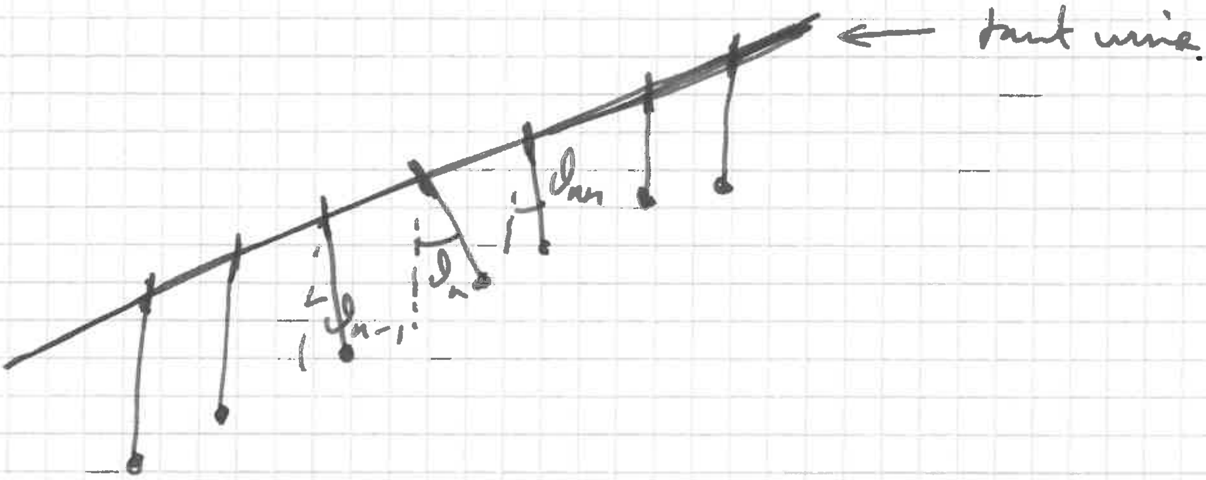
LMS course: Topological Solitons — Lecture 1

KINKS

Mechanical system: fringe of pendula.



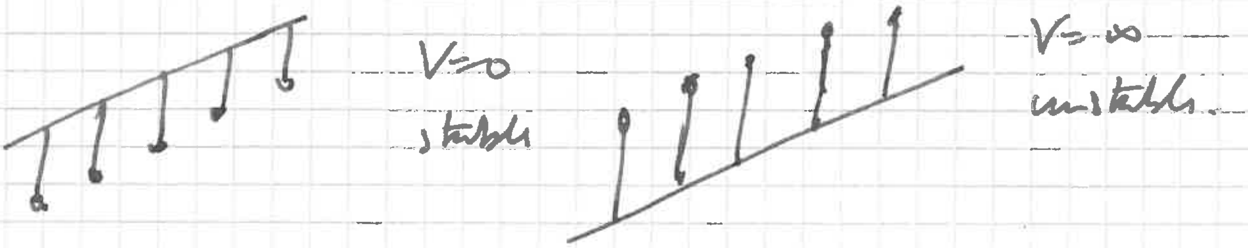
Gravitational potential energy: $V_G = \sum_{n\in\mathbb{Z}} (1-\cos\theta_n)$

Mechanical potential energy: $V_T = \sum_{n\in\mathbb{Z}} \frac{k}{2}(\theta_{n+1}-\theta_n)^2$
(from torsion in wire)

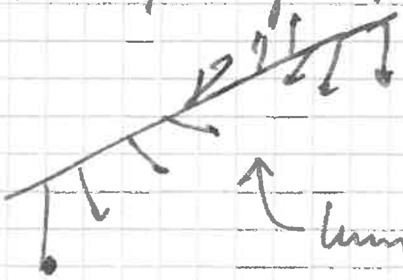
Newton's law of motion

$$\ddot{\theta}_n = -\frac{\partial V}{\partial \theta_n}$$
$$= k(\theta_{n-1} - 2\theta_n + \theta_{n+1}) - \sin\theta_n$$

Static solutions? Critical pts of V.



What if we give the wire a twist?



$\lim_{n\to-\infty} \theta_n = 0, \quad \lim_{n\to\infty} \theta_n = 2\pi$

lump of energy, can't escape. "Kink"

Hard to find exact solutions. Nice trick — take the continuum limit: $h := \frac{1}{\sqrt{k}}$ $\quad k \to \infty$

Define cts "position" coordinate $x = hn$ $\quad \tilde{\vartheta} : \mathbb{R} \to \mathbb{R}$, $\vartheta_n = \tilde{\vartheta}(nh)$

$$\frac{\vartheta_{n-1} - 2\vartheta_n + \vartheta_n}{h^2} = \tilde{\vartheta}''(nh) + O(h^2)$$

Equation of motion $\Rightarrow$ $\vartheta_{tt} - \vartheta_{xx} + \sin\vartheta = 0$

— Sine Gordon equation.

Static kink?

$$\vartheta_{xx} = \sin\vartheta$$

$$\cancel{2}\vartheta_x \vartheta_{xx} = \cancel{2}\vartheta_x \sin\vartheta$$

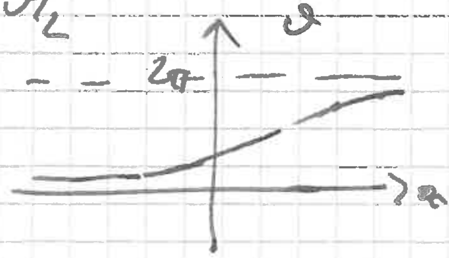
$$\Rightarrow \frac{d}{dx}\left(\frac{\vartheta_x^2}{2}\right) = -\frac{d}{dx}(\cos\vartheta)$$

$$\Rightarrow \frac{\vartheta_x^2}{2} = -\cos\vartheta + C$$

$\lim_{x \to -\infty} \vartheta(x) = 0 \quad \Rightarrow \quad C = 1$

$$\Rightarrow \frac{\vartheta_x^2}{2} = 1 - \cos\vartheta = 1 - (1 - 2\sin^2 \vartheta/2) = 2\sin^2 \vartheta/2$$

$$\Rightarrow \vartheta_x = \pm 2 \sin \vartheta/2$$



Must have + sign

Change variable: $\mu = \tan \vartheta/4$

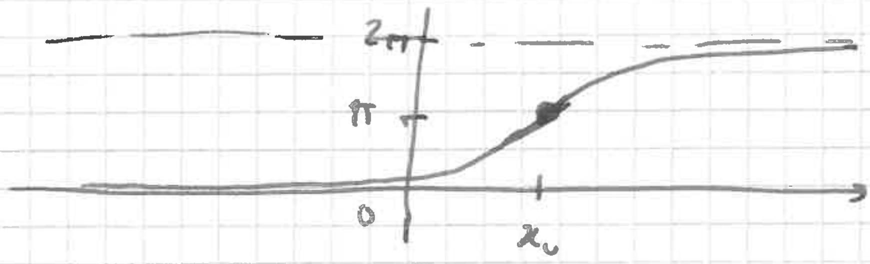
$$\mu_x = \frac{\vartheta_x}{4} \frac{1}{\cos^2 \vartheta/4} = \frac{1}{2} \frac{\sin \vartheta/2}{\cos^2 \vartheta/4}$$

$$= \frac{\sin \vartheta/4 \cos \vartheta/4}{\cos^2 \vartheta/4} = \tan \vartheta/4 = \mu$$

$$\Rightarrow \mu = Ce^x = e^{x - x_0}$$

So

$$\theta(x) = 4 \tan^{-1} e^{x-x_0}$$



Dynamics? $\qquad \theta_{tt} - \theta_{xx} + \sin\theta = 0 \qquad$ (*) $\qquad$ bc's $\theta(x \to \pm\infty) = 2\pi n_\pm$

