Vortices: moduli space and dynamics

J.M. Speight
School of Mathematics, University of Leeds
Leeds LS2 9JT, England

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Notes taken by Nora Gavrea.

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Martin Speight

Notes by Nora Gavrea.

Lectures
- moduli space
- metric
- thermodynamics
- 2nd Variation

$\varphi : \mathbb{R}^{2,1} \to \mathbb{C}$
$(+--)$

$A = A_\mu dx^\mu$

$D_\mu \varphi = \partial_\mu \varphi - i A_\mu \varphi$

$$\mathcal{L} = \frac{1}{2} \overline{D_\mu \varphi} D^\mu \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} \left(1 - |\varphi|^2\right)^2$$

$F = dA$

Gauge transf.:
$$\varphi \to e^{i\chi}\varphi$$
$$A \to A + d\chi$$

(φ, A) critical point of $S = \int_{\mathbb{R}^{2,1}} \mathcal{L}$.

EOM
$$D_\mu D^\mu \varphi - \frac{1}{2}\left(1-|\varphi|^2\right)\varphi = 0$$
$$\partial_\mu F^{\mu\nu} + \frac{1}{2}\left(\bar{\varphi} D^\nu \varphi - \varphi \overline{D^\nu \varphi}\right) = 0$$

Static en. $\varphi(x,y)$, $A = A_1 dx + A_2 dy$

$$E = \int_{\mathbb{R}^2} \frac{1}{2} D_i\varphi \overline{D_i\varphi} + \frac{1}{2}|dA|^2 + \frac{1}{8}\left(1-|\varphi|^2\right)^2 < \infty$$

Finite en. $\Rightarrow$ at $r=\infty$ $|\varphi| = 1$, $D\varphi = 0$, $dA = 0$.

$$\Rightarrow \quad \varphi \sim e^{i\chi}, \quad A = d\chi$$

$$\chi(2\pi) - \chi(0) = 2\pi m, \quad m \in \mathbb{Z}$$

$$\varphi\big|_{\partial\mathbb{R}^2} : S^1_\infty \longrightarrow S^1 \subset \mathbb{C}$$

Total magn. flux: $\displaystyle \Phi = \int_{\mathbb{R}^2} dA = \int_{S^1_\infty} A = \int_{S^1_\infty} d\chi = 2\pi m$

<u>Bog. bound (1976)</u>

$$0 \leq \int_{\mathbb{R}^2} \left[\frac{1}{2} \left| D_1\varphi + i D_2 \varphi \right|^2 + \frac{1}{2} \left| *F - \frac{1}{2}\left(1 - |\varphi|^2\right) \right|^2 \right]$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} |D\varphi|^2 + i\left(\overline{D_1\varphi} D_2\varphi - D_1\varphi \overline{D_2\varphi}\right) + |B|^2 + \frac{1}{2}\left(1-|\varphi|^2\right)$$

$$+ |\varphi|^2 B - B$$

write here $B = \bar{\varphi}[D_1, D_2]\varphi$

then cross terms are exact and using Stokes they integrate to 0.

$\Rightarrow \quad 0 \leq E - \pi n$

$\Rightarrow \quad E \geq \pi n$ with equality iff. Bog. eqns. hold

$$D_1\varphi + i D_2\varphi = 0 \qquad \text{(holomorphicity cond.)} \qquad (B1)$$

$$* B = \frac{1}{2}\left(1 - |\varphi|^2\right) \qquad (B2)$$

(B1) $\Rightarrow$ solns have associated n zeroes $\underbrace{z_1 \dots z_n}_{D} \in \mathbb{C}$

Taubes (1980)

Given any divisor D, $\exists$ sol. of B1 & B2 with φ vanishing on D. Sol. is smooth, unique up to gauge and for each $\delta \in (0,1)$ $\exists\, C > 0$ s.t. $|B|$, $||\varphi|-1|$, $|D\varphi| \leq C e^{-(1-\delta)r}$
Furthermore $|\varphi| < 1$ everywhere.

Sketch of proof $\quad D = \{z_1 \dots z_n\}$

$\varphi = e^{\frac{1}{2}h + i\chi}$ $\quad h, \chi \in \mathbb{R}$ $\quad$ h has log. sing. on D

$h, e^{i\chi}$ are well defined on $\mathbb{C} \setminus D$

B1: $\quad A = -\frac{1}{2} * dh + d\chi$

B2: $\quad -\frac{1}{2} * d * dh = \frac{1}{2}(1-e^h)$

$\Rightarrow$

$$\Rightarrow \quad \boxed{\nabla^2 h = e^h - 1 + 4\pi \sum_{n=1}^{m} \delta(z - z_n)} \quad \text{(Taubes)}$$

this uses $\nabla^2 \ln r = 2\pi \delta(r)$

i.e. δ-fnc. is Green's fnc. for Laplacian

$$h_0 = -\sum_{n=1}^{m} \log\left(1 + \frac{\mu}{|z - z_n|^2}\right) \qquad \mu >> 0$$

$\leftarrow$ log sing. on D

$$\nabla^2 h_0 = 4\pi \sum_{n=1}^{m} \delta(z - z_n) - 4 \underbrace{\sum_{n=1}^{m} \frac{\mu}{\left(|z-z_n|^2+\mu^2\right)^2}}_{g_0}$$

Define $h := h_0 + v \Rightarrow v$ smooth (h, h_0 have same sing.)

$$\Rightarrow \quad -\nabla^2 v + e^{h_0} e^{v} + (g_0 - 1) = 0 \qquad v(\infty) = 0$$

$$a : H^1 \longrightarrow \mathbb{R}$$

$$\left\{ f : \mathbb{R}^2 \to \mathbb{R}, \ \underbrace{\int_{\mathbb{R}^2} |df|^2 + |f|^2}_{\|f\|_{H^1}} < \infty \right\}$$

$$a(v) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |du|^2 + v(g_0 - 1) + e^{h_0}(e^u - 1) \right\}$$

- a is well-defined and differentiable
- a is strictly convex:

$$a(tu + (1-t)v) \leq t\,a(u) + (1-t)\,a(v) \qquad \forall\, u, v \in H^1.$$

- it's coercive $\quad (a(v) \to \infty$ as $\|v\|_{H^1} \to \infty)$

$\Rightarrow$ a has unique min v, $da_v = 0$.

