L^2 geometry of vortices

Martin Speight
Joint work with Nuno Romão and René Garcia Lara

17/5/23

University of Leeds

Nuno

René

Vortices

- Gauged sigma model on a Riemann surface Σ:
    - Gauge field A (connexion on $G \hookrightarrow P \to \Sigma$)

Vortices

- Gauged sigma model on a Riemann surface Σ:
    - Gauge field A (connexion on $G \hookrightarrow P \to \Sigma$)
    - Higgs field φ taking values in kähler mfd X

Vortices

- Gauged sigma model on a Riemann surface Σ:
  - Gauge field A (connexion on $G \hookrightarrow P \to \Sigma$)
  - Higgs field φ taking values in kähler mfd X
  - Holomorphic hamiltonian action of G on X, moment map $\mu : X \to \mathfrak{g}$

Vortices

- Gauged sigma model on a Riemann surface Σ:
  - Gauge field A (connexion on $G \hookrightarrow P \to \Sigma$)
  - Higgs field φ taking values in kähler mfd X
  - Holomorphic hamiltonian action of G on X, moment map $\mu : X \to \mathfrak{g}$
- A **vortex** is a pair $\varphi \in \Gamma(P^X)$, and $A \in \mathscr{A}(P)$, s.t.

$$
\begin{aligned}
(V1) \qquad & \bar{\partial}_A \varphi &=& \ 0 \\
(V2) \qquad & * F_A &=& \ \mu \circ \varphi
\end{aligned}
$$

Vortices

- Gauged sigma model on a Riemann surface Σ:
  - Gauge field A (connexion on $G \hookrightarrow P \to \Sigma$)
  - Higgs field φ taking values in kähler mfd X
  - Holomorphic hamiltonian action of G on X, moment map $\mu : X \to \mathfrak{g}$
- A **vortex** is a pair $\varphi \in \Gamma(P^X)$, and $A \in \mathscr{A}(P)$, s.t.

$$\begin{array}{lrcl} (V1) & \overline{\partial}_A\varphi & = & 0 \\ (V2) & *F_A & = & \mu \circ \varphi \end{array}$$

- They **minimize energy**

$$E(\varphi, A) = \frac{1}{2}\int_\Sigma |d_A\varphi|^2 + |F_A|^2 + |\mu \circ \varphi|^2$$

  in their homotopy class

Vortices

- Moduli space

$$M = \{(\varphi, A) : (V1), (V2)\}/\mathcal{G}$$

  has a natural symplectic structure (and complex structure, hence kähler **metric**)

Vortices

- Moduli space

$$M = \{(\varphi, A) : (V1), (V2)\}/\mathcal{G}$$

has a natural symplectic structure (and complex structure, hence kähler **metric**)

- Mundet i Riera (after Garcia-Prada): interpret

$$\mu^{\mathcal{G}}(\varphi, A) = \mu \circ \varphi - *F_A$$

as moment map for action of $\mathcal{G}$ on $\Gamma(X^P) \times \mathscr{A}(P)$

$$M = \mu^{\mathcal{G}}(0)/\mathcal{G}$$

Vortices

- Moduli space

$$M = \{(\varphi, A) : (V1), (V2)\}/\mathcal{G}$$

  has a natural symplectic structure (and complex structure, hence kähler **metric**)
- Mundet i Riera (after Garcia-Prada): interpret

$$\mu^{\mathcal{G}}(\varphi, A) = \mu \circ \varphi - *F_A$$

  as moment map for action of $\mathcal{G}$ on $\Gamma(X^P) \times \mathscr{A}(P)$

$$M = \mu^{\mathcal{G}}(0)/\mathcal{G}$$

- Geodesics in M: classical vortex dynamics
- $\operatorname{spec} \Delta_M$: quantum vortex dynamics

Vortices

- Moduli space

$$M = \{(\varphi, A) : (V1), (V2)\}/\mathcal{G}$$

  has a natural symplectic structure (and complex structure, hence kähler **metric**)

- Mundet i Riera (after Garcia-Prada): interpret

$$\mu^{\mathcal{G}}(\varphi, A) = \mu \circ \varphi - *F_A$$

  as moment map for action of $\mathcal{G}$ on $\Gamma(X^P) \times \mathscr{A}(P)$

$$M = \mu^{\mathcal{G}}(0)/\mathcal{G}$$

- Geodesics in M: classical vortex dynamics
- $\operatorname{spec} \Delta_M$: quantum vortex dynamics
- What can we say about this metric?