Conserved quantities:

energy $\quad E(t) = \displaystyle\int_{-\infty}^{\infty} dx \left\{ \tfrac{1}{2}\theta_t^2 + \tfrac{1}{2}\theta_x^2 + (1-\cos\theta) \right\}$

momentum $\quad P(t) = -\displaystyle\int_{-\infty}^{\infty} dx \; \theta_x \theta_t .$

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} dx \left\{ \theta_t \theta_{tt} + \theta_x \theta_{xt} + \sin\theta \, \theta_t \right\}$$

$$= \int_{-\infty}^{\infty} dx \left\{ \theta_t (\theta_{tt} + \sin\theta) + \frac{\partial}{\partial x}(\theta_x \theta_t) - \theta_{xx}\theta_t \right\}$$

$$= \Big[\theta_x \theta_t \Big]_{x=-\infty}^{x=\infty} = 0 .$$

Exercise: prove that $\dfrac{dP}{dt} = 0$ also.

Static kink: $\quad E = \displaystyle\int_{-\infty}^{\infty} dx \; \theta_x^2 = 8 , \quad P = 0 .$

~~Lorentz symmetry~~ $\quad$ ~~Wave operator~~ $\cancel{\theta_{tt} - \theta_{xx}}$

~~compare with Laplace~~ $\quad \cancel{-\theta_{xx} - \theta_{yy}}$

Lorentz symmetry (*) is a nonlinear wave equation

$$\frac{\partial^2\theta}{\partial t^2} - \frac{\partial^2\theta}{\partial x^2} - F(\theta) = 0 \qquad (*)$$

$(F = -\sin)$

cf. Nonlinear "Laplace" equation

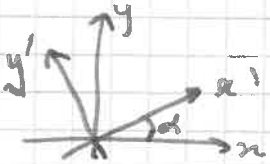
$$-\frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial y^2} - F(\theta) = 0$$

This has some obvious symmetries. If $\theta(x,y)$ is a solution, so is

$$\tilde\theta(x,y) = \theta(x-x_0, y-y_0) \quad \text{— translation invariance.}$$

$$\tilde\theta(x,y) = \theta\left(R(\alpha)\begin{pmatrix} x \\ y \end{pmatrix}\right) \qquad R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

— rotation invariance.



Slight change of viewpoint:

if $\theta(x,y)$ is a solution, so is $\theta(x',y')$ where (x',y') is any translated + rotated coordinate system.

What's the analogous symmetry for (*)?

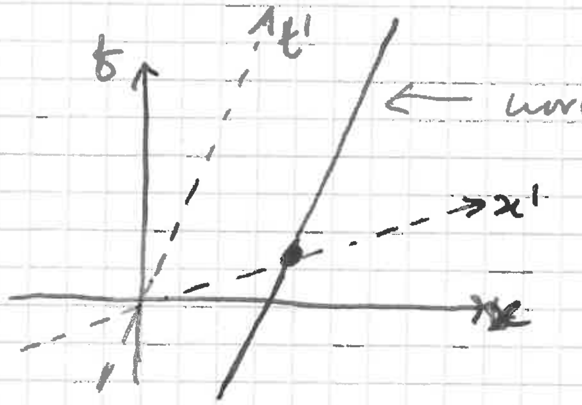
Assume $\theta(t,x)$ is a solution of (*). So is

$$\tilde\theta(t,x) = \theta(t-t_0, x-x_0) \quad \text{— spacetime translation invariance.}$$

$$\tilde\theta(t,x) = \theta\left(B(v)\begin{pmatrix} t \\ x \end{pmatrix}\right) \qquad B(v) = \gamma(v)\begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$$

— boost invariance

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}}$$

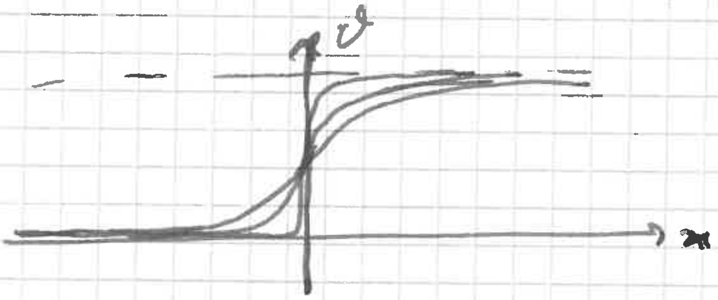


← world line of static kink in frame $(t',x') = B(-v)(t,x)$ moves at constant velocity v in frame (t,x).

So $\vartheta(t,x) = 4\tan^{-1} e^{\gamma(x-vt)}$

are solutions: moving kinks, velocity v. $\quad -1<v<1$

~~Exercise~~: Note that the kinks sharpen: $\gamma(v) = \frac{1}{\sqrt{1-v^2}} \geq 1$.



Lorentz-FitzGerald contraction.

Exercise: Show that a kink moving at velocity v has

$$E = 8\gamma \qquad\qquad P = 8\gamma v$$

so $\quad E^2 - P^2 = 8^2\gamma^2(1-v^2) = 8^2$ is velocity independent.

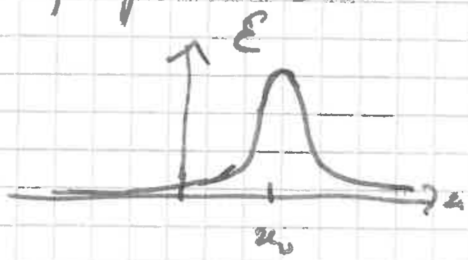
This is Einstein's famous energy-momentum relation, in units where speed of light $c=1$ (and "particle") has mass $m=8$:

$$E^2 - P^2 = m^2 c^4$$

Audacious idea: Kinks are really particles!

① They're spatially localised

$$\mathcal{E} = \frac{1}{2}\vartheta_x^2 + (1-\cos\vartheta)$$



② They obey relativistic kinematics.

③ They have ANTIPARTICLES: $\bar{\vartheta}(x) = 4\tan^{-1} e^{-x}$

Lagrangian Field Theory

Recall sine-Gordon equation: $\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0$

Can give a variational formulation...