Smoothness, localisation etc are proved separately

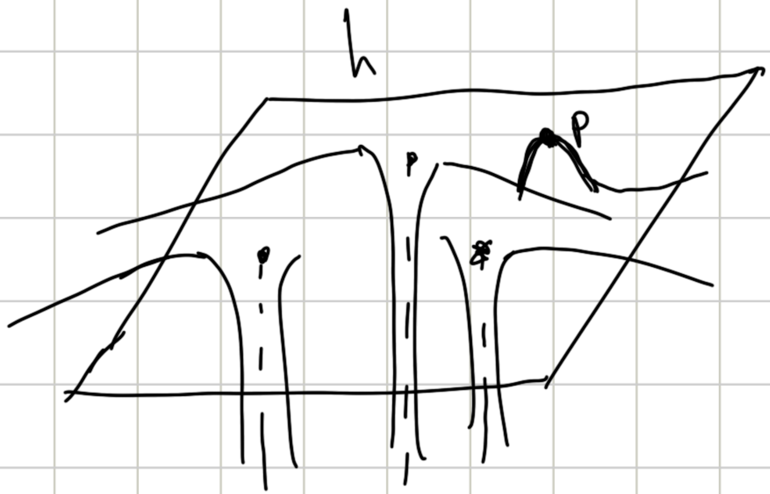
$|\varphi| < 1$ using max. principle

$\Updownarrow$

$h < 0$

Assume false. Then since $h \to 0$ as $|x| \to \infty \Rightarrow$ h attains positive max at some point, p.

$h(p) > 0$



$$\underbrace{\nabla^2 h \,\big|_p}_{} = \underbrace{e^{h(p)} - 1}_{>0} \quad , \not\#$$

$= \mathrm{Tr}(\mathrm{Hess}\, h(p)) < 0$

$\mathrm{Hess}\, h(p) = h_{xx} + h_{yy}$.

State n vortices $\underset{1 \text{ to } 1}{\longleftrightarrow}$ $\{z_1 \dots z_n\} \in \mathbb{C}^n$

So Moduli space $\mathbb{C}^n / S_n$

$\overset{1 \text{ to } 1}{\longleftrightarrow}$

Let $p(z) = (z - z_1) \dots (z - z_n)$

$= z^n + g_1 z^{n-1} + \dots + g_n$

$\overset{1 \text{ to } 1}{\longleftrightarrow}$ $\{g_1, \dots, g_n\} \in \mathbb{C}^n$ $\Rightarrow$ $\mathbb{C}^n \cong \mathbb{C}^n / S_n$

$M_n = \{\text{sol. of } B1 \,\&\, B2\} / \{\text{gauge transf}\} \cong \mathbb{C}^n$

Low energy dependence

Temporal gauge $A_0 = 0$

Euler Lagrange eqn. for A_0

$$- \partial_i \dot{A}_i + \frac{i}{2} \left(\varphi \dot{\bar{\varphi}} - \bar{\varphi} \dot{\varphi} \right) = 0$$

$$\delta \dot{A} + H (i \varphi , \dot{\varphi}) = 0 \qquad \text{(Gauss's Law)}$$

inner product $H(a,b) = \dfrac{\bar{a} b + a \bar{b}}{2}$

(φ, A) sol. then $\left(e^{i\chi(t)} \varphi , A + d\chi(t) \right)$ also sol.

$d/dt|_{t=0}$ $\Rightarrow$ $\left(i \chi(0) \varphi , d\dot{\chi}(0) \right)$

$$\langle (\dot\varphi, \dot A), (i\dot\chi, d\dot\chi) \rangle_{L^2} =$$

$$= \int_{\mathbb{R}^2} \dot\chi \, (\underbrace{\delta \dot A + H(i\varphi, \dot\varphi)}_{=0}) = 0$$

$\Rightarrow$ Gauss's law imposes initial data is L^2-orthogonal to gauge orbit.

Idea find $\underbrace{(\varphi(t), A(t))}_{q(t) \in \mathbb{C}^n} \in M_n$

geodesic motion in $\mathbb{C}^n$

Lecture 2

Metric on Moduli space

$$D_1\varphi + iD_2\varphi = 0 \qquad (B1)$$

$$*B = \frac{1}{2}\left(1-|\varphi|^2\right) \qquad (B2)$$

Taubes Solutions $\overset{1:1}{\longleftrightarrow}$ collections of pts $z_1 \dots z_n$ where $\varphi = 0$

$\overset{1:1}{\longleftrightarrow}$ monic poly $p(z) = (z-z_1)\dots\dots(z-z_n)$

$= z^n + g_1 z^{n-1} + \dots + g_n$

$\overset{1:1}{\longleftrightarrow}$ $g \in \mathbb{C}^n$

$$M_n \cong \mathbb{C}^n .$$

Temporal gauge $A_0 = 0$.

Gauss's Law $\quad \delta \dot{A} + H(i\varphi, \dot{\varphi}) = 0$.

