Skyrmions with low binding energies

Martin Speight (Leeds)
joint with
Derek Harland, Ben Maybee (Leeds)
Mike Gillard (Loughborough)
Elliot Kirk (Durham)

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The (extended) Skyrme model

$U : \mathbb{R}^3 \to SU(2),\ U(\infty) = \mathbb{I}_2,\ L_i = U^\dagger \partial_i U$

$$B(U) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \underbrace{\frac{\varepsilon_{ijk}}{12} tr(L_i L_j L_k)}_{\mathscr{B}} \in \mathbb{Z}$$

$$E(U) = \underbrace{-\frac{1}{4} \int_{\mathbb{R}^3} \mathrm{tr}(L_i L_i)}_{E_2} \underbrace{-\frac{1}{8} \int_{\mathbb{R}^3} \mathrm{tr}([L_i, L_j][L_i, L_j])}_{E_4} \underbrace{+ \int_{\mathbb{R}^3} V(U)}_{E_0} \underbrace{+ \frac{1}{2} \int_{\mathbb{R}^3} \mathscr{B}^2}_{E_6}$$

E minimizer of charge B: classical model of nucleon number B nucleus

The binding energy problem ($E = E_2 + E_4$)

Classical binding energy $= \frac{BE(U_1) - E(U_B)}{E(U_1)}$

B	Element	B.E. (Skyrme)	B.E. (experiment)
4	He	0.3639	0.0301
7	Li	0.7811	0.0414
9	Be	1.0123	0.0615
11	B	1.2792	0.0807
12	C	1.4277	0.0981
14	N	1.6815	0.1114
16	O	1.9646	0.1359
19	F	2.3684	0.1570
20	Ne	2.5045	0.1710

Naive quantization makes the problem **worse**

A solution?

- Faddeev showed that $E = E_2 + E_4$ has a topological lower bound:

$$E \geq const \times B$$

 Unfortunately (?) it's never attained
- What if we find a Skyrme energy with a bound like this which *is* attained for each B?
- Then $E(U_B) \equiv B \times E(U_1)$ so $B.E. \equiv 0$!
- Call such a model "BPS"
- Now perturb the BPS model by adding $\varepsilon(E_2 + E_4)$, ε small (for example)
- Hopefully get a "near BPS" Skyrme model with small positive BEs
- Two implementations of this idea
  - perturbed $E_6 + E_0$ model (Adam, Sanchez-Guillen, Wereszczynski)
  - perturbed $E_4 + E_0$ model (Harland)

The ASW model ($E_6 + E_0$)

- $U : M \to N$, $\Omega =$volume form on N
  ($M = \mathbb{R}^3$, $N = SU(2) = S^3$)
- Potential $V = \frac{1}{2}W^2$ where $W : N \to \mathbb{R}$ has $W(\mathbb{I}_2) = 0$

$$E(U) = \frac{1}{2}\int_M \left(|U^*\Omega|^2 + W(U)^2\right)$$

- Energy bound

$$\begin{aligned} 0 &\leq \frac{1}{2}\int_M (*U^*\Omega - W(U))^2 = E - \int_M U^*(W\Omega) \\ &= E - \underbrace{\langle W\rangle Vol(N)}_{C} B \end{aligned}$$

- So $E \geq CB$ with equality iff $U^*\Omega = *W(U)$

BPS skyrmions

$$U : M \to N, \qquad U^*\Omega = *W(U)$$

$$U : M' \to N', \qquad U^*\left(\frac{\Omega}{W}\right) = *1 = \text{volume form on } M$$

$N' = N \backslash W^{-1}(0)$ =punctured target space,
$M' = U^{-1}(N')$ = "support" of U

- BPS skyrmions are **volume preserving** maps $M' \to (N', \Omega/W)$
- Come in ∞-dim families, $U \circ \psi$, where $\psi : M \to M$ is any volume preserving map. Cf Liquid drop model.
- Compactons? Depends on W. Support of U_1 has

$$Vol = \int_{N'} \frac{\Omega}{W}$$

BPS skyrmions

$$U^*\Omega = *W(U)$$

- $B = 1$ Hedgehog (assume $W = W(\mathrm{tr}\, U)$, preserves isospin symmetry)

  $$U_H(r\mathbf{n}) = \cos f(r) + i \sin f(r)\mathbf{n}\cdot\tau \qquad f(0)=\pi, f(\infty)=0$$

  1st order ODE for f.