Define Lagrangian density $\mathcal{L} : \mathbb{R}^3 \longrightarrow \mathbb{R}$

Given a field $\varphi : \mathbb{R}^{1,1} \longrightarrow \mathbb{R}$ and a compact subset $\Omega \subset \mathbb{R}^{1,1}$ (eg $[t_0, t_1] \times [x_0, x_1]$)

define the action of φ (over Ω) to be

$$S_\Omega(\varphi) = \int_\Omega \mathcal{L}(\varphi_t, \varphi_x, \varphi)\, dt\, dx$$

φ is a solution of the theory defined by $\mathcal{L}$ if, $\forall \Omega$, and for all smooth variations of φ fixed on $\partial\Omega$,

$$\Phi : (-\delta, \delta) \times \Omega \to \mathbb{R} \quad \text{smooth},$$

$$\varphi_s : \Omega \to \mathbb{R}, \qquad \varphi_s(t,x) = \Phi(s,t,x)$$

$$\varphi_0 = \varphi$$

$$\varepsilon : \Omega \to \mathbb{R}, \qquad \varepsilon(t,x) = \partial_s\big|_{s=0} \Phi(s,t,x)$$

$$\varepsilon\big|_{\partial\Omega} = 0$$

$$\frac{d}{ds}\bigg|_{s=0} S_\Omega(\varphi_s) = 0.$$

$$S(\varphi_s) = \int_\Omega \mathcal{L}(\varphi_{s,t}, \varphi_{s,x}, \varphi_s)\, dt\, dx$$

$$\frac{d}{ds}\bigg|_{s=0} S(\varphi_s) = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \varphi_t} \varepsilon_t + \frac{\partial \mathcal{L}}{\partial \varphi_x} \varepsilon_x + \frac{\partial \mathcal{L}}{\partial \varphi} \varepsilon \right) dt\, dx$$

$$\frac{d}{ds}\Big|_{s=0} S(\varphi_s) = \int_\Omega \Big\{ \partial_t\Big(\overbrace{\frac{\partial \mathcal{L}}{\partial \varphi_t}\,\varepsilon}^{A^0}\Big) + \partial_x\Big(\overbrace{\frac{\partial \mathcal{L}}{\partial \varphi_x}\,\varepsilon}^{A^1}\Big) + \varepsilon\Big(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_t\Big(\frac{\partial \mathcal{L}}{\partial \varphi_t}\Big) - \partial_x\Big(\frac{\partial \mathcal{L}}{\partial \varphi_x}\Big)\Big)\Big\}\,dt\,dx$$

$$= \int_\Omega \operatorname{div} A \, dt\,dx + \int_\Omega \varepsilon\Big(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_t\Big(\frac{\partial \mathcal{L}}{\partial \varphi_t}\Big) - \partial_x\Big(\frac{\partial \mathcal{L}}{\partial \varphi_x}\Big)\Big)\,dt\,dx$$

$$= \underbrace{\int_{\partial\Omega} \underline{A}\cdot d\underline{S} \qquad \underline{A} = \underline{0} \text{ on } \partial\Omega!}_{0} + \int_\Omega \varepsilon\big(\;\text{———}\;\big)\,dt\,dx$$

$= 0$ for all such variations.

Fundamental Lemma of Calculus of Variations $\Rightarrow$

$$\partial_t\Big(\frac{\partial \mathcal{L}}{\partial \varphi_t}\Big) + \partial_x\Big(\frac{\partial \mathcal{L}}{\partial \varphi_x}\Big) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \quad \text{—} (EL)$$

Holds $\forall\, \Omega$, so (EL) holds on all of $\mathbb{R}^{1,1}$.

E.g. $\mathcal{L}(a_0, a_1, a) = \frac{1}{2}(a_0^2 - a_1^2) - V(a)$

(EL): $\varphi_{tt} - \varphi_{xx} + V'(\varphi) = 0$ $\qquad V(\varphi) = 1 - \cos\varphi \Rightarrow SG.$

Why is this useful? SYMMETRIES!

E.g. assume $\mathcal{L}$ takes the form $\mathcal{L}(a_0, a_1, a) = f(a_0^2 - a_1^2, a)$

Recall Lorentz boost $B(v) : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$

$$B(v)\begin{pmatrix} t \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix}$$

<u>Exercise</u>: For all $\varphi : \mathbb{R}^{1,1} \to \mathbb{R}$ and $v \in (-1,1)$

let $\tilde{\varphi}(t,x) = \varphi\left(B(v) \binom{t}{x} \right)$.

Show that, for all $\Omega \subset \mathbb{R}^{1,1}$

$$S_\Omega(\tilde{\varphi}) = S_{\tilde{\Omega}}(\varphi) \quad \text{where } \tilde{\Omega} = B(-v)\Omega.$$

So $\tilde{\varphi}$ is a critical pt of $S \iff \varphi$ is a critical pt of S.

i.e. the EL eqn <u>must</u> be boost invariant.

But it goes deeper! Noether's Principle

Continuous symmetries $\Rightarrow$ Conservation Laws.

<u>Example</u> time translation invariance

For given field $\varphi : \mathbb{R}^{1,1} \to \mathbb{R}$ consider the variation

$\varphi_s : \mathbb{R}^{1,1} \to \mathbb{R}$, $\quad \varphi_s(t,x) = \varphi(t-s,x)$.

Denote by $\mathcal{L}_\varphi : \mathbb{R}^{1,1} \to \mathbb{R}$ $\quad \mathcal{L}$ evaluated for φ

i.e. $\mathcal{L}(\varphi_t(t,x), \varphi_x(t,x), \varphi(t,x))$

Then

$$\mathcal{L}_{\varphi_s}(t,x) = \mathcal{L}_\varphi(t-s,x)$$

$$\Rightarrow \quad \partial_s\big|_{s=0} \mathcal{L}_{\varphi_s} = -\partial_t \mathcal{L}_\varphi.$$

But

$$\partial_s\big|_{s=0} \mathcal{L}_{\varphi_s} = \frac{\partial \mathcal{L}}{\partial \varphi_t} \partial_s\big|_{s=0} \varphi_{s,t} + \frac{\partial \mathcal{L}}{\partial \varphi_x} \partial_s\big|_{s=0} \varphi_{s,x} + \frac{\partial \mathcal{L}}{\partial \varphi} \partial_s\big|_{s=0} \varphi_s$$

$$= -\frac{\partial \mathcal{L}}{\partial \varphi_t} \varphi_{tt} - \frac{\partial \mathcal{L}}{\partial \varphi_x} \varphi_{xt} - \frac{\partial \mathcal{L}}{\partial \varphi} \varphi_t$$

$$= \partial_t\left(-\frac{\partial \mathcal{L}}{\partial \varphi_t} \varphi_t \right) + \partial_x \left(-\frac{\partial \mathcal{L}}{\partial \varphi_x} \varphi_x \right)$$

$$+ \varphi_t \left\{ \partial_t \left(\frac{\partial \mathcal{L}}{\partial \varphi_t} \right) + \partial_x \left(\frac{\partial \mathcal{L}}{\partial \varphi_x} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} \right\}$$

Assume now that φ is a solution of the field theory.
Then it satisfies (EL), so

$$\partial_t \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \varphi_t} \partial \varphi_t - \mathcal{L}_\varphi \right)}_{J^0} + \partial_x \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \varphi_x} \right)}_{J^1} = 0$$

i.e. the "current" (J^0, J^1) is conserved.