— $(\varphi(t), A(t))$ moves $L^2 \perp$ gauge orbit

$$L = \int_{\mathbb{R}^2} \mathcal{L}\, d^2x = \int_{\mathbb{R}^2} \frac{1}{2}\left(|\dot{\varphi}|^2 + |\dot{A}|^2 \right)$$

$E_{static}(\varphi(t), A(t))$

Idea: restrict this to fields where at each t, $(\varphi(t), A(t)) \in M$

$\varphi(z; p_1(t) \dots p_n(t))\ A(z; p_1(t) \dots p_n(t))$, where $p = \mathrm{Re}/\mathrm{Im}\ z'$s

$$L = \frac{1}{2} \int_{\mathbb{R}^2} \underbrace{\left\{ \frac{\partial \bar{\varphi}}{\partial p_i} \frac{\partial \varphi}{\partial p_j} + \frac{\partial A}{\partial p_i} \frac{\partial A}{\partial p_j} \right\}}_{g_{ij}(p) \longrightarrow \text{metric!}} \dot{p}_i \dot{p}_j$$

(Manton)

Reduced dyn. = geodesic motion in (M_n, g)

Biggest task: compute g

Direct approach: Myers, Rebbi, Strilka (1992)

Strachan - Samols localisation (1992)

Take curve of sols in M_n s.t. $(z_1(t) \dots z_n(t))$ zeroes move on curve where they stay distinct.

$$\varphi = e^{h(t)/2 + i\chi(t)}$$

Taubes eqn $\quad \nabla^2 h = e^h - 1 \quad$ on $\mathbb{C} \setminus D$ at each fixed time t

$$\nabla^2 \dot{h} = e^h \dot{h} \qquad \nabla^2 \dot{\chi} = e^h \dot{\chi}$$

B1 : $\quad A = -\frac{1}{2} * dh + d\chi$

$\overset{d/dt}{\Rightarrow} \quad \dot{A} = -\frac{1}{2} * d\dot{h} + d\dot{\chi}$

$$\delta \dot{A} = - * d * d\dot{\chi} = -\nabla^2 \dot{\chi}$$

But recall gauge cond. $\delta \dot{A} + H(i\varphi, \dot{\varphi}) = 0$

$$-\nabla^2 \dot{\chi} = -H\left(i\varphi, \left(\tfrac{1}{2}\dot{h} + i\dot{\chi}\right)\varphi\right)$$

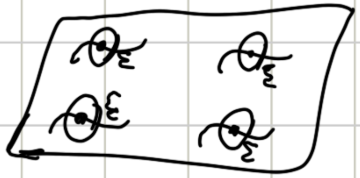
$$= -\dot{\chi}|\varphi|^2$$

Define $\quad \eta : \mathbb{C} \setminus D \to \mathbb{C} \qquad \dot{\varphi} := \varphi\eta$

Near z_n $\qquad \varphi(z) = (z - z_n)$ smooth

$$\eta = -\frac{\dot{z}_n}{z - z_n} + \text{smooth}$$

Note $\qquad \eta = \frac{1}{2}\dot{h} + i\dot{\chi} \quad \Rightarrow \quad \nabla^2 \eta = e^h \eta \quad \text{in } \mathbb{C} \setminus D$



$$T = \int_{\mathbb{R}^2} \left(|\dot{A}|^2 + |\dot{\varphi}|^2 \right)$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{2} \int_{\mathbb{R}^2 \setminus D_\varepsilon} |\dot{A}|^2 + |\dot{\varphi}|^2 + \frac{1}{2} \int_{D_\varepsilon} |\dot{A}|^2 + |\dot{\varphi}|^2 \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus D_\varepsilon} \left(|\dot{A}|^2 + |\dot{\varphi}|^2 \right) = \lim_{\varepsilon \to 0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus D_\varepsilon} \left[\|d\eta\|_{L^2}^2 + e^h \|\eta\|_{L^2}^2 \right]$$

$$\left(\|\dot{A}\|_{L^2}^2 = \| -\tfrac{1}{2} * d\dot{h} + d\dot{\chi} \|_{L^2}^2 = \|d\eta\|_{L^2}^2 \quad , \quad \eta = \tfrac{1}{2}\dot{h} + i\dot{\chi} \right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \int_{\mathbb{R}^2 \setminus D_\varepsilon} (\eta, \underbrace{-\nabla^2 \eta + e^h \eta}_{=0 \text{ on } \mathbb{R}^2 \setminus D_\varepsilon}) - \frac{1}{2} \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \underbrace{\eta \wedge * d\eta}_{\text{vanishes on } \partial\mathbb{R}^2}$$

$$\Rightarrow \quad T = -\frac{1}{2} \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \eta \wedge * d\eta \qquad \text{LF} \quad (\text{Localization formula})$$

$$\nabla^2 h = e^h - 1 \qquad\qquad h = \log |z - z_n|^2 + \text{smooth stuff}$$

$$\Downarrow \qquad\qquad \frac{\partial h}{\partial z_n} = \frac{-1}{z - z_n} + \text{smooth stuff}$$

$$\nabla^2 \frac{\partial h}{\partial z_n} = e^h \frac{\partial h}{\partial z_n}$$

$$\Rightarrow \quad \eta = \sum_{n=1}^{m} \dot{z}_n \frac{\partial h}{\partial z_n}$$

Close to z_n :

$$h = \log |z - z_n|^2 + a_n + \frac{1}{2} b_n (z - z_n) + \frac{1}{2} \bar{b}_n \overline{(z - z_n)} + \dots$$

$\mathbb{C}$-valued fnc. of $z_1 \dots z_n$

(LF) $\Rightarrow$ $$T = \frac{\pi}{2}\left(\sum_{r=1}^{n} |\dot{z}_r|^2 + 2\sum_{r,s=1}^{n} \frac{\partial b_s}{\partial \bar{z}_r}\dot{z}_r \dot{\bar{z}}_s\right)$$

$T \in \mathbb{R} \Rightarrow T = \overline{T} \Rightarrow \dfrac{\partial b_s}{\partial \bar{z}_r} = \dfrac{\partial \bar{b}_r}{\partial z_s}$ (K)

(matrix given by components $A_{rs} = \dfrac{\partial b_s}{\partial \bar{z}_r}\dot{z}_r \dot{\bar{z}}_s$ is Hermitian)

$$\Rightarrow g = \frac{\pi}{2}\left\{\sum_{r=1}^{n} dz_r\, d\bar{z}_r + 2\sum_{r,s=1}^{n} \frac{\partial \bar{b}_s}{\partial z_r} dz_r\, d\bar{z}_s\right\}$$