Two simple cases

- $X = \mathbb{C}$, $G = U(1)$, $\mu(x) = \frac{1}{2}(1 - |x|^2)$: abelian Higgs model (at critical coupling)

$$E(\varphi, A) = \frac{1}{2}\int_\Sigma |d_A\varphi|^2 + |F_A|^2 + \frac{1}{4}(1 - |\varphi|^2)^2$$

$d_A\varphi = d\varphi - iA\varphi$

Two simple cases

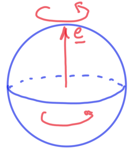
- $X = \mathbb{C}$, $G = U(1)$, $\mu(x) = \frac{1}{2}(1 - |x|^2)$: abelian Higgs model (at critical coupling)

$$E(\varphi, A) = \frac{1}{2}\int_\Sigma |d_A\varphi|^2 + |F_A|^2 + \frac{1}{4}(1 - |\varphi|^2)^2$$

$d_A\varphi = d\varphi - iA\varphi$

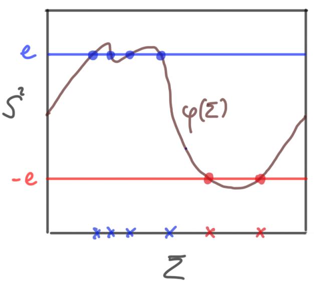
- $X = S^2$, $G = U(1)$, $\mu(\boldsymbol{x}) = \boldsymbol{e} \cdot \boldsymbol{x}$: gauged $O(3)$ sigma model

$$E(\boldsymbol{\varphi}, A) = \frac{1}{2}\int_\Sigma |d_A\boldsymbol{\varphi}|^2 + |F_A|^2 + (\boldsymbol{e} \cdot \boldsymbol{\varphi})^2$$



$d_A\boldsymbol{\varphi} = d\boldsymbol{\varphi} - A\boldsymbol{e} \times \boldsymbol{\varphi}$

Two topological charges



$$k_+ = \#(\varphi(\Sigma), \boldsymbol{e}), \qquad k_- = \#(\varphi(\Sigma), -\boldsymbol{e})$$

Constraint: $k_+ - k_- = \deg(P)$

The "Bogomol'nyi" bound (Schroers)

$$E = \frac{1}{2}\int_\Sigma |d_A\varphi(e_1) + \varphi \times d_A\varphi(e_2)|^2 + |*F_A - \boldsymbol{e}\cdot\varphi|^2 + \int_\Sigma \Xi$$

The "Bogomol'nyi" bound (Schroers)

$$
\begin{aligned}
E &= \frac{1}{2}\int_\Sigma |d_A\varphi(e_1) + \varphi \times d_A\varphi(e_2)|^2 + |*F_A - \boldsymbol{e}\cdot\varphi|^2 + \int_\Sigma \Xi \\
\int_\Sigma \Xi &= 2\pi(k_+ + k_-)
\end{aligned}
$$

The "Bogomol'nyi" bound (Schroers)

$$
\begin{aligned}
E &= \frac{1}{2}\int_\Sigma |d_A\boldsymbol{\varphi}(e_1) + \boldsymbol{\varphi}\times d_A\boldsymbol{\varphi}(e_2)|^2 + |*F_A - \boldsymbol{e}\cdot\boldsymbol{\varphi}|^2 + \int_\Sigma \Xi \\
\int_\Sigma \Xi &= 2\pi(k_+ + k_-)
\end{aligned}
$$

Hence $E \geq 2\pi(k_+ + k_-)$ with equality iff

$$
(V1) \quad d_A\boldsymbol{\varphi}\circ J_\Sigma = J_{S^2}\circ d_A\boldsymbol{\varphi}, \qquad (V2) \quad *F_A = \boldsymbol{e}\cdot\boldsymbol{\varphi}
$$

The "Bradlow" bound (Sibner^2 Yang)

$$*F_A = \boldsymbol{e} \cdot \varphi$$

The "Bradlow" bound (Sibner^2 Yang)

$$
\begin{aligned}
*F_A &= \boldsymbol{e} \cdot \varphi \\
\int_\Sigma F_A &= \int_\Sigma \boldsymbol{e} \cdot \varphi
\end{aligned}
$$

The “Bradlow” bound (Sibner^2 Yang)

$$
\begin{aligned}
*F_A &= \boldsymbol{e} \cdot \varphi \\
\int_\Sigma F_A &= \int_\Sigma \boldsymbol{e} \cdot \varphi \\
2\pi \deg(P) &\in [-|\Sigma|, |\Sigma|]
\end{aligned}
$$

Hence, if a (k_+, k_-)-vortex exists,

$$
|k_+ - k_-| \leq \frac{|\Sigma|}{2\pi}
$$

The existence theorem (Sibner^2 Yang)

Assume $|k_+ - k_-| < \frac{|\Sigma|}{2\pi}$ and choose

The existence theorem (Sibner^2 Yang)

Assume $|k_+ - k_-| < \frac{|\Sigma|}{2\pi}$ and choose

- any collection Q_+ of k_+ points on Σ (not necessarily distinct)

The existence theorem (Sibner[2] Yang)

Assume $|k_+ - k_-| < \frac{|\Sigma|}{2\pi}$ and choose

- any collection Q_+ of k_+ points on Σ (not necessarily distinct) and
- any collection Q_- of k_- points on Σ (not necessarily distinct)

The existence theorem (Sibner^2 Yang)

Assume $|k_+ - k_-| < \frac{|\Sigma|}{2\pi}$ and choose

- any collection Q_+ of k_+ points on Σ (not necessarily distinct) and
- any collection Q_- of k_- points on Σ (not necessarily distinct) such that
- $Q_+ \cap Q_- = \emptyset$.