Associated conserved charge:

$$E = \int_{\mathbb{R}} J^0(t,x)\, dx$$

$$\frac{dE}{dt} = - \int_{\mathbb{R}} \partial_x J^1 \, dx = J^1(t,-\infty) - J^1(t,\infty)$$

$$= 0 \quad \text{(if we apply the right boundary conditions)}$$

So time translation symmetry
$\Rightarrow$ conservation of <u>ENERGY</u>

<u>Exercise</u> By considering $\varphi_s(t,x) = \varphi(t, x-s)$
show that spatial translation symmetry implies
conservation of <u>momentum</u>

$$P = -\int_{\mathbb{R}} \frac{\partial \mathcal{L}}{\partial \varphi_t} \varphi_x \, dx$$

Energy of a static field: $E_{\text{static}}(\varphi) = \int_{\mathbb{R}} -\mathcal{L}(0, \varphi_x, \varphi)\, dx$
$\varphi : \mathbb{R} \to \mathbb{R}$

φ is a static solution of the theory $\iff$ φ is a critical pt of E_{static}.

The Bogomol'nyi trick (for SG theory)

$$E_{static}(\varphi) = \int_{\mathbb{R}} \left(\tfrac{1}{2}\varphi_x^2 + (1-\cos\varphi)\right)dx$$

$$= \int_{\mathbb{R}} \left(\tfrac{1}{2}\varphi_x^2 + 2\sin^2 \varphi/_2\right)dx.$$

Assume φ has kink boundary behaviour:

$$\lim_{x\to-\infty}\varphi(x) = 0, \qquad \lim_{x\to\infty}\varphi(x) = 2\pi.$$

Then

$$0 \le \tfrac{1}{2}\int_{\mathbb{R}} (\varphi_x - 2\sin\varphi/_2)^2\, dx$$

$$= E_{static}(\varphi) - 2\int_{\mathbb{R}} \varphi_x \sin\varphi/_2\, dx$$

$$= E_{static}(\varphi) + 4\left[\cos\varphi/_2\right]_{-\infty}^{\infty}$$

$$= E_{static}(\varphi) - 8$$

So $E_{static}(\varphi) \ge 8$ with equality $\Longleftrightarrow$

$$\frac{d\varphi}{dx} = 2\sin\frac{\varphi}{2}.$$

$\boxed{1^{st} \text{ order ODE}}$ $\Rightarrow$ $\varphi(x) = 4\tan^{-1} e^{(x-a)}$.

But we now have extra info:

1. $E(\varphi_{kink}) = 8$
2. φ_{kink} minimizes E_{static} among all fields in its topological class.

All these ideas generalize easily to higher dimensions.

Higher Dimensions

$\varphi : \mathbb{R}^{d,1} \to \mathbb{R}^k$ ← k interacting scalar fields on $d+1$ dimensional spacetime, $d \geq 2$

$$\mathcal{L} = \frac{1}{2}|\varphi_t|^2 - \frac{1}{2}\sum_i |\partial_i \varphi|^2 - V(\varphi)$$

Still Lorentz invariant. EL eqn. $V : \mathbb{R}^k \to [0,\infty)$

$$\left(\partial_t^2 - \sum_{i=1}^{d} \partial_i^2\right)\varphi_a + \frac{\partial V}{\partial \varphi_a} = 0$$

<u>Problem</u> This theory has no interesting static solutions at all! Why?

<u>The Derrick Scaling argument</u>:

Static solutions are critical pts of

$$E_{static}(\varphi) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2}\sum_i |\partial_i \varphi|^2 + V(\varphi) \right\} d^d x$$

So, for <u>all</u> smooth variations φ_s of $\varphi = \varphi_0$, should have

$$\left.\frac{d}{ds}\right|_{s=0} E_{static}(\varphi_s) = 0$$

In particular, this is true for the scaling variation

$$\varphi_\lambda(x) = \varphi(\lambda x) \qquad [\text{Now } \varphi_1 = \varphi]$$

Should have $\left.\frac{d}{d\lambda}\right|_{\lambda=1} E_{static}(\varphi_\lambda) = 0.$

$E_{static}(\varphi) := E_2(\varphi) + E_0(\varphi)$

$$E_2(\varphi_\lambda) = \int_{\mathbb{R}^d} \frac{1}{2}\sum_i |\partial_i \varphi_\lambda|^2(x)\, d^d\underline{x}$$

$$= \int_{\mathbb{R}^d} \frac{\lambda^2}{2}\sum_i |\partial_i \varphi|^2 (\lambda x)\, d^d \underline{x}$$

$$= \lambda^{2-d} \int_{\mathbb{R}^d} \frac{1}{2}\sum_i |\partial_i \varphi|^2 (y)\, d^d y \qquad y := \lambda x$$

$$= \lambda^{2-d} E_2(\varphi)$$

$$E_0(\varphi_\lambda) = \int_{\mathbb{R}^d} V(\varphi_\lambda(x))\, d^d x$$

$$= \int_{\mathbb{R}^d} V(\varphi(\lambda x))\, d^d x$$

$$= \lambda^{-d} E_0(\varphi)$$

So $E_{static}(\varphi_\lambda) = \lambda^{2-d} E_2(\varphi) + \lambda^{-d} E_0(\varphi)$

$$\Rightarrow \frac{d}{d\lambda}\Big|_{\lambda=1} E_{static}(\varphi_\lambda) = (2-d)\underbrace{E_2(\varphi)}_{\geq 0} - d\underbrace{E_0(\varphi)}_{\geq 0}$$
$$= 0.$$

If $d \geq 3$, this is impossible unless $E_2(\varphi) = E_0(\varphi) = 0$
— only static solutions are vacua.

If $d = 2$, need $E_0(\varphi) = 0$. Naively looks like only vacua

If $d = 1$ $\quad E_2(\varphi) = E_0(\varphi) \quad$ Virial constraint.

Many possible ways to avoid this "no go" theorem.

(1) Add to $\mathcal{L}$ terms of higher degree in
$\partial_t \varphi, \; \partial_i \varphi$

[But this requires ingenuity: eg adding

$$\mathcal{L}_4 = \frac{1}{4}\left\{ |\partial_t \varphi|^2 - \sum_i |\partial_i \varphi|^2 \right\}^2$$

doesn't work — theory has energy unbounded below.]