- Charge B solutions (ASW, Bonenfant, Marleau)

  $$\psi_B : \mathbb{R}^3\backslash\mathbb{R}_z \to \mathbb{R}^3\backslash\mathbb{R}_z, \qquad \psi_B(r,\theta,\varphi) = (B^{-1/3}r, \theta, B\varphi)$$

  $U_B = U_H \circ \psi_B$. Conical singularity along $\mathbb{R}_z$

Perturbation

- $E_6 + E_0$ is irredeemably sick (e.g. EL eqn isn't an evolution eqn)
- Need $E = E_6 + E_0 + \varepsilon E_2$
- Pion mass? Linearize about vacuum $U = \mathbb{I}_2 + i\pi \cdot \tau + \cdots$

$$V(U) = \frac{1}{2} m^2 |\pi|^2 + \cdots$$

$$E_{lin} = \int_M \left(\frac{\varepsilon}{2} \partial_i \pi \cdot \partial_i \pi + \frac{m^2}{2} |\pi|^2 \right)$$

Klein-Gordon triplet of mass $m/\sqrt{\varepsilon}$

- Better choose V with $m = 0$, else pions are heavier than nucleons!
- $V_\pi = \text{tr}(\mathbb{I}_2 - U)$ no good
- $W = [\text{tr}(\mathbb{I}_2 - U)]/2$ is OK. Compacton at $\varepsilon = 0$

Numerical results $E = E_6 + E_0 + \varepsilon E_2$

- Numerics (Harland, Gillard, JMS):
  - Start at $\varepsilon = 1$, minimize using conjugate gradient method.
  - Reduce ε, repeat
  - Check integrality of B and Derrick scaling identities

$$\begin{aligned} E(U(\lambda \mathbf{x})) &= \lambda^3 E_6 + \varepsilon \lambda^{-1} E_2 + \lambda^{-3} E_0 \\ \Rightarrow 0 &= 3E_6 - \varepsilon E_2 - 3E_0 \end{aligned}$$

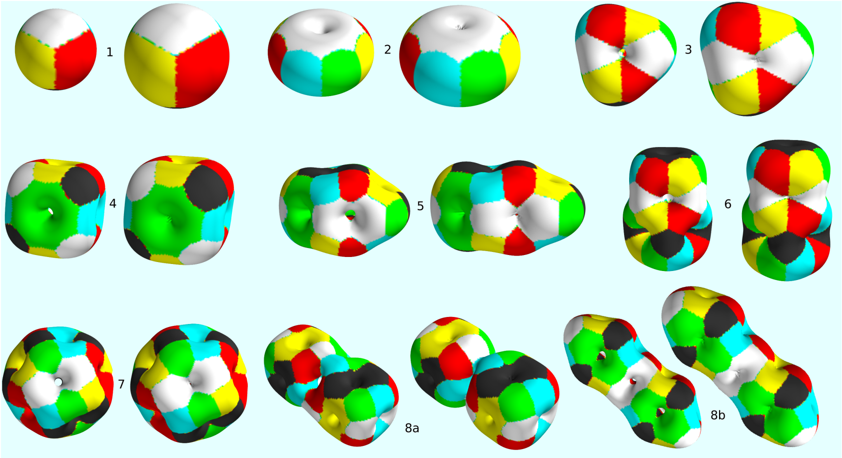
- Numerics become unstable at $\varepsilon \approx 0.2$.
- B.E.s decrease with ε, but still too large
- $B = 1, 2$ have axial symmetry: can push ε much further

