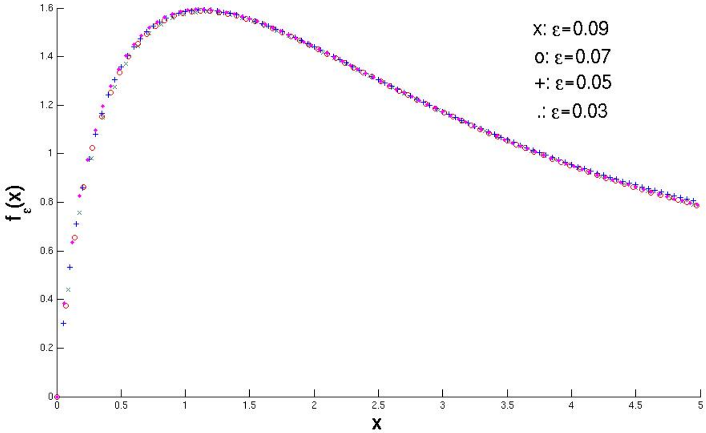

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- Unique solution (decaying at infinity)

$$f_*(re^{i\theta}) = \frac{4}{r}(1 - rK_1(r))\cos\theta$$

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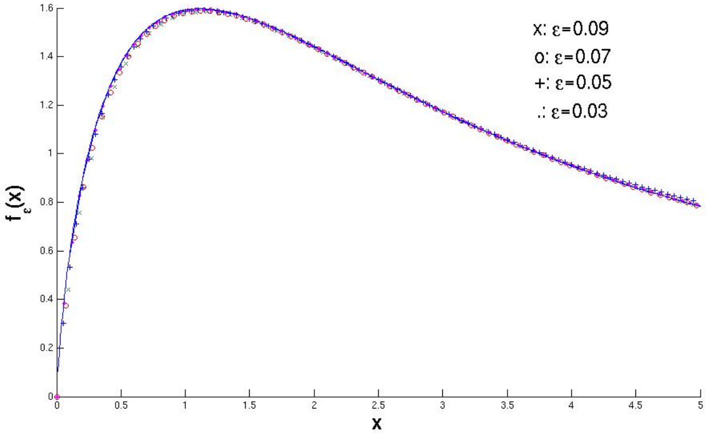

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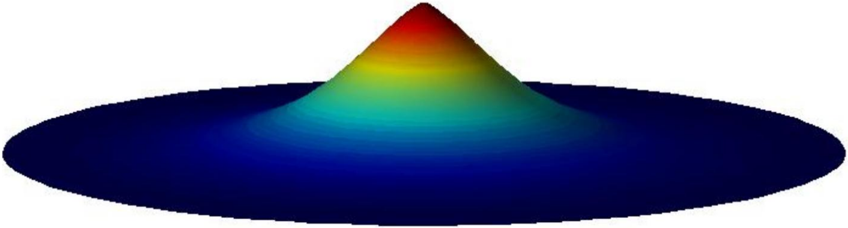
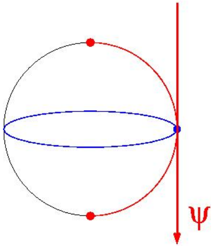
- $M_{1,1}$ is **incomplete**, with unbounded curvature



Long range vortex interactions

- $n_+ = 1$ vortex asymptotically indistinguishable from solution of *linearization* of model about vacuum $\mathbf{n} = (1, 0, 0)$



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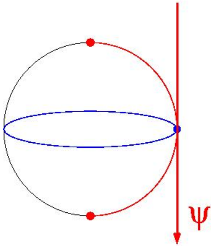

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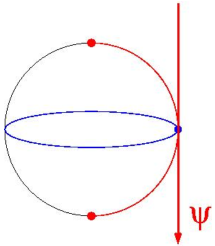

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in presence of **sources**:

$$\begin{aligned} \kappa &= q\delta(x) && \text{scalar monopole } q \\ (j^0, \mathbf{j}) &= (0, -q\mathbf{k}\times\nabla\delta(x)) && \text{magnetic dipole } q\mathbf{k} \end{aligned}$$

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Long range vortex interactions

- Point vortex: composite point particle of mass 2π, scalar charge q and magnetic dipole moment $q\mathbf{k}$
- Point antivortex: same thing, but $q \mapsto -q$
  [cf vanilla (anti)vortices: only dipole moment flips!]
- Charge q comes from numerics: $q \approx -7.1$
  [cf vanilla vortex: $q \approx -10.6$]
- Deep principle / leap of faith: physics is model independent
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- Interaction Lagrangian

$$L_{int} = \int_{\mathbb{R}^2} \left(\kappa_1 \psi_2 - j_1^\mu A_\mu^2\right)$$

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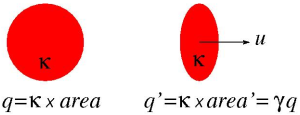

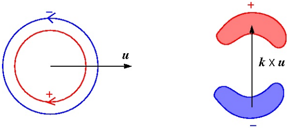

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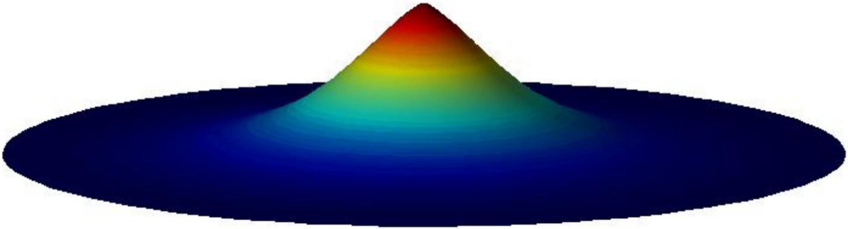
$$L = \pi(|\dot{\mathbf{x}}_1|^2 + |\dot{\mathbf{x}}_2|^2) \mp \frac{q^2}{4\pi} K_0(|\mathbf{x}_1 - \mathbf{x}_2|)|\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2|^2 \qquad \begin{cases} VV \\ V\overline{V} \end{cases}$$

Long range vortex interactions

- On $M^0_{1,1}$,

$$g^0_{L^2} = F(\varepsilon)(d\varepsilon^2 + \varepsilon^2 d\psi^2) \qquad F(\varepsilon) \sim 2\pi\left(2 + \frac{q^2}{\pi^2}K_0(2\varepsilon)\right).$$

Asymptotically negatively curved



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  - Conjecture

$$Vol(M_{n,m}(S^2)) = \frac{(2\pi)^{n+m}}{n!m!}(Vol(S^2) - \pi(n-m))^n(Vol(S^2) + \pi(n-m))^m$$