The existence theorem (Sibner^2 Yang)

Assume $|k_+ - k_-| < \frac{|\Sigma|}{2\pi}$ and choose

- any collection Q_+ of k_+ points on Σ (not necessarily distinct) and
- any collection Q_- of k_- points on Σ (not necessarily distinct) such that
- $Q_+ \cap Q_- = \emptyset$.

Then there exists a smooth solution of $(V1), (V2)$ with $\varphi^{-1}(\pm\boldsymbol{e}) = Q_\pm$ and this solution is unique up to gauge.

The existence theorem (Sibner^2 Yang)

Idea: define

$$h : \Sigma \to [-\infty, \infty], \qquad h = \log\left(\frac{1 - \boldsymbol{e} \cdot \varphi}{1 + \boldsymbol{e} \cdot \varphi}\right)$$

The existence theorem (Sibner^2 Yang)

Idea: define

$$h : \Sigma \to [-\infty, \infty], \qquad h = \log\left(\frac{1 - \boldsymbol{e}\cdot\varphi}{1 + \boldsymbol{e}\cdot\varphi}\right)$$

Observe $(V1), (V2)$ iff

$$\Delta_\Sigma h + 2\tanh\frac{h}{2} = 0$$

The existence theorem (Sibner^2 Yang)

Idea: define

$$h : \Sigma \to [-\infty, \infty], \qquad h = \log\left(\frac{1 - \boldsymbol{e}\cdot\varphi}{1 + \boldsymbol{e}\cdot\varphi}\right)$$

Observe $(V1), (V2)$ with $\varphi^{-1}(\pm\boldsymbol{e}) = Q_\pm$ iff

$$\Delta_\Sigma h + 2\tanh\frac{h}{2} = -4\pi \sum_{p\in Q_+} \delta_p + 4\pi \sum_{p\in Q_-} \delta_p$$

The existence theorem (Sibner^2 Yang)

Idea: define

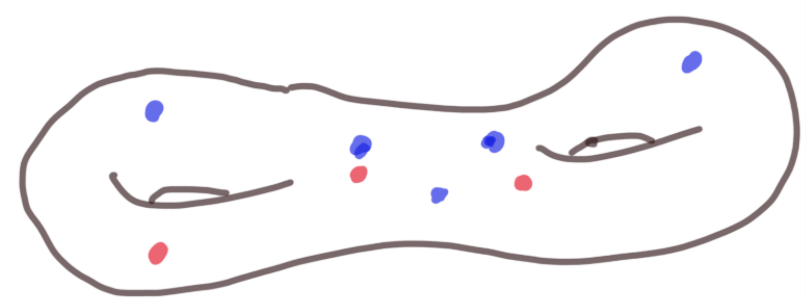
$$h : \Sigma \to [-\infty, \infty], \qquad h = \log\left(\frac{1 - \boldsymbol{e}\cdot\varphi}{1 + \boldsymbol{e}\cdot\varphi}\right)$$

Observe $(V1), (V2)$ with $\varphi^{-1}(\pm\boldsymbol{e}) = Q_\pm$ iff

$$\Delta_\Sigma h + 2\tanh\frac{h}{2} = -4\pi\sum_{p\in Q_+}\delta_p + 4\pi\sum_{p\in Q_-}\delta_p$$

Regularize, convert into $T(h_{reg}) = h_{reg}$ for a nonlinear operator $T : H^1_0(\Sigma) \to H^1_0(\Sigma)$. Apply Leray-Schauder.