$$\frac{2E(1) - E(2)}{2E(1)} \approx 0.01$$

requires $\varepsilon = 0.014$, way too small for 3D numerics

Numerical results $E = E_6 + E_0 + \varepsilon E_2$

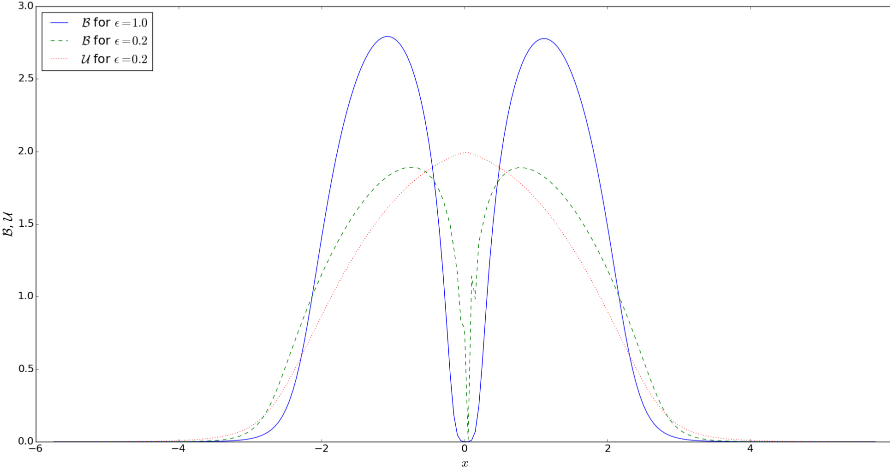
$$\mathscr{B} = 0.5\mathscr{B}_{max}$$



Left $\varepsilon = 1$, right $\varepsilon = 0.2$

Numerical results $E = E_6 + E_0 + \varepsilon E_2$

Covergence to BPS skyrmions? $B = 4$



Restricted harmonicity

$$E(U) = E_6(U) + E_0(U) + \varepsilon E_2(U)$$

- $E(U)$ should be stationary for all smooth variations U_t of U
- Choose $U_t = U \circ \psi_t$, ψ_t a curve through Id in $SDiff(M)$
- E_6 and E_0 are $SDiff$ invariant! So E_2 must be stationary for these variations: geometric language, $U : M \to N$ must be **restricted harmonic**
- Strain tensor $\mathscr{D}_{ij} = -\frac{1}{2}\,\mathrm{tr}(L_i L_j)$, or better $\mathscr{D} = \mathscr{D}_{ij} dx_i dx_j$
- U restricted harmonic iff $div\mathscr{D}$ is *exact*

$$\partial_i \partial_k \mathscr{D}_{kj} dx_i \wedge dx_j = 0$$

- True for all $\varepsilon > 0$. So if $U \xrightarrow{\varepsilon \to 0} U_{BPS}$, this should be RH also
- Bad news: $U_B = U_H \circ \psi_B$ isn't (failure gets worse as B increases)

Harland's unbound model ($E = E_4 + E_0$)

$$E = -\frac{1}{16}\int_{\mathbb{R}^3} \mathrm{tr}([L_i, L_j][L_i, L_j]) + \int_{\mathbb{R}^3} V(U)$$

- Harland's energy bound:

$$E \geq \underbrace{4(2\pi^2)\langle V^{1/4}\rangle}_{C} B$$

Proof uses AM-GM and Hölder inequalities

- Crucial step (Manton): rewrite E_4 in terms of eigenvalues of $\mathscr{D}$

$$E_4 = \int_M (\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2)$$

- Bound attained iff

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = V^{1/2}$$

everywhere: $U : \mathbb{R}^3 \to S^3$ must be conformal with conformal factor $\sqrt{V(U)}$

Harland's unbound model ($E = E_4 + E_0$)

- Essentially unique solution: $U : \mathbb{R}^3 \to S^3$ is inverse stereographic projection, and

$$V(U) = V_{quartic}(U) = \left(\frac{1}{2}\,\mathrm{tr}(\mathbb{I}_2 - U)\right)^4$$