Local formula for g on $M_n \setminus \Delta$

$\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n / S_n = (M_n, g)$ smooth

$\pi^* g$ - $(0,2)$ tensor on $\mathbb{C}^n$

$(z_1 \ldots z_n)$

Nice fine $\quad J : TM_n \longrightarrow TM_n$

$\omega(X,Y) = g(JX, Y) \qquad g(JX, JY) = g(X,Y)$

2-form
(Kähler form)

$$\omega = i \frac{\pi}{2} \left(\sum_{r,s} dz_r \wedge d\bar{z}_r + 2 \sum_{r,s} \frac{\partial \bar{l}_s}{\partial z_r} dz_r \wedge d\bar{z}_s \right)$$

Exercise: $\text{(K)} \Rightarrow d\omega = 0$

g is a Kähler metric on M_m.

- $\bar{b}_r = \bar{b}_r(\bar{z}_1, \bar{z}_2 \ldots \bar{z}_m)$

$$\Rightarrow 0 = \sum_{r,s} \frac{\partial}{\partial z_s} \bar{b}_r \overset{\text{(K)}}{=} \frac{\partial}{\partial \bar{z}_r} \left(\sum_{s=1}^{m} b_s \right)$$

$$\Rightarrow \sum_{s=1}^{m} b_s = \text{const on } M_m \setminus \Delta$$

$m = 2 \Rightarrow b_2 = -b_1$ ($b_1 + b_2 = 0$ centre of mass frame)

Lecture 3

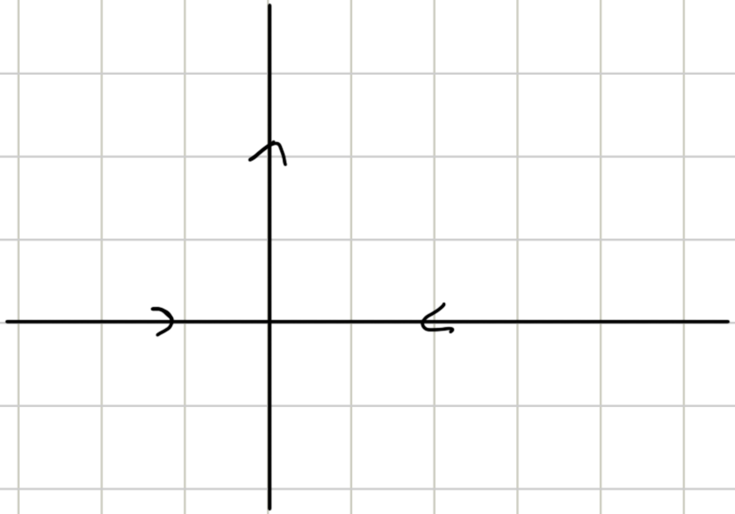
2-vortex scattering (Ruback 1988, Samols 1992)

$$p(z) = (z-z_1)(z-z_2) = z^2 \underbrace{-(z_1+z_2)}_{q_1} z + \underbrace{z_1 z_2}_{q_2}$$

$$(z_1, z_2) \longrightarrow (-z_1, -z_2) \iff \begin{array}{l} q_1 \to -q_1 \\ q_2 \to q_2 \end{array}$$

$$(z_1, z_2) \longrightarrow (\bar{z}_1, \bar{z}_2) \iff (q_1, q_2) \longrightarrow (\bar{q}_1, \bar{q}_2)$$

Fixed point set $p(z) = z^2 + t$ $\quad t \in \mathbb{R}$ (unparametrised) geodesic



$z_1(t) = \sqrt{-t} \qquad t \le 0$

$|\dot{z}_1| = \infty!$ (not in contradiction with SR or etc)

$$z = \frac{1}{2}(z_1 + z_2)$$

$$\zeta = \frac{1}{2}(z_1 - z_2) \equiv -\zeta$$

↑ identify ζ with $-\zeta$ as this doesn't change metric

Most general $E(2)$ (rotation) invariant Hermitian metric on M_2

$$g = g_{z\bar{z}}(|\zeta|)\, dz\, d\bar{z} + g_{\zeta\bar{z}}(|\zeta|)\, d\zeta\, d\bar{z} + g_{z\bar{\zeta}}(|\zeta|)\, dz\, d\bar{\zeta} + g_{\zeta\bar{\zeta}}(|\zeta|)\, d\zeta\, d\bar{\zeta}$$

Kähler : $\omega = \frac{1}{2} g_{ij}\, dw_i \wedge d\bar{w}_j$ $\quad d\omega = 0$

$$\Rightarrow \quad \frac{\partial g_{ij}}{\partial w_k} = \frac{\partial g_{ik}}{\partial w_j} \qquad \frac{\partial g_{ij}}{\partial w_k} = \frac{\partial g_{kj}}{\partial w_i}$$

$$\frac{\partial}{\partial \xi} g_{zz} = \frac{\partial}{\partial z} g_{\xi z} = 0$$

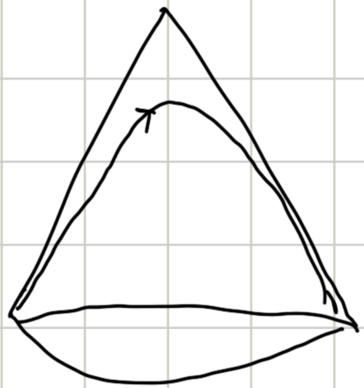
g_{zz}, $g_{z\xi}$, $g_{\xi z}$ constant!

$$\frac{\partial}{\partial \xi} g_{z\xi} = \frac{\partial}{\partial z} g_{\xi z} = 0$$