The moduli space



The moduli space

$$M_{k_+,k_-}(\Sigma) \equiv [\mathrm{Sym}_{k_+}\Sigma \times \mathrm{Sym}_{k_-}\Sigma] \backslash \Delta_{k_+,k_-}$$

- $\mathrm{Sym}_k\,\Sigma = \Sigma^k / S_k$

The moduli space

$$M_{k_+,k_-}(\Sigma) \equiv [\mathrm{Sym}_{k_+}\Sigma \times \mathrm{Sym}_{k_-}\Sigma]\backslash\Delta_{k_+,k_-}$$

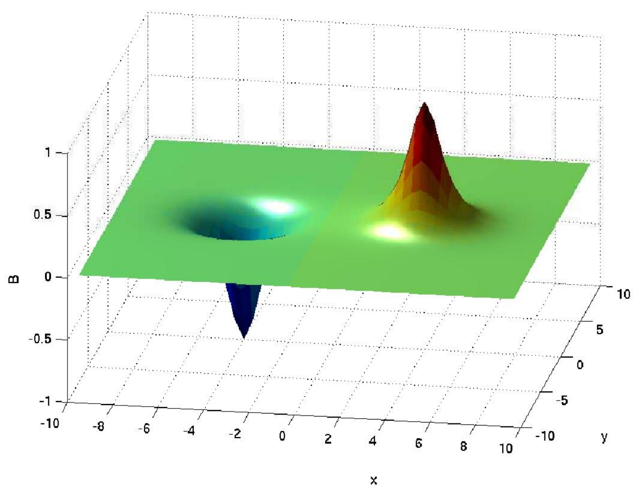
- $\mathrm{Sym}_k\,\Sigma = \Sigma^k/S_k$
- Looks singular on coincidence set but **in dimension 2** has a canonical smooth structure

$$\begin{aligned} p(z) &= (z-z_1)(z-z_2)\cdots(z-z_k) \\ &= z^k + a_1 z^{k-1} + \cdots + a_k \end{aligned}$$

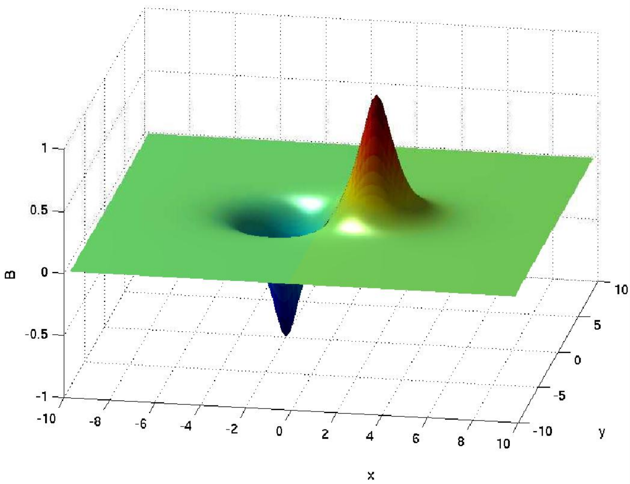
$(a_1, a_2, \ldots, a_k)$ define local complex coords on $\mathrm{Sym}_k\,\Sigma$

- $M_{k_+,k_-}(\Sigma)$ is noncompact (if $k_+, k_- > 0$)