- Bound only saturated for $B = 1$. For $B \geq 2$, $E(U) > CB$, so model is **unbound**

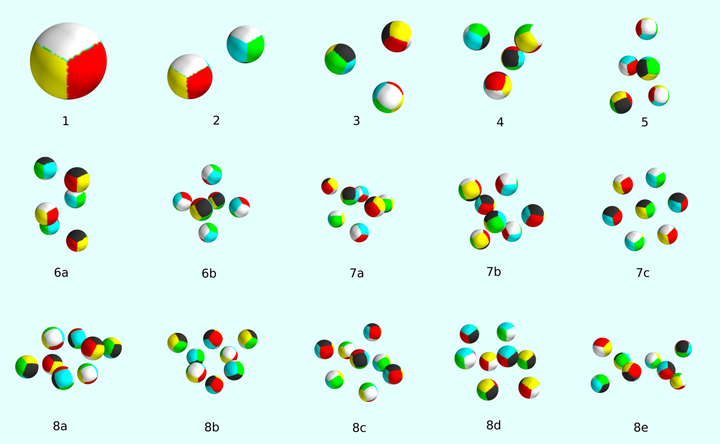
Perturbation (Harland, Gillard, JMS)

$$E_\varepsilon(U) = E_4 + (1-\varepsilon)E_0^{quartic} + \varepsilon(E_2 + E_0^{pion})$$

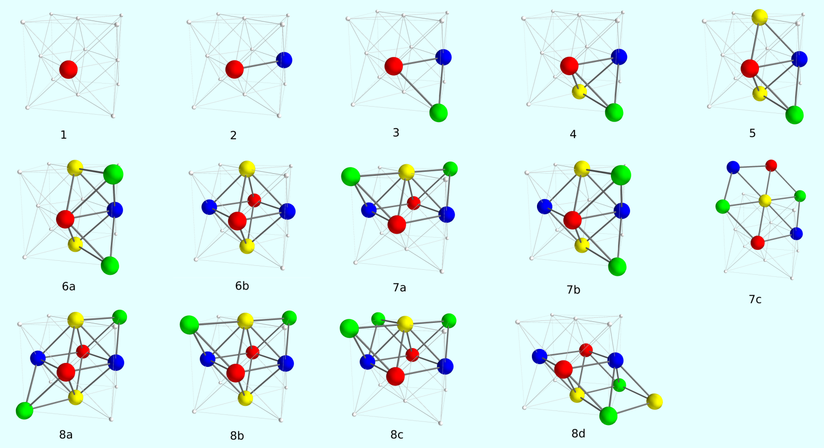
- $\varepsilon = 0$ Harland's unbound model, $\varepsilon = 1$ usual model with massive pions
- Numerics: minimize using conjugate gradient method, start at $\varepsilon = 1$, reduce ε
- Get "realistic" binding energies for $\varepsilon \approx 0.05$, easily accessible to numerics
- Skyrmions are **lightly bound**: $B = 1$ units occupying subsets of FCC lattice in maximally attractive internal orientation
- Many nearly degenerate local minima
- Minima tend to have much less symmetry than in usual $E_2 + E_4$ model