Long $|\xi|$ dist

$g_{zz} = 2\pi$, $g_{z\xi} = g_{\xi z} = 0$

$$g = 2\pi\, dz\, d\bar{z} + F(|\xi|)\, d\xi\, d\bar{\xi}$$



$$g = F(|\xi|)\, d\xi\, d\bar{\xi}$$

$$F(\rho) = 2\pi\left(1 + \frac{1}{\rho}\frac{d}{d\rho}\big(\rho\, b(\rho)\big)\right)$$

$$b(\rho) = b_1(\rho, -\rho) \in \mathbb{R}$$

Vortices in $S^2 = \Sigma$

$\varphi \in \Gamma(L)$ section of line bundle

A = connection on L

on overlap of patches, φ and A have to agree up to gauge

$$\bar{\partial}_A \varphi = 0 \qquad \text{B1}$$

$$* B = \frac{1}{2}\left(1 - |\varphi|^2\right) \qquad \text{B2}$$

$$\int_\Sigma \text{B2} \Rightarrow \underbrace{\int * B}_{2\pi m} = \int \frac{1}{2}\left(1 - |\varphi|^2\right)$$

$$= \frac{1}{2}\Big(\underbrace{|\Sigma|}_{\text{area of } \Sigma} - \|\varphi\|^2_{L^2}\Big)$$

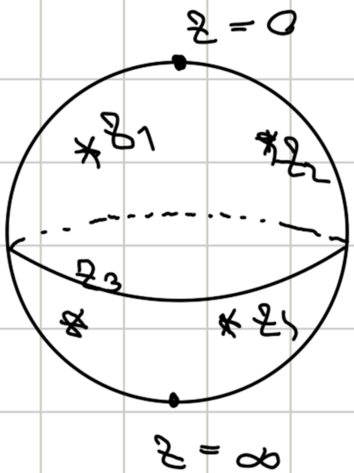
$$\Rightarrow \underbrace{\|\varphi\|^2_{L^2}}_{\geq 0} = |\Sigma| - 4\pi m \geq 0$$

$\Rightarrow$ BPS Vortices exist iff $|\Sigma| \geq 4\pi m$.

Bradlow, Garcia Prada $\to$ Taubes thm. still holds

$$M_n = \Sigma^n / S_n$$

$z=\infty$ is root iff $z=0$ is root of $p\left(\frac{1}{z}\right)$



$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$

$$p(z_i) = 0$$

Degree $\leq n$ since $a_n z^n$ is ignored if $z=\infty$ is a root.

$$\overset{1:1}{\longleftrightarrow} \quad [a_0, a_1, \dots, a_n] \in \mathbb{CP}^n \Rightarrow M_n \in \mathbb{CP}^n.$$

Thermodynamics

volume form of Kähler Manifold

$$|M_n| = \int_{M_n} \frac{\omega^n}{n!}$$

$$\omega \to \tilde{\omega} = \omega + d\mu \qquad \mu \in \Omega^1(M_n)$$

$$\omega^n \to \tilde{\omega}^n = \omega^n + d(\ V\ )$$

(ω, $\tilde{\omega}$ are in the same cohomology class)

$$\int_{M_n} \tilde{\omega}^n = \int_{M_n} \omega^n + \underbrace{\int_{\partial M_n} V}_{=0 \text{ since } \partial M_n = \emptyset}$$

as M_n is compact

$$M_n = \mathbb{C}P^n \qquad H^2(\mathbb{C}P^n) \cong \mathbb{R}$$

this says that if ω is 2-form then $\tilde{\omega}$ is also 2-form iff they are related by a translation

2nd de Rham Cohomology group

Choose ω_0 s.t. $d\omega_0 = 0$ and

$$\mathbb{C}P^n \supset \mathbb{C}P^1_n = \{ [z_1, z_2, 0, \dots 0] , [z_1, z_2] \in \mathbb{C}P^1 \}$$

$$\int_{\mathbb{C}P^1_n} \omega_0$$

$$|M_n| = \frac{1}{n!} \int_{M_n} \omega^n = \frac{1}{n!} \int_{M_n} (\alpha \omega_0)^n = \frac{\alpha^n}{n!} \underbrace{\int_{\mathbb{C}P^n} \omega_0^n}_{=1} = \frac{\alpha^n}{n!}$$

$= 1$ (special fact of $\mathbb{C}P^n$)

for higher genus this would be very different

Define $M_n^\circ \subset M_n$ — coincident n vortices

$$p(z) = (z-t)^n = z^n - n t z^{n-1} + \dots + (-t)^n$$

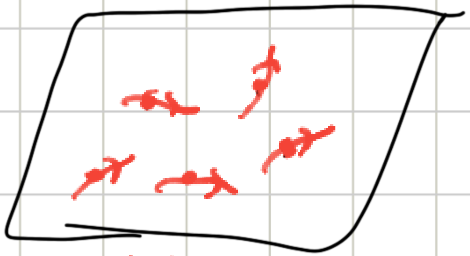
$t \in \mathbb{C} \cup \{\infty\}$ c.f. $z^n - t z^{n-1}$ $\quad M_n^{(0)} \underset{\text{homotopy}}{\simeq} n\,\mathbb{CP}^1_0$

$$\Rightarrow \int_{M_n^\circ} \omega = n \int_{\mathbb{CP}^1_0} \omega = n\alpha$$

Localisation formula

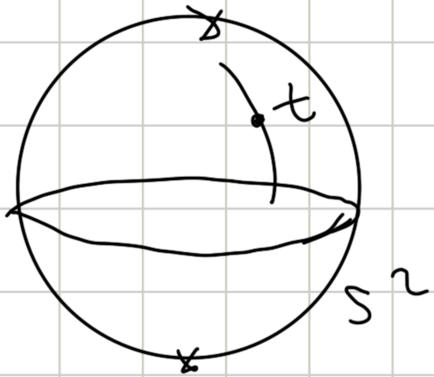
Calculation yesterday:

$\mathbb{C}$



n distinct vortices moving in $\mathbb{C}$

Calculation today:



$$g_\Sigma = \Omega(z,\bar{z})\, dz \wedge d\bar{z}$$

$t \in \mathbb{C}$

n coincident vortices moving on S^2.