SKYRMIONS

(2) Add "gauge fields"
Vector fields (or 1-forms, or connections)
that interact with φ in a very constrained way
Beautiful, ~~…~~ really relevant for fundamental physics
— VORTICES, MONOPOLES, INSTANTONS

(3) Add a constraint to φ (and live on a knife edge)
LUMPS

$$\varphi : \mathbb{R}^{2,1} \longrightarrow S^2 \subset \mathbb{R}^3 \quad \text{i.e. demand } |\varphi| = 1 \;\; \forall (t,x) \in \mathbb{R}^{2,1}$$

$$\mathcal{L} = \frac{1}{2}\left(|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2 \right)$$

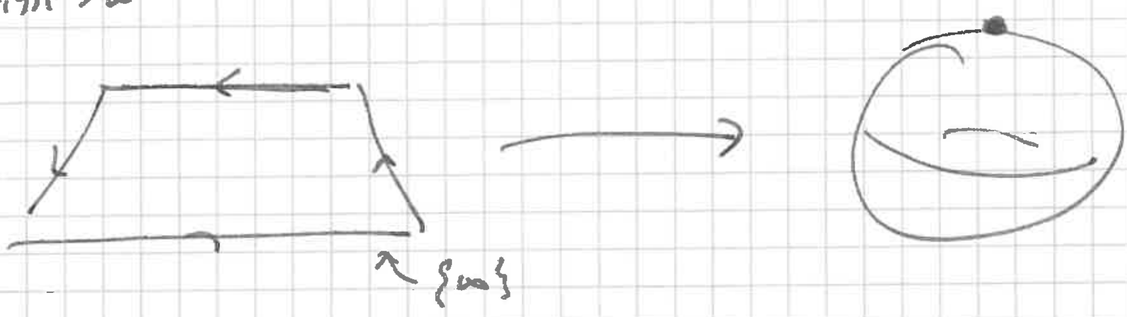
— no potential
— evades Derrick (just)

$$E = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\varphi_t|^2 + |\varphi_x|^2 + |\varphi_y|^2 \right) dx\,dy.$$

Static: $\quad E(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\varphi_x|^2 + |\varphi_y|^2 \right) dx\,dy.$

Demand E is finite. Motivates choice of b.c.

$$\lim_{|(x,y)| \to \infty} \varphi(x,y) = \varphi_0 \quad \text{constant} = (0,0,1) \quad \text{WLOG.}$$



Extends ctsly to a map $\varphi : \underbrace{\mathbb{R}^2 \cup \{\infty\}}_{S^2} \to S^2$

Such maps are classified topologically by their degree

$$n = \deg\varphi = \text{\# times } \varphi \text{ wraps } S^2 \text{ around } S^2 \text{ (counted with orientation)}$$

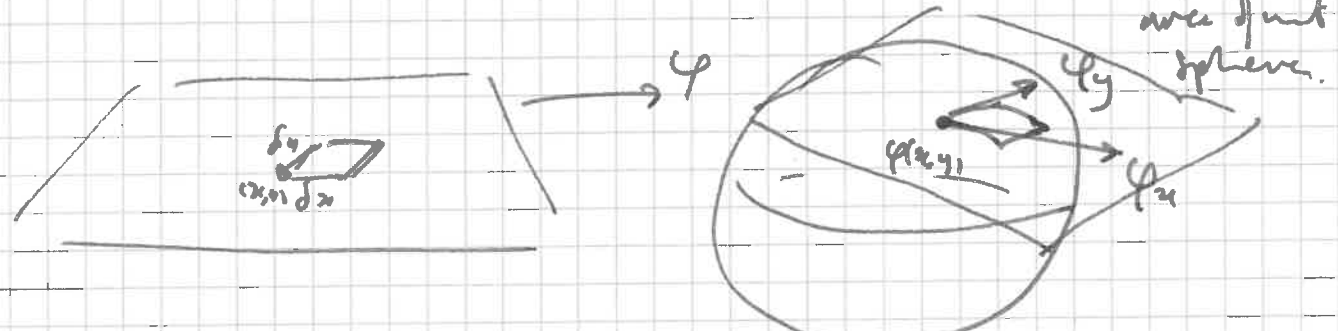
i.e. If $\deg\varphi \neq \deg\psi$ $\nexists$ cts deformation $\varphi \rightsquigarrow \psi$

If $\deg\varphi = \deg\psi$ there does.

Two ways to compute degree of φ

(i) Compute the signed area of $\varphi(\mathbb{R}^2) = n \cdot 4\pi$

area of unit sphere.

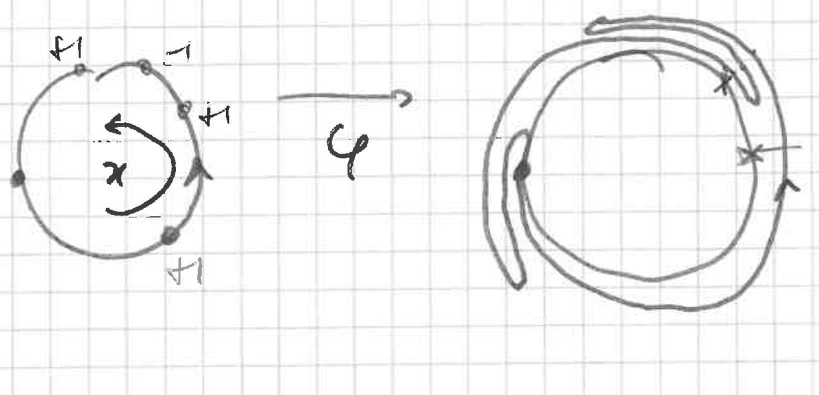


Signed Area of image of $\square \approx \varphi\cdot(\varphi_x \times \varphi_y)\,\delta x\,\delta y$

So $$n = \frac{1}{4\pi}\int_{\mathbb{R}^2} \varphi\cdot(\varphi_x \times \varphi_y)\,dx\,dy$$

(ii) ~~transport the~~ Count signed preimages of regular values of φ

eg one dimension lower



Pick a pt in target $p \in S^1$.

Assign signs to preimages $x_i \in \varphi^{-1}(p)$

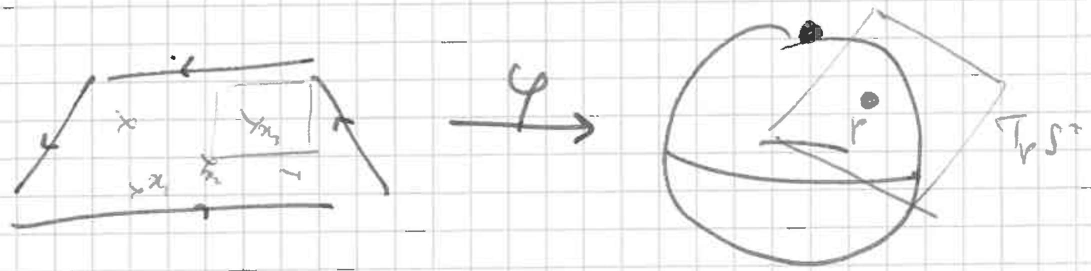
$$\text{sign}(x_i) = \begin{cases} +1 & \text{if } \varphi_x(x_i) > 0 \\ -1 & \text{if } \varphi_x(x_i) < 0 \end{cases}$$

(If $\exists$ preimage with $\varphi_x(x_i) = 0$ choose a different p)

Then $$n = \sum_{x_i \in \varphi^{-1}(p)} \text{sign}(x_i)$$

— counts # times φ wraps S^1 around S^1.