Lightly bound skyrmions

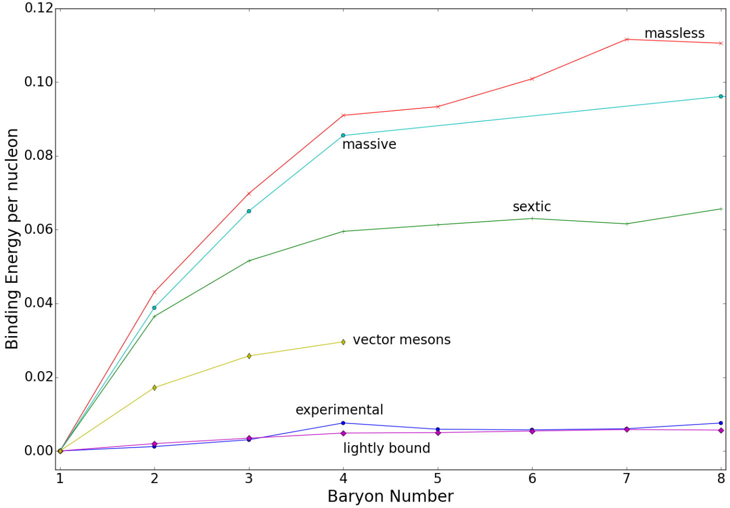
$$\mathscr{B} = 0.1\mathscr{B}_{max}$$



Lightly bound skyrmions



Classical binding energies: summary



Point skyrmion model

- General unit skyrmion

$$U(\mathbf{x}) = U_H(R(\mathbf{x} - \mathbf{x}_0))$$

position $\mathbf{x}_0$, orientation $R \in SO(3)$

- Interaction energy of Skyrmion pair at $(\mathbf{x}_1, R_1)$, $(\mathbf{x}_2, R_2)$ depends only on $\mathbf{X} = \mathbf{x}_1 - \mathbf{x}_2$ and $R = R_1^{-1} R_2$
- Assumption/approximation

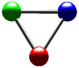
$$V_{int} = V_0(|\mathbf{X}|) + V_1(|\mathbf{X}|)\,\mathrm{tr}\,R + V_2(|\mathbf{X}|)\frac{\mathbf{X} \cdot R\mathbf{X}}{|\mathbf{X}|^2}$$

- Find V_0, V_1, V_2 by fitting to classical scattering solutions
- Very simple point particle approximation to Skyrme energy

$$E_{pp}(\mathbf{x}_1, \ldots, \mathbf{x}_B, R_1, \ldots, R_B) = BE(U_H) + \sum_{1 \le a < b \le B} V_{int}(|\mathbf{x}_a - \mathbf{x}_b|, R_a^{-1} R_b)$$

- Does remarkably well: for $1 \le B \le 8$ reproduces all local minima, with correct energy ordering **except** reverses $6a$ and $6b$

Point skyrmion model (H+G+JMS+Maybee+Kirk)



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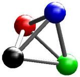

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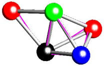
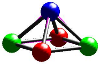
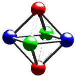


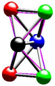
5(a)



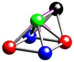
5(b)*



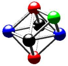
6(a)



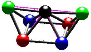
6(b)



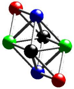
6(c)*



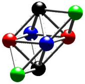
7(a)*



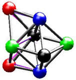
7(b)



8(a)



8(b)



8(c)

Point skyrmion model

- Let it loose on $9 \leq B \leq 23$
- Can automate rigid body quantization procedure (not entirely trivial)
- Results modest: binding energies get inflated (as usual), spin/isospin predictions often unphysical