$$h = \log|\varphi|^2 = n \log|z-t|^2 + a(t) + \tfrac{1}{2} b(t)(z-t) +$$
$$+ \tfrac{1}{2}\bar{b}(t)(\bar{z}-\bar{t}) + \dots$$

$$T = \frac{1}{2} n\pi \left(\Omega(t,\bar{t}) + 2\frac{db}{d\bar{t}} \right) |\dot{t}|^2$$

$$\omega = n\pi\,\omega_\Sigma - i n\pi\, d\beta \qquad \beta := b(t,\bar{t})\,dt$$

a $(1,0)$ form on $S^2 \setminus (0,0,-1)$

Now repeat the calculation on $S^2 \setminus (0,0,1)$ with the other stereographic coordinate $\tilde{z} := 1/z$, $t := 1/s$

$$h = n\log|\tilde{z}-s|^2 + \tilde{a} + \frac{1}{2}\tilde{b}(\tilde{z}-s) + \frac{1}{2}\bar{\tilde{b}}(\bar{\tilde{z}}-\bar{s}) + \dots$$

$$= n\log\left|\frac{1}{z}-\frac{1}{t}\right|^2 + \tilde{a} + \frac{1}{2}\tilde{b}\left(\frac{1}{z}-\frac{1}{t}\right) + \frac{1}{2}\bar{\tilde{b}}\left(\frac{1}{\bar{z}}-\frac{1}{\bar{t}}\right) + \dots \quad (*)$$

$$\equiv n\log|z-t|^2 + a + \frac{1}{2}b(z-t) + \frac{1}{2}\bar{b}(\bar{z}-\bar{t}) + \dots \quad (\dagger)$$

Expand $(*)$ around $z=t$, compare with $(\dagger)$. Deduce that

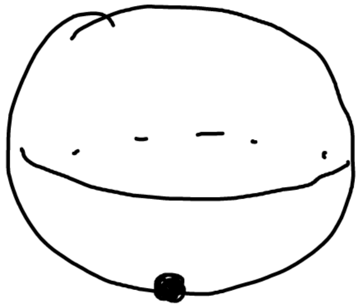
$$\tilde{b} = 2nt - t^2 b$$

$\Rightarrow$ Associated $(1,0)$ form $\tilde{\beta} = \beta + \frac{2n}{t}\,dt$

So what do we have?

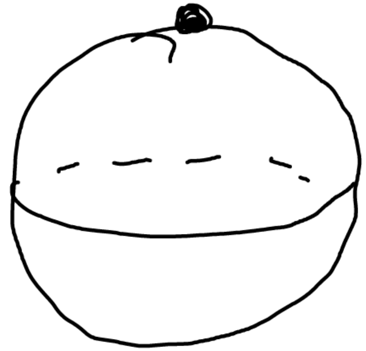
A pair of local complex 1-forms defined on

β



$S^2 \setminus (0,0,-1)$

$\tilde{\beta}$



$S^2 \setminus (0,0,1)$

which, on their overlap $S^2 \setminus (0,0,\pm)$, are related by

$$\tilde{\beta} = \beta + \frac{2n}{t}\, dt$$

Precisely the data of a connection on a degree $2n$ line bundle over S^2

Call the bundle $\mathbb{L}$, the connection $\mathbb{B}$. Note that this has curvature $iF_{\mathbb{B}} = i d\beta = i d\hat{\beta}$, whose integral over $S^2 = \mathcal{M}_n^0$ is a topological invariant:

$$\int_{S^2} i F_{\mathbb{B}} = 2\pi . (2n) = 4\pi n .$$

So our localization formula for ω_{L^2} globalizes on $\mathcal{M}_n^0$:

$$\omega_{L^2}\Big|_{\mathcal{M}_n^0} = \tau n \pi \omega_{\Sigma} - n\pi i F_{\mathbb{B}}$$

$$\Rightarrow n\alpha = \int_{\mathcal{M}_n^0} \omega_{L^2} = n\pi \left(\tau |\Sigma| - 4\pi n \right)$$

Hence $\quad [\omega_{L^2}] = \pi\,(2|\Sigma| - 4\pi n)\,[\omega_0]$

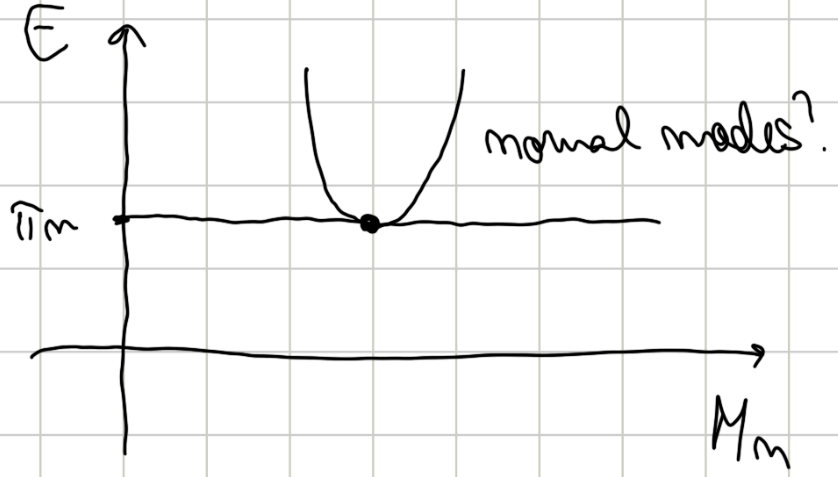
and so $\quad \boxed{|M_n| = \frac{\pi^n}{n!}\,(2|\Sigma| - 4\pi n)^n} = \frac{\pi^n}{n!}\,\varepsilon^n.$

Argument can be generalized to case genus $(\Sigma) > 0$, but it gets a lot more complicated, since the cohomology ring of $M_n = \Sigma^n / S_n$ is much more elaborate in that case.