Extends to maps $\varphi : \underset{\mathbb{R}^2 \cup \{\infty\}}{\overset{\shortparallel}{S^2}} \to S^2$



$T_p S^2$ = set of velocity vectors of curves through p in S^2 (at ~~the~~ the pt p)

$T_x \mathbb{R}^2$ = set of velocity vectors of curves in $\mathbb{R}^2$ through x

$$d\varphi_x : T_x \mathbb{R}^2 \longrightarrow T_p S^2 \qquad \text{linear map}$$

$\Downarrow v$

take a curve $\gamma(t)$ in $\mathbb{R}^2$ with $\gamma(0) = x$, $\dot\gamma(0) = v$

$$d\varphi_x(v) := \frac{d}{dt}\bigg|_{t=0} \varphi(\gamma(t)) \in T_p S^2$$

Assign signs to preimages $x_i \in \varphi^{-1}(p)$

$$\text{sign}(x_i) = \begin{cases} +1 & \text{if } \cancel{d\varphi}\ d\varphi_{x_i} \text{ is orientation preserving} \\ -1 & \text{if } d\varphi_{x_i} \text{ is orientation reversing} \end{cases}$$

(If rank $d\varphi_{x_i} < 2$, choose a different p)

$$n = \sum_{x_i \in \varphi^{-1}(p)} \deg_{x_i}(\varphi).$$

<u>Exercise</u> Consider $\varphi : \mathbb{R}^2 \to S^2$

$$\varphi(r,\theta) = (\sin f(r) \cos k\theta, \sin f(r) \sin k\theta, \cos f(r))$$

where $f : [0,\infty) \to \mathbb{R}$ is smooth, decreasing and has

$$f(0) = \pi, \qquad \lim_{r\to\infty} f(r) = 0. \qquad k \in \mathbb{Z}^*.$$

Compute the degree of φ.

<u>The Bogomolny Bound</u> Let $\varphi : \mathbb{R}^2 \to S^2$
$\varphi(\infty) = (0,0,1)$ (say)
degree $n \geq 0$.

Then $E(\varphi) \geq 4\pi n$ and

$$E(\varphi) = 4\pi n \iff \varphi_x + \varphi \times \varphi_y = 0$$

Proof:
$$0 \leq \frac{1}{2}\int_{\mathbb{R}^2} |\varphi_x + \varphi\times\varphi_y|^2 \, dx\,dy$$
$$= \frac{1}{2}\int_{\mathbb{R}^2} (|\varphi_x|^2 + |\varphi\times\varphi_y|^2)\, dx\,dy + \int_{\mathbb{R}^2} \varphi_x\cdot(\varphi\times\varphi_y)\, dx\,dy$$
$$= \frac{1}{2}\int_{\mathbb{R}^2} (|\varphi_x|^2 + |\varphi_y|^2)\, dx\,dy - 4\pi n.$$
$\square$

Cf static field equation: smooth variation φ_s of $\varphi = \varphi_0$

$\partial_s|_{s=0} \varphi_s = \varepsilon : \mathbb{R}^2 \to \mathbb{R}^3$ N.B. $\varphi\cdot\varepsilon \equiv 0$
$(\varepsilon(x,y) \in T_{\varphi(x,y)}S^2)$

Assume ε has compact support $\Omega \subset \mathbb{R}^2$ say.

$$\frac{d}{ds}E(\varphi_s)\Big|_{s=0} = \frac{d}{ds}\Big|_{s=0} \frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\frac{\partial\varphi_s}{\partial x}\right|^2 + \left|\frac{\partial\varphi_s}{\partial y}\right|^2\right) dx\,dy$$
$$= \int_{\Omega} (\varphi_x\cdot\varepsilon_x + \varphi_y\cdot\varepsilon_y)\, dx\,dy$$

$$\frac{d}{ds} E(\varphi_s)\Big|_{s=0} = \int_\Omega \{\partial_x(\varphi_x\cdot\varepsilon)+\partial_y(\varphi_y\cdot\varepsilon) - \varepsilon\cdot(\varphi_{xx}+\varphi_{yy})\}\,dx\,dy$$

$$= \int_\Omega \varepsilon\cdot(-\varphi_{xx}-\varphi_{yy})\,dx\,dy$$

$= 0$ for all variations i.e. all ε.

$\Rightarrow -(\varphi_{xx}+\varphi_{yy}) = 0$?? $\underline{\underline{\text{NO}}}$.

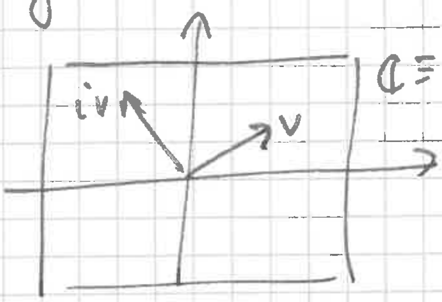
Recall $\varepsilon\cdot\varphi \equiv 0$, so we can only conclude that the component of $\varphi_{xx}+\varphi_{yy}$ $\perp \varphi$ vanishes:

$$\Delta\varphi := -\varphi_{xx} - \varphi_{yy}$$

Then $\Delta\varphi - (\varphi\cdot\Delta\varphi)\,\varphi = 0$ — 2nd order PDE.

cf. $\varphi_x + \varphi\times\varphi_y = 0$ — 1st order PDE

But it gets better. Can identify $T_{(x,y)}\mathbb{R}^2$ with $\mathbb{C}$.



$\mathbb{C} \equiv \mathbb{R}^2$

Have natural linear map

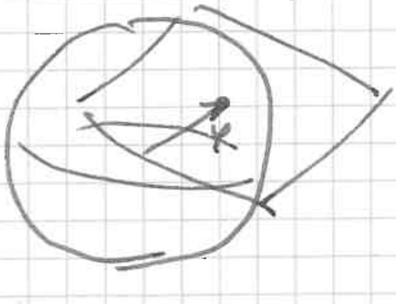
$$J_{\mathbb{R}^2} : \underset{\mathbb{C}}{\underset{\shortparallel}{T_{(x,y)}\mathbb{R}^2}} \to \underset{\mathbb{C}}{\underset{\shortparallel}{T_{(x,y)}\mathbb{R}^2}}$$

$$v \longmapsto iv$$

= rotation by $\pi/2$ anticlockwise.

Note $J_{\mathbb{R}^2}^2 = -1$.

Can also identify $T_\varphi S^2$ with $\mathbb{C}$ (will see how shortly).