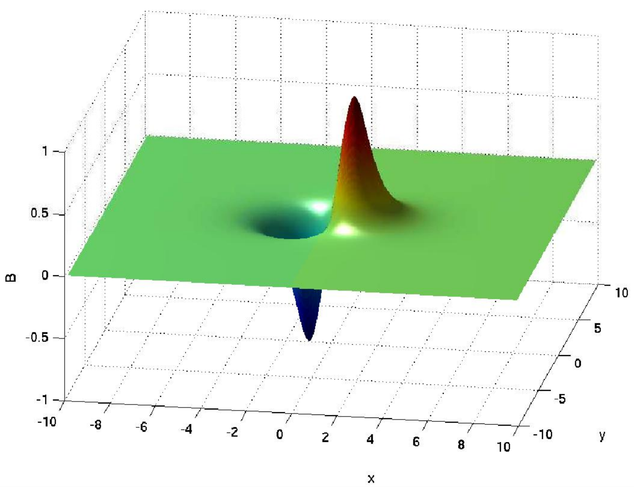
Noncompactness



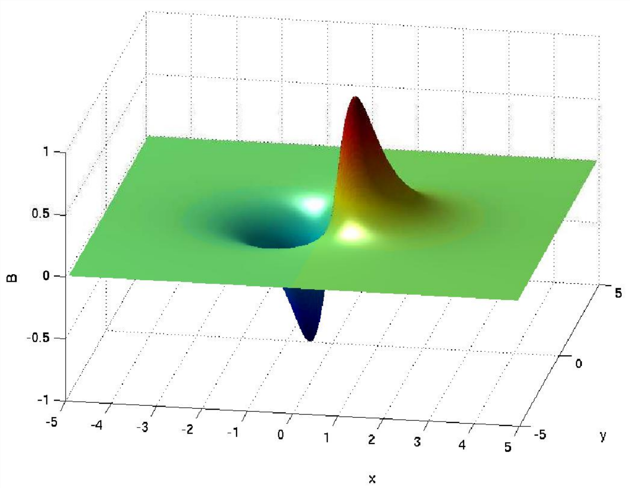
Noncompactness



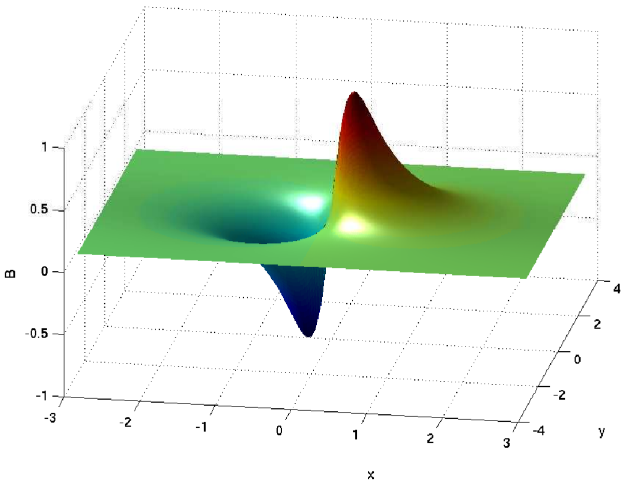
Noncompactness



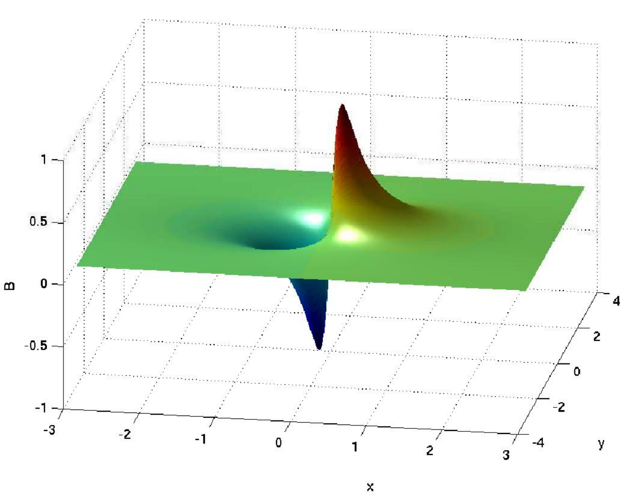
Noncompactness



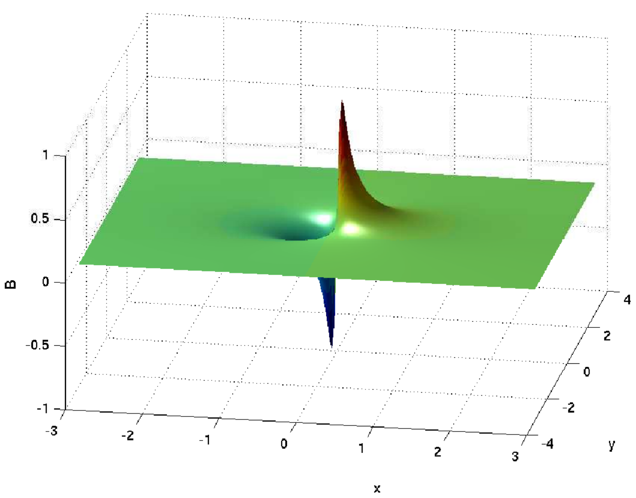
Noncompactness



Noncompactness

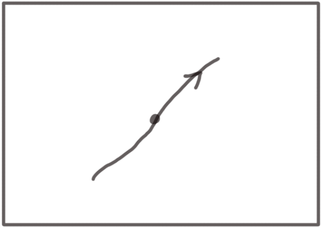


Noncompactness



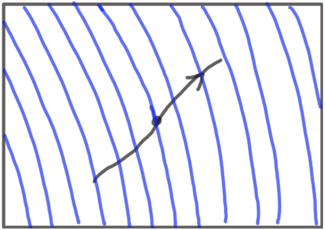
The metric on $M_{k_+,k_-}(\Sigma)$

- Take a curve $(\varphi(t), A(t))$ of solutions of $(V1), (V2)$



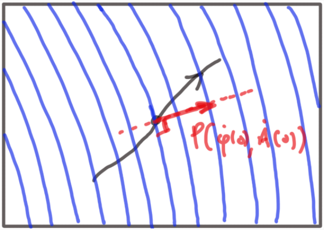
The metric on $M_{k_+,k_-}(\Sigma)$

- Take a curve $(\varphi(t), A(t))$ of solutions of $(V1), (V2)$



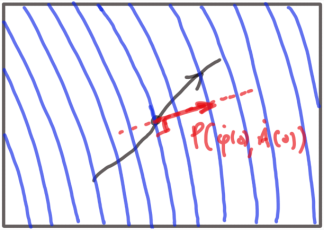
The metric on $M_{k_+,k_-}(\Sigma)$

- Take a curve $(\varphi(t), A(t))$ of solutions of $(V1), (V2)$



The metric on $M_{k_+,k_-}(\Sigma)$

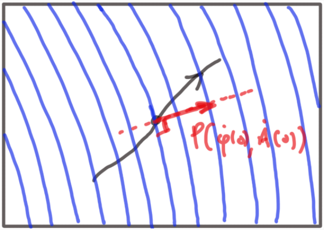
- Take a curve $(\varphi(t), A(t))$ of solutions of $(V1), (V2)$