So even though we can't compute g_{L^2} exactly, we can compute the volume of (M_n, g_{L^2}) exactly!

Lecture 4

The 2nd Variation of $E[\varphi, A]$



$$V : \mathbb{R}^n \to \mathbb{R}$$

$$M \subset \mathbb{R}^n \quad \text{s.t.} \quad V(x) = 0 \text{ for } x \in M$$

$$V(x) = V(x_*) + \frac{\partial V}{\partial x_i}\bigg|_{x_*} (x - x_0) + \underbrace{\frac{\partial^2 V}{\partial x_i \partial x_j}}_{J_{ij}} (x - x_0)_i (x - x_0)_j$$

(Hessian)

Normal modes = efunctions of J = $\begin{cases} T_{x_*} M = \ker J \\ \text{normal modes} \end{cases}$

Want to do this in field theory:

$\mathbb{R}^n \longrightarrow$ space of (φ, A)

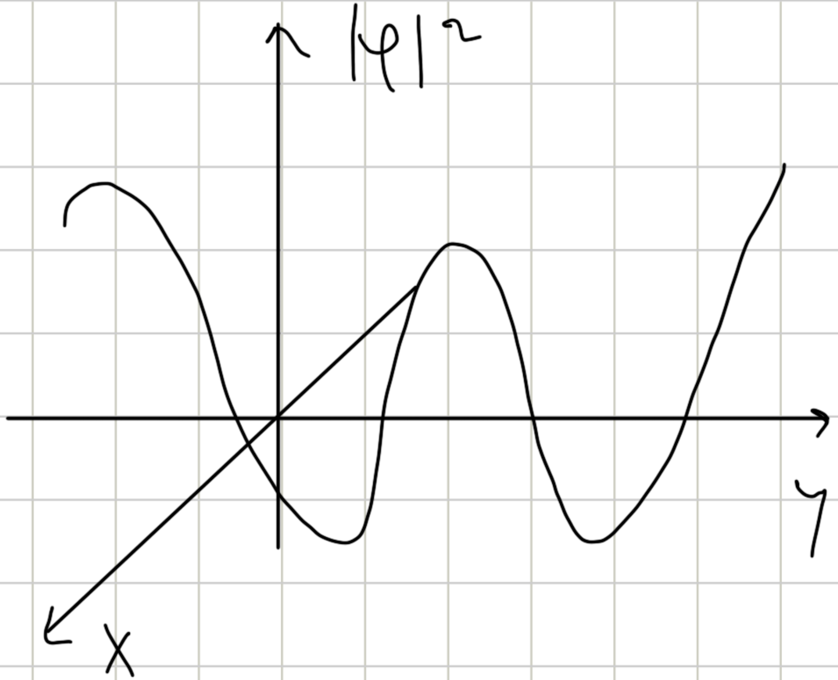
$V \longrightarrow E[\varphi, A]$

$M \longrightarrow M_n$

$x_* \longrightarrow (\varphi, A)$ vortex

<u>Beautiful fact</u> (Alberto-Izquierdo)

$|\varphi|^2 < 1$



$$H = \Delta + |\varphi|^2 = -\partial_i \partial_i + |\varphi|^2$$ has at least one bound state.

$$H\chi = \lambda \chi$$

To any such χ $\exists$ bound state of vortex

$$\chi : \mathbb{R}^2 \to \mathbb{R}$$

Inf. gauge transf : $(i\varphi\chi,\ d\chi)$

Map by S_1 (almost complex structure)

$$(i\varphi\chi,\ d\chi) \xrightarrow{S_1} (-\varphi\chi,\ * d\chi)$$

bound state.

What is J : $E : \Gamma(L) \times \mathcal{A}(L) \longrightarrow \mathbb{R}$

$$E = \frac{1}{2}\|D\varphi\|_{L^2}^2 + \frac{1}{2}\|F_A\|_{L^2}^2 + \frac{1}{2}\|1-|\varphi|^2\|_{L^2}^2$$

Take 2 param. family var. $\varphi_{s,t}$, $A_{s,t}$

s.t. $(\varphi_{0,0}, A_{0,0}) = (\varphi, A)$

defines 2 perturbations

$$(\hat{\varepsilon}, \hat{\alpha}) = (\partial_s \varphi_{s,t}, \partial_s A_{s,t})\big|_{s=t=0}$$

$$(\varepsilon, \alpha) = (\partial_t \varphi_{s,t}, \partial_t A_{s,t})\big|_{s=t=0}$$

$$(\varepsilon, \alpha), (\hat{\varepsilon}, \hat{\alpha}) \in \Gamma(L) \oplus \Omega^1(\Sigma)$$

$$\frac{\partial^2 \mathcal{E}}{\partial s \partial t}(\varphi_{s,t}, A_{s,t})\Big|_{s=t=0} := \mathrm{Hess}\big((\hat{\varepsilon}, \hat{\alpha}), (\varepsilon, \alpha)\big)$$

symmetric bilinear form

$$\mathrm{Hess} : (\Gamma(L) \oplus \Omega^1(\Sigma)) \times (\Gamma(L) \oplus \Omega^1(L)) \longrightarrow \mathbb{R}$$

$$= \langle (\hat{\varepsilon}, \hat{\alpha}), J(\varepsilon, \alpha) \rangle_{L^2}$$

↑ Jacobi operator

$$J : \Gamma(L) \oplus \Omega^1(\Sigma) \to \Gamma(L) \oplus \Omega^1(\Sigma)$$

$J^\dagger = J$ since Hessian is sym.