Point skyrmion model: rigid body quantization

Name	Bonds	Colour count	Classical energy	Symmetry group	I	J	Quantum energy	Experiment
2a	1	1,1,0,0	-0.310	D_2	0	1	3.813	$^{2}H_1$
3a	3	1,1,1,0	-0.931	C_3	1/2	1/2	1.106	$^{3}He_2$
4a	6	1,1,1,1	-1.862	T	0	0	-1.862	$^{4}He_2$
5a	8	2,1,1,1	-2.338	1	1/2	1/2	-1.167	
5b	8	2,2,1,0	-2.185	C_4	1/2	3/2	-0.700	$^{5}He_2$
6a	12	2,2,2,0	-3.229	O	2	1	4.275	
6b	11*	2,2,1,1	-3.117	D_2	0	1	-2.973	$^{6}Li_3$
6c	11*	2,2,1,1	-3.046	1	0	0	-3.046	
7a	15	2,2,2,1	-4.057	C_3	1/2	1/2	-3.210	
8a	18	2,2,2,2	-4.889	D_3	0	0	-4.889	$^{8}Be_4$
8b	18	2,2,2,2	-4.869	C_2	0	1	-4.769	
9a	21	3,2,2,2	-5.664	C_3	1/2	1/2	-5.024	
9b	21	3,2,2,2	-5.598	1	1/2	1/2	-4.956	
10a	25	3,3,2,2	-6.443	D_2	0	1	-6.352	
10b	24*	4,2,2,2	-6.442	T	0	0	-6.442	
11a	28	3,3,3,2	-7.261	1	1/2	1/2	-6.736	
12a	31*	3,3,3,3	-8.081	C_2	0	0	-8.081	$^{12}C_6$
12b	32	3,3,3,3	-8.066	1	0	0	-8.066	
13a	36	4,3,3,3	-9.016	C_3	1/2	1/2	-8.575	$^{13}C_6$
14a	39*	4,4,3,3	-9.821	1	0	0	-9.821	
15a	43*	4,4,4,3	-10.653	1	1/2	1/2	-10.272	$^{15}N_7$
15b	42**	4,4,4,3	-10.627	1	1/2	1/2	-10.247	$^{15}N_7$
15c	43*	4,4,4,3	-10.584	1	1/2	1/2	-10.202	$^{15}N_7$
16a	48	4,4,4,4	-11.771	T	0	0	-11.771	$^{16}O_8$
17a	51*	5,4,4,4	-12.563	C_3	1/2	1/2	-12.228	
18a	54**	5,5,4,4	-13.356	C_2	0	0	-13.356	
18b	56	6,4,4,4	-13.340	C_4	0	0	-13.340	
19a	60	5,5,5,4	-14.251	C_3	1/2	1/2	-13.951	$^{19}F_9$
19b	60	7,4,4,4	-14.244	O	1/2	1/2	-13.946	$^{19}F_9$
19c	58**	5,5,5,4	-14.178	1	1/2	1/2	-13.879	$^{19}F_9$
19d	59*	5,5,5,4	-14.164	1	1/2	1/2	-13.864	$^{19}F_9$

Summary

- Near BPS model ($E_6 + E_0 + \varepsilon E_2$)
  - Skyrmions seem to keep conventional symmetries
  - Has (approx) *SDiff* invariance: liquid drop model
  - Need $\varepsilon \approx 0.014$ to get realistic B.E.s, much too small for reliable numerics
  - No idea what limit BPS skyrmions actually are. $U_H \circ \psi_B$ certainly wrong.
- Lightly bound model ($E_4 + E_0 + \varepsilon(E_2 + E_0^\pi - E_0)$)
  - Numerically tractable at very low ε
  - $\varepsilon \approx 0.05$ yields realistic B.E.s
  - Skyrmions resemble molecules, subsets of FCC lattice
  - Lose symmetries, many nearly degenerate minima
  - Simple and reliable point particle model
  - Has inspired new initial data for conventional model at high B (Manton et al)
  - Good laboratory for more advanced quantization techniques

Summary: other approaches

- **Loosely** bound model (Gudnason): $E = E_4 + E_0 + \varepsilon(E_2 + E_0^\pi)$ but with

$$E_0 = \int_M [\mathrm{tr}(\mathbb{I}_2 - U)]^2 \qquad \text{instead of} \qquad E_0 = \int_M [\mathrm{tr}(\mathbb{I}_2 - U)]^4.$$

  Gets low classical B.E. without losing as much symmetry as lightly bound model.
- Holography (Sutcliffe):
  - Interpret pure YM on M^4 as Skyrme model (on $\mathbb{R}^3$) coupled to infinite tower of vector mesons
  - Get near BPS theory by truncating meson tower
  - $N = 1$ modest reduction in B.E.s
  - $N = 2$ a lot better (Sutcliffe, Naya)
  - Price: extremely complicated numerical problem
  - Advantage: vector meson coupling interesting for other reasons