- $\|[\dot{\varphi}(0), \dot{A}(0)]\|^2 := \int_\Sigma \left(|P\dot{\varphi}(0)|^2 + |P\dot{A}(0)|^2 \right)$

The metric on $M_{k_+,k_-}(\Sigma)$

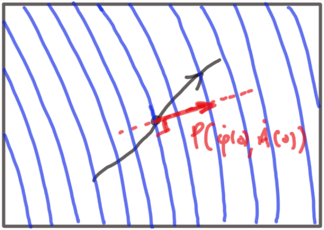
- Take a curve $(\varphi(t), A(t))$ of solutions of $(V1), (V2)$



- $\|[\dot{\varphi}(0), \dot{A}(0)]\|^2 := \int_\Sigma \left(|P\dot{\varphi}(0)|^2 + |P\dot{A}(0)|^2 \right)$
- Defines a riemannian metric g_{L^2} on $M_{k_+,k_-}(\Sigma)$

The metric on $M_{k_+,k_-}(\Sigma)$

- Take a curve $(\varphi(t), A(t))$ of solutions of $(V1), (V2)$



- $\|[\dot\varphi(0), \dot A(0)]\|^2 := \int_\Sigma \left(|P\dot\varphi(0)|^2 + |P\dot A(0)|^2\right)$
- Defines a riemannian metric g_{L^2} on $M_{k_+,k_-}(\Sigma)$
- Coincides with metric defined by symplectic quotient

The localization formula (JMS-Romão)

- Consider case $k_+ = k_- = 1$: $M = \Sigma \times \Sigma \backslash \Delta$

The localization formula (JMS-Romão)

- Consider case $k_+ = k_- = 1$: $M = \Sigma \times \Sigma \backslash \Delta$
- Choose local coord z on Σ

The localization formula (JMS-Romão)

- Consider case $k_+ = k_- = 1$: $M = \Sigma \times \Sigma \backslash \Delta$
- Choose local coord z on Σ
- Pt in $M \leftrightarrow (z_+, z_-)$, $z_+ \neq z_-$

The localization formula (JMS-Romão)

- Consider case $k_+ = k_- = 1$: $M = \Sigma \times \Sigma \backslash \Delta$
- Choose local coord z on Σ
- Pt in $M \leftrightarrow (z_+, z_-)$, $z_+ \neq z_-$
- Consider Yang function $h = \log\left(\frac{1-\boldsymbol{e}\cdot\varphi}{1+\boldsymbol{e}\cdot\varphi}\right)$ for this $(1,1)$-vortex

The localization formula (JMS-Romão)

- Consider case $k_+ = k_- = 1$: $M = \Sigma \times \Sigma \backslash \Delta$
- Choose local coord z on Σ
- Pt in $M \leftrightarrow (z_+, z_-)$, $z_+ \neq z_-$
- Consider Yang function $h = \log\left(\frac{1 - \boldsymbol{e}\cdot\boldsymbol{\varphi}}{1 + \boldsymbol{e}\cdot\boldsymbol{\varphi}}\right)$ for this $(1,1)$-vortex
- Close to $z_\pm$

$$\pm h(z) = \log|z - z_\pm|^2 + a_\pm + \frac{1}{2} b_\pm (z - z_\pm) + \frac{\overline{b_\pm}}{2}(\overline{z} - \overline{z_\pm}) + \cdots$$

Defies complex functions $b_\pm(z_+, z_-)$

The localization formula (JMS-Romão)

- Consider case $k_+ = k_- = 1$: $M = \Sigma \times \Sigma \backslash \Delta$
- Choose local coord z on Σ
- Pt in $M \leftrightarrow (z_+, z_-)$, $z_+ \neq z_-$
- Consider Yang function $h = \log\left(\frac{1-\boldsymbol{e}\cdot\boldsymbol{\varphi}}{1+\boldsymbol{e}\cdot\boldsymbol{\varphi}}\right)$ for this $(1,1)$-vortex
- Close to $z_\pm$

  $$\pm h(z) = \log|z - z_\pm|^2 + a_\pm + \frac{1}{2}b_\pm(z - z_\pm) + \frac{\overline{b_\pm}}{2}(\overline{z} - \overline{z_\pm}) + \cdots$$

  Defies complex functions $b_\pm(z_+, z_-)$
- $b := b_+ dz_+ + b_- dz_-$ locally defined $(1,0)$ form

The localization formula (JMS-Romão)