$T_\varphi S^2$

S^2 also has a natural map $J_{S^2} : T_\varphi S^2 \to T_\varphi S^2$ s.t. $J_{S^2}^2 = -1$

Namely $v \mapsto \varphi \times v$

Polyakov eqn

$$\varphi_x + \varphi \times \varphi_y = 0 \qquad -① $$

$$\Rightarrow \quad \varphi_y - \varphi \times \varphi_x = 0 \qquad -②$$

①: $\quad d\varphi\left(\frac{\partial}{\partial x}\right) = \varphi \times d\varphi\left(\frac{\partial}{\partial y}\right)$

i.e. $\quad d\varphi\left(J_{\mathbb{R}^2} \frac{\partial}{\partial y}\right) = J_{S^2}\, d\varphi\left(\frac{\partial}{\partial y}\right)$

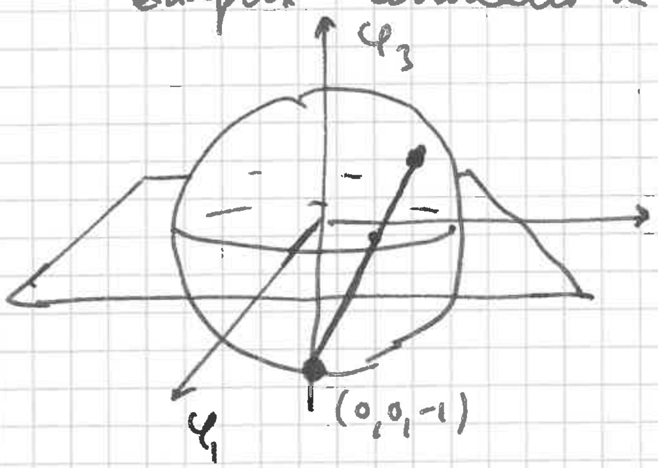
②: $\quad d\varphi\left(\frac{\partial}{\partial y}\right) = \varphi \times d\varphi\left(\frac{\partial}{\partial x}\right)$

i.e. $\quad d\varphi\left(J_{\mathbb{R}^2} \frac{\partial}{\partial x}\right) = J_{S^2}\, d\varphi\left(\frac{\partial}{\partial x}\right)$

$\therefore$ Polyakov $\iff d\varphi \circ J_{\mathbb{R}^2} = J_{S^2} \circ d\varphi$

$\iff \varphi$ is HOLOMORPHIC!

But how can we identify $T_\varphi S^2 \equiv \mathbb{C}$? By defining complex coordinates on S^2:



$$\omega = \frac{\varphi_1 + i\varphi_2}{1+\varphi_3}$$

Exercise: Show that

$$\varphi = \frac{(2\,\mathrm{Re}\,\omega,\ 2\,\mathrm{Im}\,\omega,\ 1-|\omega|^2)}{1+|\omega|^2}$$

Exercise: Show that, w.r.t. complex coords $z = x+iy$ on $\mathbb{R}^2 \cong \mathbb{C}$ and ω on $S^2 \equiv \mathbb{C} \cup \{\infty\}$ then

Polyakov eqn is $\dfrac{\partial w}{\partial \bar z} = 0$: Cauchy-Riemann eqn.

General solution of degree n

$$w(z) = \frac{a_0 z^n + a_1 z^{n-1} + \dots + a_n}{b_0 z^n + b_1 z^{n-1} + \dots + b_n} = \frac{p(z)}{q(z)}$$

max{deg p, deg q} $= n$, no common roots.

b.c. $w(\infty) = 0 \Rightarrow a_0 = 0.$

$\Rightarrow b_0 \neq 0.$

$$w(z) = \frac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{z^n + b_1 z^{n-1} + \dots + b_n}$$

Space of solutions $M_n \subset \mathbb{C}^{2n}$

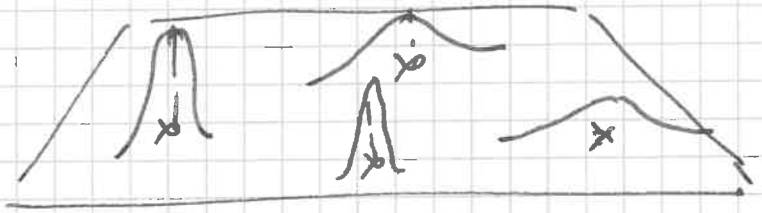
E.g. $n=1$ $\quad w(z) = \dfrac{a_1}{z+b_1}$ $\quad (a_1, b_1) \in \mathbb{C}^\times \times \mathbb{C}$

Exercise show that the energy density

$$\mathcal{E} = \tfrac{1}{2}\left(|\varphi_x|^2 + |\varphi_y|^2\right)$$ of this soliton

is a lump centred at $z = -b_1$, with width $\sim |a_1|$.

Generically:



Exercise Describe $\mathcal{E}$ for the "coincident" 2 lump solution

$$w(z) = \frac{a_2}{z^2}.$$

Is it just a bigger lump?

$$\mathcal{E} = |\varphi_z|^2 = \frac{4|w_z|^2}{(1+|w|^2)^2} = \frac{4\left|\frac{-a_1}{(z+b_1)^2}\right|^2}{\left(1+\left|a_1/z+b_1\right|^2\right)^2} = \frac{4|a_1|^2}{\left(|a_1|^2 + |z+b_1|^2\right)^2}$$

$$= \frac{4/|a_1|^2}{\left(1 + |(z+b_1)/a_1|^2\right)^2}$$

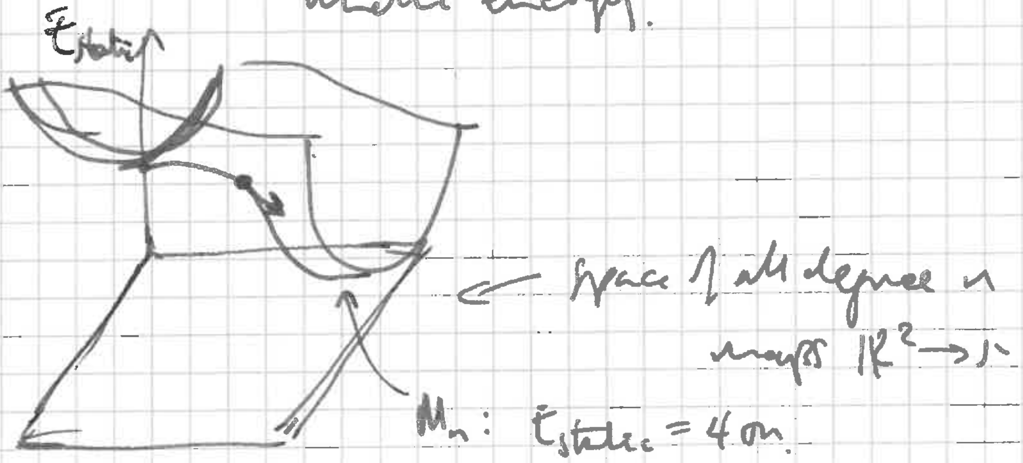
Dynamics of Lumps

$$\mathcal{L} = \tfrac{1}{2}\left(|\varphi_t|^2 - |\varphi_x|^2 - |\varphi_y|^2\right)$$

$$E = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\varphi_t|^2 + |\varphi_x|^2 + |\varphi_y|^2\right) dx\,dy.$$

$$= T + E_{static}$$

$\uparrow$ kinetic energy.