$$* D\varphi + i D\varphi = 0$$

$$* B = \frac{1}{2}(1 - |\varphi|^2)$$

$$\mathrm{Bog} : \Gamma(L) \times \mathcal{A}(L) \to \Omega^1(L) \times C^\infty(\Sigma)$$

$$\mathrm{Bog}(\varphi, A) = \begin{pmatrix} \frac{1}{\sqrt{2}}(* + i) D\varphi \\ * B - \frac{1}{2}(1 - |\varphi|^2) \end{pmatrix}$$

$$E[\varphi, A] = \frac{1}{4}\| * D\varphi + i D\varphi \|_{L^2}^2 + \frac{1}{2}\| *B - \frac{1}{2}(1-|\varphi|^2)\|_{L^2}^2 + \pi m$$

$$= \frac{1}{2}\| \mathrm{Bog}(\varphi, A)\|_{L^2}^2 + \text{const}.$$

$$\Rightarrow \quad E[\varphi_{s,t}, A_{s,t}] = \frac{1}{2}\left\langle \mathrm{Bog}(\varphi_{s,t}, A_{s,t}), \mathrm{Bog}(\varphi_{s,t}, A_{s,t})\right\rangle_{L^2}$$

$$\frac{\partial^2 E}{\partial s \partial t}[\varphi_{s,t}, A_{s,t}]\Big|_{s=t=0} = \frac{1}{2}\Big\langle \underbrace{d\,\mathrm{Bog}_{(\varphi,A)}}_{\mathcal{B}}(\hat{\varepsilon}, \tilde{\alpha}), d\mathrm{Bog}_{(\varphi,A)}(\varepsilon, \alpha)\Big\rangle_{L^2}$$

$$\mathcal{B} : (\varepsilon, \alpha) \longrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}(*+i)(D\varepsilon - i\alpha\varphi) \\ *d\alpha + H(\varphi, \varepsilon) \end{pmatrix}$$

$$\mathcal{B} : \Gamma(L) \oplus \Omega^1(\Sigma) \rightarrow \Omega^1(L) \oplus C^\infty(\Sigma)$$

$$\frac{\partial^2 E}{\partial s \partial t}[\varphi_{s,t}, A_{s,t}]\Big|_{s=t=0} = \frac{1}{2}\left\langle \mathcal{B}(\hat{\varepsilon}, \hat{\alpha}), \mathcal{B}(\varepsilon, \alpha)\right\rangle_{L^2}$$

$$= \frac{1}{2}\left\langle (\hat{\varepsilon}, \hat{\alpha}), \mathcal{B}^+\mathcal{B}(\varepsilon, \alpha)\right\rangle_{L^2}$$

Define $\quad G: \quad C^\infty(\Sigma) \to \Gamma(L) \oplus \Omega^1(\Sigma)$

$$\chi \longrightarrow \begin{pmatrix} i\varphi\chi \\ d\chi \end{pmatrix}$$

$$(\varepsilon, \alpha) \in \mathcal{G}_\infty^\perp \Rightarrow \langle \underbrace{G\chi}_{\in \mathcal{G}_\infty}, (\varepsilon, \alpha)\rangle_{L^2} = 0 \quad \forall \chi$$

$$\Rightarrow \langle \chi, G^+(\varepsilon, \alpha)\rangle_{L^2} = 0 \quad \forall \chi$$

$$\Rightarrow (\varepsilon, \alpha) \in \ker G^+$$

Also $\quad J G = 0 \quad (\mathcal{G}_\infty \subset \ker J)$

and hence $\quad G^+ J = 0$.

$$G^+(\varepsilon, \alpha) = \delta\alpha + H(i\varphi, \varepsilon)$$

Clever trick: extend $\mathcal{B}$:

$$\hat{\mathcal{B}} : \Gamma(L) \oplus \Omega^1(\Sigma) \to \Omega^1(\Sigma) \oplus C^\infty(\Sigma) \oplus C^\infty(\Sigma)$$

$$\hat{\mathcal{B}} = \begin{pmatrix} \mathcal{B} \\ G^+ \end{pmatrix}$$

$$\hat{J} = \hat{\mathcal{B}}^+ \hat{\mathcal{B}} = J + G G^+$$

Fact $\quad \operatorname{spec} J = \operatorname{spec} \hat{J}$

$\ker \hat{J} = \ker J \cap \mathcal{G}_{\infty}^{-1} \equiv T_{(\varphi, A)} \mathcal{M}_m$!!!

Define $\quad S_1 : \Gamma(L) \oplus \Omega^1(\Sigma) \longrightarrow \Gamma(L) \oplus \Omega^1(\Sigma)$

$$S_1(\varepsilon, \alpha) = (i\varepsilon, *\alpha)$$

$$S_1^2 = -1 \qquad S_1^+ = -S_1$$

$$\hat{\mathcal{B}} S_1 = S_2 \hat{\mathcal{B}}$$

$$S_2 : \Omega^1(\Sigma) \oplus C^\infty(\Sigma) \oplus C^\infty(\Sigma) \circlearrowleft$$

$$(\xi, f_1, f_2) \longmapsto (i\xi, -f_2, f_1)$$

$$\hat{\mathcal{B}} S_1 = S_2 \hat{\mathcal{B}}$$

$$\hat{\mathcal{B}}^+ S_2 = S_1 \hat{\mathcal{B}}^+$$

$$\hat{J} S_1 = \hat{\mathcal{B}}^+ \hat{\mathcal{B}} S_1 = \dots = S_1 \hat{J} \Rightarrow \xi, S_1\xi$$ are both evals of $\hat{J}$ with same eval.

Fact $\quad S_1 \xi \in \mathcal{G}_\infty^\perp$

So $\Leftrightarrow \quad S_1 G = 0 \quad \Leftrightarrow \quad G^+ S_1 = 0$.

Let $\quad G^+ G \chi = \lambda \chi$

$$\hat{J} S_1 G\chi = S_1 \hat{J} G\chi = S_1 G \underbrace{G^+ G\chi}_{=\lambda\chi} = \lambda S_1 G \chi$$

$\Rightarrow S_1 G \chi$ is emode of $\hat{J}$, and also of J since $S_1 G\chi \in \mathcal{G}_\infty^\perp$.