- Consider case $k_+ = k_- = 1$: $M = \Sigma \times \Sigma \backslash \Delta$
- Choose local coord z on Σ
- Pt in $M \leftrightarrow (z_+, z_-)$, $z_+ \neq z_-$
- Consider Yang function $h = \log\left(\frac{1-\boldsymbol{e}\cdot\varphi}{1+\boldsymbol{e}\cdot\varphi}\right)$ for this $(1,1)$-vortex
- Close to $z_\pm$

  $$\pm h(z) = \log|z - z_\pm|^2 + a_\pm + \frac{1}{2}b_\pm(z - z_\pm) + \frac{\overline{b_\pm}}{2}(\overline{z} - \overline{z_\pm}) + \cdots$$

  Defies complex functions $b_\pm(z_+, z_-)$
- $b := b_+ dz_+ + b_- dz_-$ locally defined $(1,0)$ form
- Amazing fact:

  $$\omega_{L^2} = 2\pi(\pi_+^*\omega_\Sigma + \pi_-^*\omega_\Sigma + \overline{\partial}b)$$

  where $\pi_\pm : \Sigma \times \Sigma \backslash \Delta \to \Sigma$ are the projection maps
  $\pi_\pm(p_+, p_-) = p_\pm$

The localization formula (JMS-Romão)

- Form b globalizes as a connexion on K^{-1}, the dual canonical bundle on $M_{1,1}$

The localization formula (JMS-Romão)

- Form b globalizes as a connexion on K^{-1}, the dual canonical bundle on $M_{1,1}$
- ω_{L^2} is a real form by definition, so $\partial b = 0$. Hence $\overline{\partial} b = db = F_b$!

$$\omega_{L^2} = 2\pi(\pi_+^* \omega_\Sigma + \pi_-^* \omega_\Sigma + F_b)$$

- Can use this formula to prove (JMS, Romão, Garcia Lara) that
  - For all compact Σ, $M_{1,1}$ is geodesically incomplete

The localization formula (JMS-Romão)

- Form b globalizes as a connexion on K^{-1}, the dual canonical bundle on $M_{1,1}$
- ω_{L^2} is a real form by definition, so $\partial b = 0$. Hence $\overline{\partial} b = db = F_b$!

$$\omega_{L^2} = 2\pi(\pi_+^*\omega_\Sigma + \pi_-^*\omega_\Sigma + F_b)$$

- Can use this formula to prove (JMS, Romão, Garcia Lara) that
  - For all compact Σ, $M_{1,1}$ is geodesically incomplete
  - If $\Sigma = S^2_{round}$, $|M_{1,1}| = (2\pi|\Sigma|)^2$

The localization formula (JMS-Romão)

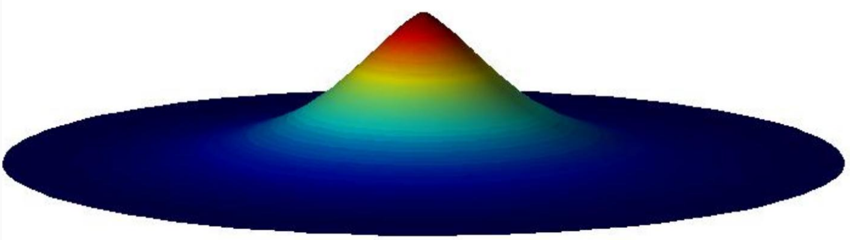
- Form b globalizes as a connexion on K^{-1}, the dual canonical bundle on $M_{1,1}$
- ω_{L^2} is a real form by definition, so $\partial b = 0$. Hence $\overline{\partial} b = db = F_b$!

$$\omega_{L^2} = 2\pi(\pi_+^* \omega_\Sigma + \pi_-^* \omega_\Sigma + F_b)$$

- Can use this formula to prove (JMS, Romão, Garcia Lara) that
  - For all compact Σ, $M_{1,1}$ is geodesically incomplete
  - If $\Sigma = S^2_{round}$, $|M_{1,1}| = (2\pi|\Sigma|)^2$
  - If $\Sigma = T^2_{flat}$, $|M_{1,1}| = (2\pi|\Sigma|)^2 + 16\pi^3|\Sigma|$

The localization formula (JMS-Romão)

- Can also use formula to do numerics



Isometric embedding of the space of centred $(1,1)$ vortices on euclidean $\mathbb{R}^2$

Compactification

- $G = T^2$, $X = \mathbb{C}^2$,
  $(e^{i\theta_1}, e^{i\theta_2}) : (x_+, x_-) \mapsto (e^{i(\theta_1+\theta_2)}x_+, e^{i\theta_2}x_-)$