M_n = flat valley bottom in potential energy landscape

Say $\varphi(0) \in M_n$, $\dot\varphi(0) \in T_{\varphi(0)}M_n$ small

$\Rightarrow T$ small.

$E = \underset{\substack{\uparrow \\ \text{small}}}{T} + E_{static}$ conserved.

$\Rightarrow \varphi(t)$ can't get very far from M_n.

Geodesic Approximation (Manton): Constrain φ s.t., at each t, $\varphi(t) \in M_n$. Dynamics of φ determined by restricted action functional

$$S\Big| = \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dx\,dy \left(\tfrac{1}{2}|\dot\varphi|^2 - \tfrac{1}{2}\left(|\varphi_x|^2 + |\varphi_y|^2\right)\right)$$
$$= \int_{\mathbb{R}} dt \left\{\frac{1}{2}\int_{\mathbb{R}^2}|\dot\varphi|^2 dx\,dy - 4\pi n\right\}$$

Recall $\varphi \leftrightarrow \omega(z) = \dfrac{a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}{z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}$

$$a_1 = q_1 + i q_2, \quad a_2 = q_3 + i q_4, \dots, \quad b_n = q_{4n-1} + i q_{4n}$$

Write ω schematically as $\omega(z;q)$

Now let q_i depend on t, compute $\omega_t = \frac{\partial \omega}{\partial t}$

$$\omega_t(z) = \frac{\partial \omega}{\partial q_i} \dot{q}_i$$

$$T = \frac{1}{2}\int_{\mathbb{R}^2} |\varphi_t|^2 \, dx\,dy = \frac{1}{2}\int_{\mathbb{R}^2} \frac{4|\omega_t|^2}{(1+|\omega|^2)^2} \, dx\,dy$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} \sum_{i,j=1}^{4n} \frac{4}{(1+|\omega(z;q)|^2)^2} \frac{\partial \bar{\omega}}{\partial q_i}\frac{\partial \omega}{\partial q_j} \dot{q}_i \dot{q}_j \, dx\,dy$$

$$=: \frac{1}{2}\sum_{i,j=1}^{4n} g_{ij}(q)\,\dot{q}_i \dot{q}_j$$

$$g_{ij}(q) = \mathrm{Re}\int_{\mathbb{R}^2} \frac{4}{(1+|\omega(z;q)|^2)^2} \frac{\partial \bar{\omega}}{\partial q_i}\frac{\partial \omega}{\partial q_j} \, dx\,dy.$$

Action governing evolution of curve $q(t) \in M_n$:

$$S(q) = \int_{\mathbb{R}} \frac{1}{2}\sum_{i,j} g_{ij}(q)\,\dot{q}_i \dot{q}_j \, dt$$

Can think of g as defining an inner product on each tangent space $T_\varphi M_n$

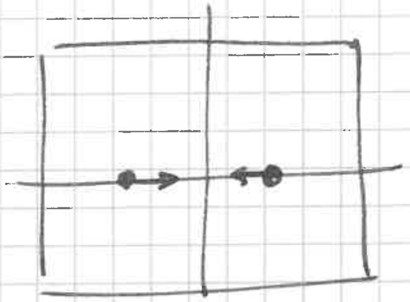
$$v = \sum_i v_i \frac{\partial}{\partial q_i} \qquad u = \sum_j u_j \frac{\partial}{\partial q_j}$$

$$\langle v, u \rangle = \sum_{i,j} g_{ij}(q)\, v_i u_j$$ i.e. equip M_n with a Riemannian metric g.

Critical pts of S are geodesics in (M_n, g) (locally length minimizing curves, parametrised with constant speed).

2-lump scattering

Say at $t=0$ $\qquad w(z) = \dfrac{1}{z^2 - b}$ $\qquad b(0) = 1$, $\quad \dot{b}(0) = -1$



What is the subsequent motion in M_2?

M_2: $\qquad w(z) = \dfrac{a_1 z + a_2}{z^2 + b_1 z + b_2}$

Symmetries: $\qquad w(z) \longmapsto \overline{w}(\bar{z})$

$\big((x,y) \mapsto (x,-y), \quad (\varphi_1, \varphi_2, \varphi_3) \mapsto (\varphi_1, -\varphi_2, \varphi_3) \big)$

Fixed pt set: $\qquad a_1, a_2, b_1, b_2 \in \mathbb{R}$.

$$w(-z) = w(z)$$

$\big((x,y) \mapsto (-x,-y), \quad \varphi \longmapsto \varphi \big)$

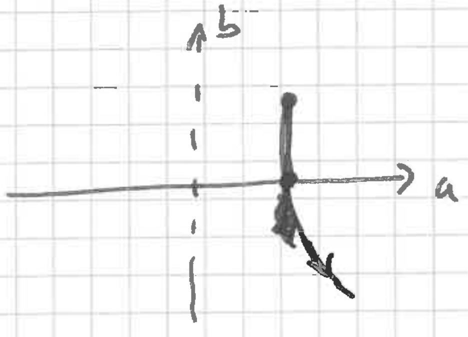
Fixed pt set: $\qquad a_1 = b_1 = 0$.

Together, generate K_4 $\qquad$ Fix $\qquad M_2^{K_4} \subset M_2$

$w(z) = \dfrac{a}{z^2 - b}$ $\qquad (a,b) \in \mathbb{R}^\times \times \mathbb{R}$.

Initial data in and tangent to $M_2^{K_4}$.

But Fix of any group of isometries is a totally geodesic submanifold $\Rightarrow$ geodesic stays in $M_2^{K_4}$.

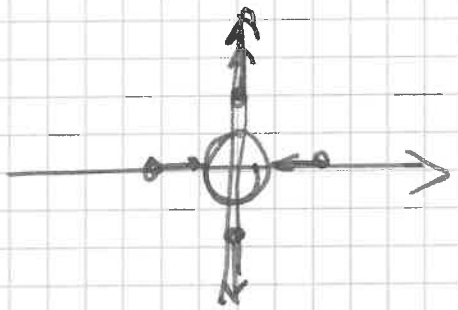


Assume geodesic crosses the a axis (certainly does if $b(0)$ is small enough).

What happens to the lump positions?

$b(t) > 0$ lumps at $z = \pm\sqrt{b(t)}$

$b(t) < 0$ lumps at $z = \pm i\sqrt{-b(t)}$



Lumps scatter through $90°$!!

Very robust feature of topological soliton dynamics
(vortices and monopoles behave analogously)

N.B. caveat: g isn't really well defined

eg $n=1$ $w(z) = \dfrac{q_1 + iq_2}{z + q_3 + iq_4}$

What is $g_{11}(q)$?