Compactification

- $G = T^2$, $X = \mathbb{C}^2$,
  $(e^{i\theta_1}, e^{i\theta_2}) : (x_+, x_-) \mapsto (e^{i(\theta_1+\theta_2)}x_+, e^{i\theta_2}x_-)$
- $g_{\mathfrak{g}} = d\theta_1^2 + e^{-2}d\theta_2^2$

Compactification

- $G = T^2$, $X = \mathbb{C}^2$,
  $(e^{i\theta_1}, e^{i\theta_2}) : (x_+, x_-) \mapsto (e^{i(\theta_1+\theta_2)}x_+, e^{i\theta_2}x_-)$
- $g_{\mathfrak{g}} = d\theta_1^2 + e^{-2}d\theta_2^2$
- $\mu(x_+, x_-) = (1 - |x_-|^2)\partial_{\theta_1} + e^2(2 - |x_+|^2 - |x_-|^2)\partial_{\theta_2}$

Compactification

- $G = T^2$, $X = \mathbb{C}^2$,
  $(e^{i\theta_1}, e^{i\theta_2}) : (x_+, x_-) \mapsto (e^{i(\theta_1+\theta_2)}x_+, e^{i\theta_2}x_-)$
- $g_{\mathfrak{g}} = d\theta_1^2 + e^{-2}d\theta_2^2$
- $\mu(x_+, x_-) = (1 - |x_-|^2)\partial_{\theta_1} + e^2(2 - |x_+|^2 - |x_-|^2)\partial_{\theta_2}$
- $M^{\mathbb{C}^2}_{k_+,k_-}(\Sigma) = \mathrm{Sym}_{k_+}\Sigma \times \mathrm{Sym}_{k_-}\Sigma$

Compactification

- $G = T^2$, $X = \mathbb{C}^2$,
  $(e^{i\theta_1}, e^{i\theta_2}) : (x_+, x_-) \mapsto (e^{i(\theta_1+\theta_2)}x_+, e^{i\theta_2}x_-)$
- $g_{\mathfrak{g}} = d\theta_1^2 + e^{-2}d\theta_2^2$
- $\mu(x_+, x_-) = (1 - |x_-|^2)\partial_{\theta_1} + e^2(2 - |x_+|^2 - |x_-|^2)\partial_{\theta_2}$
- $M^{\mathbb{C}^2}_{k_+,k_-}(\Sigma) = \mathrm{Sym}_{k_+}\Sigma \times \mathrm{Sym}_{k_-}\Sigma$
- Canonical embedding $\iota : M^{S^2}_{k_+,k_-}(\Sigma) \to M^{\mathbb{C}^2}_{k_+,k_-}(\Sigma)$

Compactification

- $G = T^2$, $X = \mathbb{C}^2$,
  $(e^{i\theta_1}, e^{i\theta_2}) : (x_+, x_-) \mapsto (e^{i(\theta_1+\theta_2)}x_+, e^{i\theta_2}x_-)$
- $g_{\mathfrak{g}} = d\theta_1^2 + e^{-2}d\theta_2^2$
- $\mu(x_+, x_-) = (1 - |x_-|^2)\partial_{\theta_1} + e^2(2 - |x_+|^2 - |x_-|^2)\partial_{\theta_2}$
- $M^{\mathbb{C}^2}_{k_+,k_-}(\Sigma) = \mathrm{Sym}_{k_+}\,\Sigma \times \mathrm{Sym}_{k_-}\,\Sigma$
- Canonical embedding $\iota : M^{S^2}_{k_+,k_-}(\Sigma) \to M^{\mathbb{C}^2}_{k_+,k_-}(\Sigma)$
- **Conjecture** $\iota^* g^{\mathbb{C}^2}_{L^2}$ converges uniformly to $g^{S^2}_{L^2}$ in the limit $e \to \infty$

Compactification

- $G = T^2$, $X = \mathbb{C}^2$,
  $(e^{i\theta_1}, e^{i\theta_2}) : (x_+, x_-) \mapsto (e^{i(\theta_1+\theta_2)}x_+, e^{i\theta_2}x_-)$
- $g_{\mathfrak{g}} = d\theta_1^2 + e^{-2}d\theta_2^2$
- $\mu(x_+, x_-) = (1 - |x_-|^2)\partial_{\theta_1} + e^2(2 - |x_+|^2 - |x_-|^2)\partial_{\theta_2}$
- $M^{\mathbb{C}^2}_{k_+,k_-}(\Sigma) = \mathrm{Sym}_{k_+}\Sigma \times \mathrm{Sym}_{k_-}\Sigma$
- Canonical embedding $\iota : M^{S^2}_{k_+,k_-}(\Sigma) \to M^{\mathbb{C}^2}_{k_+,k_-}(\Sigma)$
- **Conjecture** $\iota^* g^{\mathbb{C}^2}_{L^2}$ converges uniformly to $g^{S^2}_{L^2}$ in the limit $e \to \infty$
- Consistent with volume computations